THE HEFFERNAN GROUP

P.O. Box 1180
Surrey Hills North VIC 3127
ABN 20 607 374 020
Phone 9836 5021
Fax 9836 5025

SPECIALIST MATHS TRIAL EXAMINATION 1 SOLUTIONS 2006

Question 1

a. $v = \sin(3t) - \frac{t}{2}$

$a = 3\cos(3t) - \frac{1}{2}$

(1 mark)

b. $R = ma$ where R is the magnitude of the resultant force.

$R = 2\left(3\cos(3t) - \frac{1}{2}\right)$

$= 6\cos(3t) - 1$ **(1 mark)**

R is a maximum when $\cos(3t) = 1$; that is, when $\cos(3t)$ equals its maximum value.
So, the maximum value of R is given by

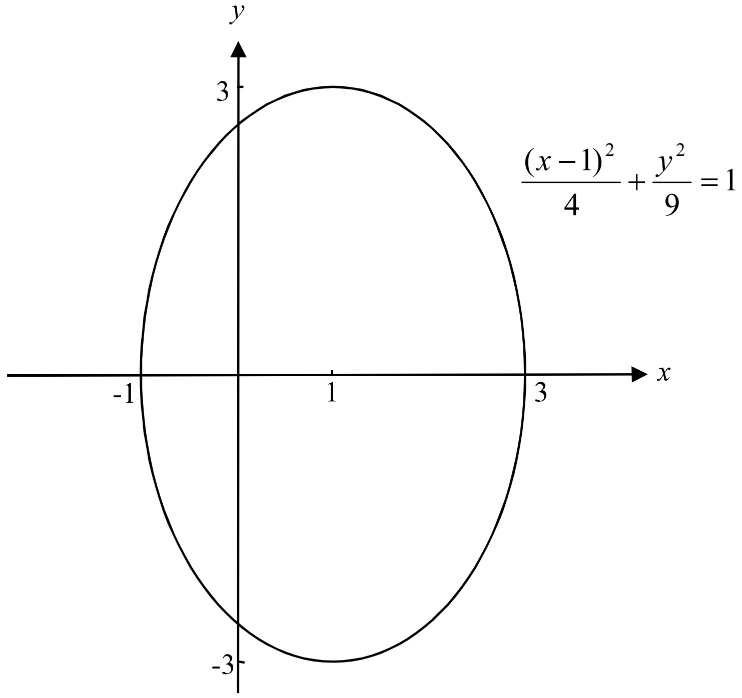
$R = 6 \times 1 - 1$

$= 5$ Newtons

(1 mark)

Question 2

a.



(1 mark) correct shape with points $(-1,0)(3,0),(1,-3)(1,3)$ shown
(1 mark) correct centre

b. $\frac{(x-1)^2}{4}+\frac{y^2}{9}=1$

$9(x-1)^2+4y^2=36$

$18(x-1)+8y\frac{dy}{dx}=0$ (implicit differentiation)

(1 mark) for differentiating x term and constant term
(1 mark) for differentiating y term

$8y\frac{dy}{dx}=-18(x-1)$

$\frac{dy}{dx}=\frac{-18(x-1)}{8y}$

$=\frac{-9(x-1)}{4y}$ **(1 mark)**

c. If $y>0$ then $4y>0$ __________(A)

For $x\in(-1,1)$, $x-1<0$
so $-9(x-1)>0$ __________(B) **(1 mark)**
Using (A) and (B), we have

$\frac{-9(x-1)}{4y}>0$

Therefore $\frac{dy}{dx}>0$ since $\frac{dy}{dx}=\frac{-9(x-1)}{4y}$

(1 mark)

So for $y>0,\frac{dy}{dx}>0$ for $x\in(-1,1)$.

Question 3

a. Let $y = \arctan\left(e^{2x}\right)$

$= \arctan(u)$ where $u = e^{2x}$

$$\frac{dy}{du} = \frac{1}{1+u^2} \qquad \frac{du}{dx} = 2e^{2x}$$

$$= \frac{1}{1+e^{4x}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{(chain rule)}$$

$$= \frac{1}{1+e^{4x}} \times 2e^{2x}$$

$$= \frac{2e^{2x}}{1+e^{4x}}$$

as required. **(1 mark)**

b. From **a.** $\frac{d}{dx}\left(\arctan\left(e^{2x}\right)\right) = \frac{2e^{2x}}{1+e^{4x}}$

so, $\int \frac{d}{dx}\left(\arctan\left(e^{2x}\right)\right)dx = \int \frac{2e^{2x}}{1+e^{4x}}\,dx$

$\arctan\left(e^{2x}\right) + c = 2\int \frac{e^{2x}}{1+e^{4x}}\,dx \quad c$ is a constant

Now $\int_0^{\log_e(5)} \frac{e^{2x}}{1+e^{4x}}\,dx = \frac{1}{2}\left[\arctan\left(e^{2x}\right)\right]_0^{\log_e 5}$ **(1 mark)**

$$= \frac{1}{2}\left\{\arctan\left(e^{2\log_e(5)}\right) - \arctan\left(e^0\right)\right\}$$

$$= \frac{1}{2}\left\{\arctan\left(e^{\log_e\left(5^2\right)}\right) - \arctan(1)\right\}$$ **(1 mark)**

$$= \frac{1}{2}\left(\arctan(25) - \frac{\pi}{4}\right)$$

(1 mark)

Question 4

a. If $z=\sqrt{3}i$ is a solution to the equation $z^4-2z^3+5z^2-6z+a=0$ then

$$\left(\sqrt{3}i\right)^4-2\left(\sqrt{3}i\right)^3+5\left(\sqrt{3}i\right)^2-6\left(\sqrt{3}i\right)+a=0$$
$$9+6\sqrt{3}i-15-6\sqrt{3}i+a=0$$
$$-6+a=0$$
$$a=6$$

(1 mark)

b. Since all the coefficients of the equation are real, one other solution is $z=-\sqrt{3}i$ since the solutions occur in conjugate pairs (conjugate root theorem).

Now $\left(z-\sqrt{3}i\right)\left(z+\sqrt{3}i\right)=z^2+3$ is a quadratic factor.

(1 mark)

<u>Method 1</u>

Let $p(z)=z^4-2z^3+5z^2-6z+6$

$$=\left(z^2+3\right)z^2+\left(z^2+3\right)\left(-2z\right)+\left(z^2+3\right)(2)$$
$$=\left(z^2+3\right)\left(z^2-2z+2\right)$$

(1 mark)

<u>Method 2</u>

$$\begin{array}{r|l} & z^2-2z+2 \\ \hline z^2+3 & z^4-2z^3+5z^2-6z+6 \\ & z^4 \qquad\;\; +3z^2 \\ \hline & -2z^3+2z^2-6z \\ & -2z^3 \qquad\;\; -6z \\ \hline & 2z^2 \quad +6 \\ & 2z^2 \quad +6 \end{array}$$

$z^4-2z^3+5z^2-6z+6$

$=\left(z^2+3\right)\left(z^2-2z+2\right)$

(1 mark)

Now z^2-2z+2

$=\left(\left(z^2-2z+1\right)-1+2\right)$

$=\left(z-1\right)^2+1$

$=\left(z-1\right)^2-i^2$

$=\left(z-1-i\right)\left(z-1+i\right)$

All the solutions to $p(z)=0$ are therefore $z=\pm\sqrt{3}i$ and $z=1\pm i$.

(1 mark)

Question 5

a. $$\int\left(\frac{\sec(2x)}{\tan(2x)}\right)^2 dx = \int \frac{\sec^2(2x)}{\tan^2(2x)}dx$$

$$= \frac{1}{2}\int \frac{du}{dx}u^{-2}\,dx$$

$u = \tan(2x)$

$\frac{du}{dx} = 2\sec^2(2x)$

$$= \frac{1}{2}\int u^{-2}du$$

$$= -\frac{1}{2}u^{-1} + c$$

$$= \frac{-1}{2\tan(2x)} + c$$

(1 mark)

b. $$\int_0^1 \frac{x}{\sqrt{2-x}}dx = \int_2^1 u^{-\frac{1}{2}} \times -1\frac{du}{dx} \times (2-u)dx$$

$u = 2 - x$

$\frac{du}{dx} = -1$

$x = 2 - u$

$x = 1,\ u = 1$

$x = 0,\ u = 2$

$$= -1\int_2^1\left(2u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right)du$$ **(1 mark)** for integrand

$$= \left[4u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^2$$ **(1 mark)** for terminals

$$= \left\{\left(4\sqrt{2} - \frac{2}{3}2^{\frac{3}{2}}\right) - \left(4 - \frac{2}{3}\right)\right\}$$

$$= \sqrt{2}\left(4 - \frac{4}{3}\right) - \frac{10}{3}$$

$$= \frac{8\sqrt{2}}{3} - \frac{10}{3}$$

$$= \frac{2\left(4\sqrt{2} - 5\right)}{3}$$

(1 mark)

Question 6

$$\underset{\sim}{u} = \underset{\sim}{i} + \sqrt{2}\,\underset{\sim}{j} + \underset{\sim}{k}$$

$$\underset{\sim}{v} = \underset{\sim}{i} + a\,\underset{\sim}{j} - \underset{\sim}{k}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\left(\frac{\pi}{3}\right)$$ **(1 mark)**

$$1 + \sqrt{2}a - 1 = \sqrt{1+2+1}\sqrt{1+a^2+1} \times \frac{1}{2}$$

$$\sqrt{2}a = \sqrt{2+a^2} \quad -(*)$$ **(1 mark)**

$$2a^2 = a^2 + 2 \quad \text{(Square both sides)}$$

$$a^2 = 2$$

$$a = \pm\sqrt{2}$$

Check $a = \sqrt{2}$ in $-(*)$

$$LS = \sqrt{2} \times \sqrt{2}$$
$$= 2$$
$$RS = \sqrt{2+2}$$
$$= 2$$

Check $a = -\sqrt{2}$ in $-(*)$

$$LS = \sqrt{2} \times -\sqrt{2}$$
$$= -2$$
$$RS = \sqrt{2+2}$$
$$= 2$$

$LS \neq RS$ so reject $a = -\sqrt{2}$

So $a = \sqrt{2}$

(1 mark)

(1 mark) for rejecting $a = -\sqrt{2}$.
(Note – when you square both sides of an equation it is important that you verify any resulting solutions.)

Question 7

$$\frac{dy}{dx} = \frac{x^2 + 7}{x^2 + 4}$$

$$y = \int \frac{x^2 + 7}{x^2 + 4}\, dx$$

$$= \int \left(\frac{x^2 + 4}{x^2 + 4} + \frac{3}{x^2 + 4} \right) dx$$

$$= \int \left(1 + \frac{3}{2} \times \frac{2}{x^2 + 4} \right) dx$$ **(1 mark)**

$$y = x + \frac{3}{2}\arctan\left(\frac{x}{2}\right) + c$$ **(1 mark)**

Now $y(0) = 0$

so $c = 0$

$$y = x + \frac{3}{2}\arctan\left(\frac{x}{2}\right)$$

(1 mark)

Question 8

$$\text{volume} = \pi \int_0^{\frac{\pi}{2}} y^2\, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(2 - 2\sin(x)\right)^2 dx$$ **(1 mark)**

$$= \pi \int_0^{\frac{\pi}{2}} \left(4 - 8\sin(x) + 4\sin^2(x)\right) dx$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \left(1 - 2\sin(x) + \frac{1}{2}\left(1 - \cos 2x\right) \right) dx$$ **(1 mark)**

$$= 4\pi \left[\frac{3x}{2} + 2\cos(x) - \frac{1}{4}\sin(2x) \right]_0^{\frac{\pi}{2}}$$ **(1 mark)**

$$= 4\pi \left\{ \left(\frac{3\pi}{4} + 0 - 0 \right) - \left(0 + 2 - 0\right) \right\}$$

$$= 4\pi \left(\frac{3\pi}{4} - 2 \right)$$

$$= \pi(3\pi - 8)\ \text{cubic units}$$

(1 mark)

Question 9

a. **i.**

$$y = \frac{1}{x^2 - 2x - 3}$$
$$= \left(x^2 - 2x - 3\right)^{-1}$$
$$\frac{dy}{dx} = -1\left(x^2 - 2x - 3\right)^{-2} \times (2x - 2)$$
$$= \frac{2(1 - x)}{\left(x^2 - 2x - 3\right)^2}$$

(1 mark)

For a stationary point $\frac{dy}{dx} = 0$

$$2(1 - x) = 0$$
$$x = 1$$

When $x = 1$,

$$y = \frac{1}{(1)^2 - 2(1) - 3}$$
$$= \frac{1}{-4}$$

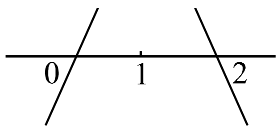
$\left(1, -\frac{1}{4}\right)$ is the stationary point.

(1 mark)

ii. For $x = 0$, $\frac{dy}{dx} = \frac{2}{9}$
> 0

For $x = 2$, $\frac{dy}{dx} = \frac{-2}{9}$
< 0

There is a maximum turning point at $\left(1, -\frac{1}{4}\right)$.



(1 mark)

b. $$\frac{1}{x^2-2x-3}=\frac{1}{(x-3)(x+1)}$$

Let $\frac{1}{(x-3)(x+1)}\equiv\frac{A}{(x-3)}+\frac{B}{(x+1)}$

$$\equiv\frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

True iff $\quad 1\equiv A(x+1)+B(x-3)$ **(1 mark)**

Put $x=-1,\quad 1=-4B,\quad B=-\frac{1}{4}$

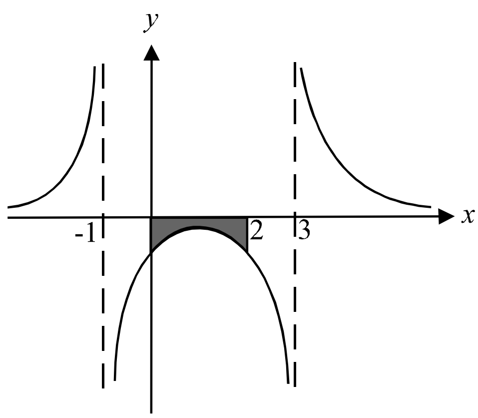
Put $x=3,\quad 1=4A,\quad A=\frac{1}{4}$

So, $\frac{1}{x^2-2x-3}=\frac{1}{4(x-3)}-\frac{1}{4(x+1)}$ **(1 mark)**

(Check $\frac{1}{4(x-3)}-\frac{1}{4(x+1)}=\frac{x+1-(x-3)}{4(x-3)(x+1)}$

$=\frac{1}{(x-3)(x+1)}$)

c. Do a fast sketch.
From **a.** we know that
there is a max. tp. at $\left(1,-\frac{1}{4}\right)$
and no other stationary points.
There are asymptotes at $x=3$ and $x=-1$.
The area required is shaded in the diagram.



$$\text{area}=-\int_0^2\frac{1}{(x-3)(x+1)}\,dx$$

$$=-\frac{1}{4}\int_0^2\left(\frac{1}{x-3}-\frac{1}{x+1}\right)dx \text{ from part b.}$$ **(1 mark)**

$$=-\frac{1}{4}\left[\log_e|x-3|-\log_e|x+1|\right]_0^2$$

$$=-\frac{1}{4}\{(\log_e(1)-\log_e(3))-(\log_e(3)-\log_e(1))\}$$

$$=-\frac{1}{4}(-2\log_e(3))$$

$$=\frac{1}{2}\log_e(3)\text{square units}$$

(1 mark)