Year 2006 VCE Specialist Mathematics Examination 2 Suggested Solutions



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Specialist Mathematics Examination 2 Solutions 2006 SECTION 1

Question 1 Answer C

The hyperbola $\frac{(x-2)^2}{9} - \frac{(y-1)^2}{4} = 1$ has asymptotes when the right hand side is zero, $\frac{(x-2)^2}{9} - \frac{(y-1)^2}{4} = 0$ taking the square root of both sides $\frac{(x-2)}{3} = \pm \frac{(y-1)}{2}$ $2(x-2) = \pm 3(y-1)$ 2x-4 = 3y-3 2x-4 = -3y+32x-3y=1 and 2x+3y=7 as the equation of the asymptotes.

Question 2 Answer E

The function $f(x) = \frac{1}{(x-4)(x+2)}$ has vertical asymptotes when the denominator is zero, that is when (x-4)(x+2) = 0 at x = 4 and x = -2now $(x-4)(x+2) = x^2 - 2x - 8$ so f'(x) = 0 when 2x - 2 = 0 at x = 1and $f(1) = \frac{1}{-3 \times 3} = -\frac{1}{9}$, there is a turning point at $\left(1, -\frac{1}{9}\right)$

Question 3 Answer D

r(t) = (1+t)i + (1-t)j is the position vector, the parametric equations are (1) x = 1+t (2) y = 1-t adding (1)+(2) gives x+y=2 so y = -x+2

Question 4 Answer C

Let z = a + bi where $a, b \in R$ and a > b > 0now $-i(a+bi) = i^3(a+bi)$ is a rotation of z by 270° anticlockwise from z, or a rotation 90° clockwise from z.

Question 5 Answer B

 $z^{5} = -a = a \operatorname{cis}(\pi) = a \operatorname{cis}(\pi + 2k\pi)$ $z = a^{\frac{1}{5}} \operatorname{cis}\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right) \quad \text{when} \quad k = 0 \quad \text{one solution is} \quad z = a^{\frac{1}{5}} \operatorname{cis}\left(\frac{\pi}{5}\right)$

when k = 2 another one of the five solutions is $z = a^{\frac{1}{5}} cis(\pi) = -a^{\frac{1}{5}}$

Question 6 Answer B

The required region is the inside of a circle with centre c = a + 2i and radius r = 1

$$|z-c| \le r$$
 so $|z-(a+2i)| \le 1$

Question 7 Answer E

Checking each alternative

A. If z = x + iy, $\overline{z} = x - iy$, $z + \overline{z} = 0$ so 2x = 0

B. If
$$z = x + iy$$
 $\operatorname{Re}(z) = x$, $\operatorname{Im}(z) = y$ $3\operatorname{Re}(z) = \operatorname{Im}(z) \implies 3x = y$

C.
$$z = i\overline{z}$$
, $x + iy = i(x - iy) = ix - i^2 y = ix + y$

D.
$$x - 2y = 0$$

all of A.B.C. and D. pass through the origin, x = 0 y = 0

E. x + y = 1 does not pass through the origin.

Question 8 Answer A

Given $2x^3 - y^2 = 7$ using implicit differentiation $\frac{d}{dx}$

$$\frac{d}{dx}(2x^3) - \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

 $6x^2 - 2y\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = \frac{6x^2}{2y} = \frac{3x^2}{y}$ now at the point where y = -3, $2x^3 - 9 = 7$

$$2x^3 = 16$$
, $x^3 = 8$, $x = 2$ the point is $(2, -3)$ $\frac{dy}{dx}\Big|_{(2, -3)} = \frac{3 \times 4}{-3} = -4$

Question 9 Answer C

$$\int_{a}^{b} x (x^{2} + 1)^{5} dx \quad \text{let } u = x^{2} + 1 \quad \frac{du}{dx} = 2x$$

change terminals when x = b $u = b^2 + 1$ and when x = a $u = a^2 + 1$

 $\int_{a}^{b} x \left(x^{2}+1\right)^{5} dx = \frac{1}{2} \int_{a^{2}+1}^{b^{2}+1} u^{5} du$

Question 10 Answer B

x kg has dissolved, 8-x kg is undissolved, $\frac{dx}{dt} = \frac{5}{100}(8-x) = \frac{8-x}{20}$ for 0 < x < 8

Question 11 Answer C

At x=0 the gradient of the curves is zero, at the point $\left(\frac{1}{2},\frac{1}{2}\right)$ the gradient is negative, and less than one, option C. is the only one which satisfies this.

Question 12 Answer D The velocity $v = \sin(2x)$ then $\frac{dv}{dx} = 2\cos(2x)$ then acceleration $a = v\frac{dv}{dx} = 2\sin(2x)\cos(2x) = \sin(4x)$

Question 13 Answer E

 $\underline{r}(t) = (2t - 10)\underline{i} + 3\underline{j} \qquad \underline{s}(t) = 2\underline{i} + (t - 1)\underline{j} \text{ for } t \ge 0$

checking each alternative

$\underline{r}(1) = -8\underline{i} + 3$	į	$\underline{s}(1) = 2\underline{i}$
r(4) = -2i + 3	j Ž	$\underline{s}(4) = 2\underline{i} + 3\underline{j}$
r(5) = 3	j	$\underline{s}(5) = 2\underline{i} - 4\underline{j}$
$\underline{r}(6) = 2\underline{i} + 3\underline{j}$		$\underline{s}(6) = 2\underline{i} + 5\underline{j}$

so R and S are never in the same position.

Question 14 Answer B

 $\underline{r}(t) = (3-t)\underline{i} - 6\sqrt{t}\underline{j} + 5\underline{k}$ differentiating the position vector with respect to t gives the velocity vector $\dot{r}(t) = -\underline{i} - \frac{3}{\sqrt{t}}\underline{j}$ at t = 9 $\dot{r}(9) = -\underline{i} - \underline{j}$, this is the direction of motion of the particle when t = 9

Question 15 Answer D

In the parallelogram a + b + c + d = 0and b = -d and a = -c so a + c = 0

Question 16 Answer D

Let
$$\underline{a} = 5\underline{i} + \underline{j} - 2\underline{k}$$
 and $\underline{b} = \frac{1}{\sqrt{29}} \left(2\underline{i} - 4\underline{j} + 3\underline{k} \right)$
 $\underline{a} \cdot \underline{b} = 10 - 4 - 6 = 0$ and $|\underline{b}| = \frac{\sqrt{4 + 16 + 9}}{\sqrt{29}} = 1$

Option **D.** is correct, the vectors in **A. B.** and **C.** are not unit vectors and the vector in **E.** is not perpendicular to a

Question 17 Answer B

If u = i + j v = i + 2j + 2k $|u| = \sqrt{2}$ $|v| = \sqrt{1 + 4 + 4} = 3$ $u \cdot v = 1 + 2 = 3$ $\cos(\theta) = \frac{u \cdot v}{|u||v|} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$ so $\theta = 45^{\circ}$

Question 18

Answer A



so
$$R = mg \cos(\theta)$$
 now $m = 20 \text{ kg}$, $\theta = 60^{\circ}$
 $R = 20g \cos(60^{\circ}) = 10g$

resolving perpendicular to the plane $R - mg\cos(\theta) = 0$

Qu	estion 19	Answer A
A.	$12\cos(30^\circ) = 10a$	$a = 1.04 \mathrm{m/s^2}$
B.	$10\cos(10^\circ) = 10a$	$a = 0.98 \mathrm{m/s^2}$
C.	10 = 10a	$a = 1.0 \mathrm{m/s^2}$
D.	$10\cos(20^\circ) = 10a$	$a = 0.94 \mathrm{m/s^2}$
E.	$12\cos(45^\circ) = 10a$	$a = 0.85 \mathrm{m/s^2}$

the largest is **A**.

Question 20 Answer D

By Lami's Theorem

$$\frac{F_3}{\cos(150^\circ)} = \frac{F_2}{\cos(150^\circ)} = \frac{F_1}{\cos(60^\circ)}$$
$$F_3 = F_2 = \frac{F_1 \cos(150^\circ)}{\cos(60^\circ)} = \frac{\frac{1}{2}F_1}{\frac{\sqrt{3}}{2}} = \frac{F_1}{\sqrt{3}} = \frac{\sqrt{3}}{3}F_1$$

Question 21 Answer A

resolving perpendicular to the plane $N - 8g\cos(30^\circ) = 0 \implies N = 8g\cos(30^\circ) = 4g\sqrt{3}$ resolving parallel to the plane $F - 8g\sin(30^\circ) = 0 \implies F = 8g\sin(30^\circ) = 4g$ for equilibrium to be maintained $F \le \mu N$

$$4g \le \mu 4g\sqrt{3}$$
$$\mu \ge \frac{1}{\sqrt{3}}$$

Question 22 Answer E

resolving around the 2 kg block, moving upward (1) T - 2g = 2a

resolving around the 5kg block, moving downwards (2) 5g - T = 5a adding (1)+(2)

gives
$$3g = 7a$$
 so $a = \frac{3g}{7}$

SECTION 2

Question 1

a.
$$V = \pi \int_{0}^{5} \frac{36x^{2}}{1+x^{3}} dx$$

b. let $u = 1 + x^{3} \frac{du}{dx} = 3x^{2}$ when $x = 5$ $u = 126$ and if $x = 0$ $u = 1$
 $V = 12\pi \int_{1}^{126} \frac{1}{u} du$

c. $V = 12\pi \left[\log_e(u) \right]_1^{126} = 12\pi \left[\log_e(126) - \log_e(1) \right]$ $V = 12\pi \log_e(126) = 182.32$ $V = 182 \,\mathrm{cm}^3$

d. given
$$\frac{dx}{dt} = 2 \text{ cm/sec}$$
 and $\frac{dy}{dt} = \frac{6-3x^3}{(1+x^3)^{\frac{3}{2}}}$
then $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{2(6-3x^3)}{(1+x^3)^{\frac{3}{2}}}$
e. $A = \pi y^2$ then $\frac{dA}{dy} = 2\pi y = \frac{12\pi x}{\sqrt{1+x^3}}$
 $\frac{dA}{dt} = \frac{dA}{dy} \cdot \frac{dy}{dt} = \frac{24\pi x (6-3x^3)}{(1+x^3)^2}$ so $a = 24\pi$, $b = 1$, $c = 2$

f. A is a maximum when
$$\frac{dA}{dx} = 0$$
 or when $\frac{dy}{dx} = 0$
so $6 - 3x^3 = 0$ when $3x^3 = 6$ $x^3 = 2$ so $x = \sqrt[3]{2}$

so the angles ADC and ABC add to 180° , they are supplementary angles.

d.
$$p = \overrightarrow{OP} = 2i$$

$$\overrightarrow{PA} = a - p = -3i - 4j$$

$$|\overrightarrow{PA}| = \sqrt{9 + 16} = 5$$

$$\overrightarrow{PC} = c - p = 3i - 4j$$

$$|\overrightarrow{PC}| = \sqrt{9 + 16} = 5$$

$$\cos(\angle APC) = \frac{\overrightarrow{PA} \cdot \overrightarrow{PC}}{|\overrightarrow{PA}||\overrightarrow{PC}|} = \frac{-9 + 16}{25} = \frac{7}{25}$$

$$\det \alpha = \angle APC \text{ and } \beta = \angle ADC \text{ to show}$$

$$\cos(\alpha) = \cos(2\beta)$$

$$\cos(\alpha) = \frac{7}{25} \cos(2\beta) = \cos^2(\beta) - \sin^2(\beta) = \frac{16}{25} - \frac{9}{25} = \frac{7}{25} \text{ shown}$$

a. i.
$$F = 105,600N$$
 , $m = 48,000 \text{ kg}$ using $F = ma$
 $a = \frac{F}{m} = \frac{105,600}{48,000} = 2.2 \text{ ms}^{-2}$
ii. $u = 0$, $a = 2.2 \text{ ms}^{-2}$, $v = 70 \text{ ms}^{-1}$ $t = ?$ using $v = u + at$
 $70 = 0 + 2.2t$
 $t = \frac{70}{2.2} = 31.8 \text{ s}$
iii. using $v^2 = u^2 + 2as$
 $70^2 = 0 + 2 \times 2.2 \times 5$
 $s = \frac{70^2}{2 \times 2.2} = 1113.64 \text{ m} = 1114 \text{ m}$



ii. resolving (1) $T - R - W \sin(10^\circ) = 0$

$$(2) \quad L - W \cos(10^\circ) = 0$$

iii.
$$L = W \cos(10^\circ) = 48,000 \times 9.8 \times \cos(10^\circ) = 463253.57 = 463,254N$$

c. i.
$$48,000a = -5v^2 - 500(80 - v) - 80,000$$
 all three forces oppose the motion of the plane.

ii.
$$48,000a = -5v^2 - 40,000 + 500v - 80,000$$

 $= -5v^2 + 500v - 120,000$

$$a = v \frac{dv}{dx} = \frac{-5v^2 + 500v - 120,000}{48,000}$$

$$x = \int_{80}^{10} \frac{48,000 \ v \ dv}{-5v^2 + 500v - 120,000}$$

$$x = \int_{80}^{10} \frac{48,000 \ v \ dv}{-5v^2 + 500v - 120,000}$$

$$x = \int_{80}^{10} \frac{48,000 \ v \ dv}{-5v^2 + 500v - 120,000}$$

iii. 1385 m

a.
$$\frac{dy}{dt} = 1 - y$$
 inverting $\frac{dt}{dy} = \frac{1}{1 - y}$ integrating with respect to y
 $t = \int \frac{1}{1 - y} dy = -\log_e |(1 - y)| + c \implies \log_e |1 - y| = c - t$ where c is the constant of integration
b. $\dot{x}(t) = \frac{1}{y(t)} \dot{x} + (1 - y(t)) \dot{y} = \dot{x}(t) \dot{x} + \dot{y}(t) \dot{y}$ so $\frac{dx}{dt} = \frac{1}{y}$ now $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = y(1 - y)$
c. i. Given $\frac{dy}{dx} = y(1 - y)$ using implicit differentiation

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left[y(1-y)\right] = \frac{d}{dy}\left[\left(y-y^2\right)\right]\frac{dy}{dx} = (1-2y)\frac{dy}{dx} = (1-2y)y(1-y)$$

ii. For
$$0 < y < 1$$
 if $\frac{d^2 y}{dx^2} = 0$ then $y = \frac{1}{2}$ for an inflexion point.
if $y < \frac{1}{2}$ then $\frac{dy}{dx} > 0$ and if $y > \frac{1}{2}$ then $\frac{dy}{dx} > 0$ so $\left(0, \frac{1}{2}\right)$ is an inflexion point

d.



e.
$$\frac{dy}{dx} = y(1-y)$$
 $y(0) = 2$ $h = \frac{1}{4}$
 $y_1 = y_0 + hf(x_0, y_0) = 2 + \frac{1}{4}(2)(1-2) = \frac{3}{2}$
 $y_2 = y_1 + hf(x_1, y_1) = \frac{3}{2} + \frac{1}{4}(\frac{3}{2})(1-\frac{3}{2}) = \frac{21}{16}$

a. i.

c.



ii.
$$\left\{ z: \left|z-\overline{z_1}\right| = \left|z+\overline{z_1}\right| \right\}$$

b. Using $\cos(2A) = 2\cos^2(A) - 1$ let $A = \frac{\pi}{8}$

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = 2\cos^{2}\left(\frac{\pi}{8}\right) - 1$$

$$2\cos^{2}\left(\frac{\pi}{8}\right) = 1 + \frac{\sqrt{2}}{2} = \frac{2 + \sqrt{2}}{2}$$

$$\cos^{2}\left(\frac{\pi}{8}\right) = \frac{2 + \sqrt{2}}{4}$$

$$\cos\left(\frac{\pi}{8}\right) = \pm \frac{\sqrt{2 + \sqrt{2}}}{2} \quad \text{but } \cos\left(\frac{\pi}{8}\right) > 0 \quad \text{so we need to take the positive}$$

$$\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2 + \sqrt{2}}}{2}$$

$$\sin^{2}\left(\frac{\pi}{8}\right) + \cos^{2}\left(\frac{\pi}{8}\right) = 1$$

$$\sin^{2}\left(\frac{\pi}{8}\right) = 1 - \cos^{2}\left(\frac{\pi}{8}\right) = 1 - \frac{2 + \sqrt{2}}{4} = \frac{4 - \left(2 + \sqrt{2}\right)}{4} = \frac{2 - \sqrt{2}}{4} \quad \text{and } \sin\left(\frac{\pi}{8}\right) > 0$$

$$\sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2 - \sqrt{2}}}{4}$$

$$\mathbf{d.} \quad \left(\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2}i\right)^{\prime} = \left[\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right]^{7} = \operatorname{cis}\left(\frac{7\pi}{8}\right)$$
$$\mathbf{e.} \quad \operatorname{if}\left(\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2}i\right)^{n} \text{ is a real number}$$
$$= \left[\operatorname{cis}\left(\frac{\pi}{8}\right)\right]^{n} = \operatorname{cis}\left(\frac{n\pi}{8}\right) = \pm 1 + 0i = \cos\left(\frac{n\pi}{8}\right) + i\sin\left(\frac{n\pi}{8}\right)$$
$$\operatorname{so} \quad \sin\left(\frac{n\pi}{8}\right) = 0 \qquad \frac{n\pi}{8} = k\pi$$
$$n = 8k \text{ where } k \in J \text{ or } n = 0, \pm 8, \pm 16, \dots$$

f. The roots of $z^8 = 1$ are all on a circle of radius one, all the roots are equally separated by $\frac{2\pi}{8} = 45^\circ$, there are 8 roots.





Mathematics 2007

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