SPECIALIST MATHEMATICS EXAM 2: SOLUTIONS

Section 1: Multiple Choice

Answers

1. A	2. E	3. D	4. B	5. A
6. C	7. C	8. A	9. E	10. B
11. E	12. E	13. C	14. B	15. D
16. B	17. A	18. C	19. B	20. E
21. D	22. D			

Question 1 **Answer A**

$(x+1)^2 - (y+2)^2 = 1$

Hyperbola of the form $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ where $a = 1, \quad b = 1, \quad h = -1, \quad k = -2$

Asymptotes have equations $y = \pm\dfrac{b}{a}(x-h) + k$, $\Rightarrow \quad y = \pm 1(x+1) - 2$

x-intercepts occur where $y = 0$

$$0 = \pm(x+1) - 2$$

$-x - 1 = 2$ or $x + 1 = 2$

$x = -3$ or $x = 1$

Question 2 **Answer E**

$$\begin{aligned} u + \bar{v} &= (-1+i) + (1+i) \\ &= 2i \quad \text{not real} \end{aligned}$$

$$\begin{aligned} u - v &= (-1+i) - (1-i) \\ &= -2 + 2i \quad \text{not real} \end{aligned}$$

$$\begin{aligned} uv &= (-1+i)(1-i) \\ &= -1 + i + i - i^2 \\ &= 2i \quad \text{not real} \end{aligned}$$

$$\begin{aligned} \frac{1}{v} &= \frac{1}{1-i} \\ &= \frac{1}{1-i} \times \frac{1+i}{1+i} \\ &= \frac{1+i}{2} \quad \text{not real} \end{aligned}$$

$$\begin{aligned} \frac{u}{v} &= \frac{-1+i}{1-i} \\ &= \frac{(-1+i)(1+i)}{(1-i)(1+i)} \\ &= \frac{-1 - i + i + i^2}{1 - i^2} \\ &= \frac{-2}{2} \\ &= -1 \quad \text{real number} \end{aligned}$$

Question 3 **Answer D**

The roots of a complex number are equally spaced around the circumference of a circle. The angle between each fourth root will be $\frac{2\pi}{4} = \frac{\pi}{2}$.

Given the argument of one root is $\frac{\pi}{3}$, the arguments of the three other roots will be:

$\theta = \frac{\pi}{3} + \frac{2k\pi}{4}$ where $k = 1, -1, -2$

$\theta = \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}$ $\quad\quad$ $\theta = \frac{\pi}{3} - \frac{\pi}{2} = -\frac{\pi}{6}$ $\quad\quad$ $\theta = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

The modulus of each root must be the same, therefore $r = 16$

The other roots of z are $16\operatorname{cis}\left(\frac{5\pi}{6}\right)$, $16\operatorname{cis}\left(-\frac{\pi}{6}\right)$ and $16\operatorname{cis}\left(-\frac{2\pi}{3}\right)$.

Question 4 **Answer B**

Equation is $(\bar{z} - i)(z + i) = 4$

$$(x - iy - i)(x + iy + i) = 4$$
$$(x - i(y+1))(x + i(y+1)) = 4$$
$$x^2 - i^2(y+1)^2 = 4$$
$$x^2 + (y+1)^2 = 4$$

Circle centre (0, -1), radius 2

Question 5 **Answer A**

$f(x) = \arccos(x)$ has domain $[-1,\ 1]$

This is translated right *m* units and upwards *n* units to give $f(x) = \arccos(x - m) + n$

Since $-1 \le x - m \le 1$

then $m - 1 \le x \le m + 1$

The domain is $[m-1,\ m+1]$

Question 6 **Answer C**

Graph is of the form $a\operatorname{cosec}(nx) + k$

Period is π : $\frac{2\pi}{n} = \pi$

$n = 2$

minimum at $\left(\frac{\pi}{4}, 4\right)$, maximum at $\left(\frac{3\pi}{4}, 0\right)$

Graph has been translated 2 units vertically, $k = 2$

$a = 2$

Equation of curve is $y = 2\operatorname{cosec}(2x) + 2$

Question 7 **Answer C**

Graphs of the form $f(x) = x^m + x^{-n}$ where $m \in \{1, 2\}$ will have a vertical asymptote at $x = 0$.

Graphs A, B, D and E have vertical asymptote at $x = 0$, but graph C does not.

Question 8 **Answer A**

$x = 2\sin(t)$ is translated 1 unit in the negative x direction $\Rightarrow \quad x = 2\sin(t) - 1 \quad ....(1)$

$y = \cos(t)$ is translated 2 units in the positive y direction $\Rightarrow \quad y = \cos(t) + 2 \quad ....(2)$

From (1) and (2)

$$\sin(t) = \frac{x+1}{2} \qquad \text{and} \qquad \cos(t) = y - 2$$

$$\Rightarrow \sin^2(t) = \frac{(x+1)^2}{4} \quad ....(3) \qquad \Rightarrow \cos^2(t) = (y-2)^2 \quad ....(4)$$

Add (3) and (4)

$$\frac{(x+1)^2}{4} + (y-2)^2 = \sin^2(t) + \cos^2(t)$$

$$\frac{(x+1)^2}{4} + (y-2)^2 = 1$$

$$(x+1)^2 + 4(y-2)^2 = 4$$

Question 9 **Answer E**

$$x = 2\sqrt{t+1} \qquad y = t^2 + 1.$$

$$\frac{dx}{dt} = 2 \times \frac{1}{2}(t+1)^{-\frac{1}{2}} \times 1 \qquad \frac{dy}{dt} = 2t$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{t+1}}$$

$$\frac{dt}{dx} = \sqrt{t+1}$$

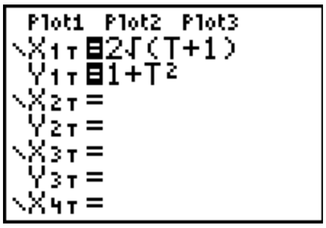
Using the chain rule $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

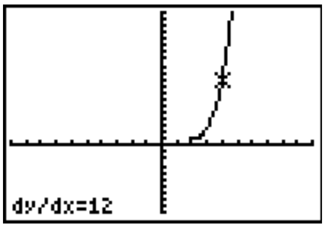
$$\frac{dy}{dx} = 2t \times \sqrt{t+1}$$

When $t = 3$ $\frac{dy}{dx} = 2 \times 3 \times \sqrt{3+1}$

$$\frac{dy}{dx} = 12$$

Can also be done on the calculator using parametric mode





Question 10 **Answer B**

$2y - x^2 e^y + 1 = 0$

Using implicit differentiation

$$2\frac{dy}{dx} - \left(2x e^y + x^2 \frac{dy}{dx} e^y\right) + 0 = 0$$

$$2\frac{dy}{dx} - x^2 \frac{dy}{dx} e^y = 2x e^y$$

$$\frac{dy}{dx}\left(2 - x^2 e^y\right) = 2x e^y$$

$$\frac{dy}{dx} = \frac{2x e^y}{2 - x^2 e^y}$$

Finding the gradient of the tangent at the point (1, 0)

$$\frac{dy}{dx} = \frac{2 \times 1 \times e^0}{2 - 1^2 e^0} = \frac{2}{1} = 2$$

Equation of tangent

$$y - 0 = 2(x - 1)$$

$$y - 2x + 2 = 0$$

Question 11 **Answer** E

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3(2x)\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(2x)\left(1 - \sin^2(2x)\right)dx \qquad \text{Let } u = \sin(2x) \Rightarrow \frac{du}{dx} = 2\cos(2x)$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(1 - \sin^2(2x)\right)\left(2\cos(2x)\,dx\right)$$

Terminals:

When $x = \frac{\pi}{2}$, $u = \sin(2 \times \frac{\pi}{2})$, $u = 0$ When $x = \frac{\pi}{4}$, $u = \sin(2 \times \frac{\pi}{4})$, $u = 1$

$$= \frac{1}{2}\int_{1}^{0} (1 - u^2)\,du$$

$$= -\frac{1}{2}\int_{0}^{1} (1 - u^2)\,du$$

$$= \frac{1}{2}\int_{0}^{1} (u^2 - 1)\,du$$

Question 12 **Answer E**

Rotating shaded region is around the x-axis

$$V = \pi\int_0^2 (y_1)^2\, dx - \pi\int_0^2 (y_2)^2\, dx \qquad \text{where } y_1 = 1 \text{ and } y_2 = e^{x-2}$$

$$V = \pi\int_0^2 1^2\, dx - \pi\int_0^2 \left(e^{x-2}\right)^2 dx$$

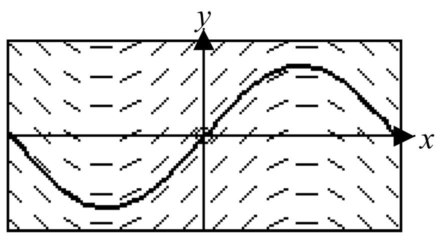
$$V = \pi\left[x\right]_0^2 - \pi\int_0^2 \left(e^{x-2}\right)^2 dx$$

$$V = \pi\left[2-0\right] - \pi\int_0^2 \left(e^{x-2}\right)\left(e^{x-2}\right) dx$$

$$V = 2\pi - \pi\int_0^2 e^{2x-4}\, dx$$

$$V = 2\pi - \pi\int_0^2 e^{2(x-2)}\, dx$$

Question 13 **Answer C**



$y = \sin(x)$

Question 14 **Answer B**

$$y = \int \frac{1}{\log_e(x+1)}\, dx$$
$$= f(x) + c$$

When $x = 1, \quad y = -2$

$$-2 = f(1) + c$$
$$c = -f(1) - 2 \qquad (1)$$

When $x = 2, \quad y = f(2) + c \ \ldots \qquad (2)$

Substitute (1) into (2)

$$y = f(2) - f(1) - 2$$

$$y = \int_1^2 \frac{1}{\log_e(x+1)}\, dx - 2$$

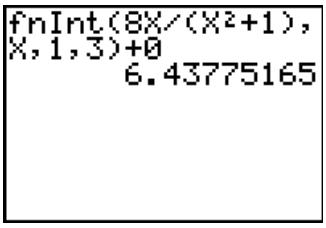
Numerical solution using calculator

```
fnInt(1/ln(X+1),
X,1,2)-2
        -.8815751855
```

Question 15 **Answer D**

$$a = v\frac{dv}{dx}$$
$$= \frac{v^2+1}{8}$$
$$\frac{dv}{dx} = \frac{v^2+1}{8v}$$
$$\frac{dx}{dv} = \frac{8v}{v^2+1}$$
$$x = \int \frac{8v}{v^2+1}\,dv$$

Integral can be evaluated numerically on calculator



$$x = 4\int \frac{2v}{v^2+1}\,dv$$
$$x = 4\log_e(v^2+1) + \text{C}$$

When $x = 0$ $v = 1$ $\Rightarrow$ $0 = 4\log_e(1^2+1) + \text{C}$

$\text{C} = -4\log_e 2$

$$x = 4\log_e(v^2+1) - 4\log_e 2$$
$$x = 4\log_e\left(\frac{v^2+1}{2}\right)$$

When $v = 3$ $x = 4\log_e\left(\frac{3^2+1}{2}\right)$

$x = 4\log_e 5$

$x = 6.4$ metres

Question 16 **Answer B**

Area of trapezium above horizontal axis

$$A = \frac{1}{2}(10+20)\times 20$$
$$= 300$$

Area of trapezium below horizontal axis

$$A = \frac{1}{2}(40+10)\times -10$$
$$= -250$$

After 60 seconds the particle is $300 - 250 = 50$ metres from its starting point.

Question 17 **Answer A**

Toy is travelling under constant acceleration

$u = 4,\ s = 3,\ t = 1.2$

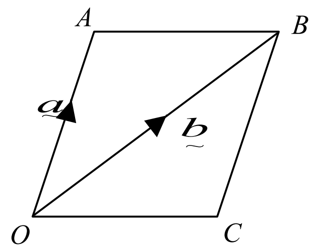
$$s = \frac{1}{2}(u+v)t$$

$$3 = \frac{1}{2}(4+v)\times 1.2$$

$$\frac{3}{0.6} = 4 + v$$

$$v = 1 \text{ m/s}$$

Question 18 **Answer C**



The diagonals of a rhombus are perpendicular, therefore $\vec{AC}.\vec{OB} = 0$

Find a vector expression for $\vec{AC}$ given $\vec{OA} = \underset{\sim}{a}$ and $\vec{OB} = \underset{\sim}{b}$.

$$\begin{aligned}\vec{AC} &= \vec{AO} + \vec{OC} \\ &= -\underset{\sim}{a} + (\underset{\sim}{b} - \underset{\sim}{a}) \\ &= \underset{\sim}{b} - 2\underset{\sim}{a}\end{aligned}$$

$$\vec{AC}.\vec{OB} = (\underset{\sim}{b} - 2\underset{\sim}{a}).\underset{\sim}{b}$$

$$\therefore\quad (\underset{\sim}{b} - 2\underset{\sim}{a}).\underset{\sim}{b} = 0$$

Question 19 **Answer B**

$$\begin{aligned}\underset{\sim}{p}.\hat{\underset{\sim}{q}} &= \frac{\left(\underset{\sim}{i} + 0\underset{\sim}{j} + 2\underset{\sim}{k}\right).\left(5\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right)}{\sqrt{5^2 + 1^2 + 1^2}} \\ &= \frac{5 + 0 - 2}{\sqrt{27}} \\ &= \frac{3}{3\sqrt{3}} \\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

Question 20 **Answer E**

$\underset{\sim}{F} = m\underset{\sim}{a}$

$$\left(3\underset{\sim}{i} - 2\underset{\sim}{j}\right) + \left(\underset{\sim}{i} + 5\underset{\sim}{j}\right) = 0.5\underset{\sim}{a}$$

$$4\underset{\sim}{i} + 3\underset{\sim}{j} = 0.5\underset{\sim}{a}$$

$$\underset{\sim}{a} = \frac{4\underset{\sim}{i} + 3\underset{\sim}{j}}{0.5}$$

$$\underset{\sim}{a} = 8\underset{\sim}{i} + 6\underset{\sim}{j}$$

Question 21 **Answer D**

The forces acting on the decoration are in equilibrium.
Therefore the resultant force is zero.

$$\underset{\sim}{T} + \underset{\sim}{P} + \underset{\sim}{W} = \underset{\sim}{0}$$

$$\underset{\sim}{T} + \underset{\sim}{P} = -\underset{\sim}{W}$$

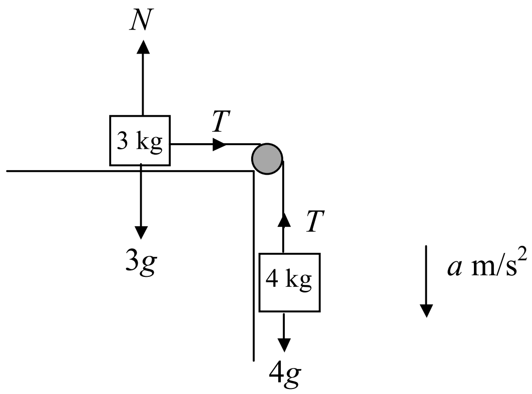
Question 22 **Answer D**

Show forces on diagram
Let T be the tension in the string and a the acceleration of the system

Resolve 3 kg mass horizontally:

$T = 3a$

$a = \frac{T}{3} \quad \ldots(1)$



System is moving downwards
Resolve 4 kg mass vertically:

$4g - T = 4a \quad \ldots(2)$

Substitute (1) into (2)

$$4g - T = 4\left(\frac{T}{3}\right)$$

$$4g = \frac{4T}{3} + T$$

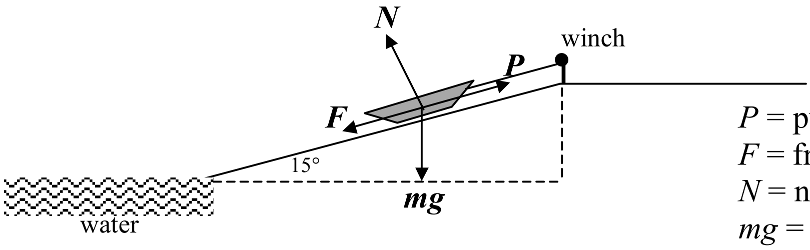
$$4g = \frac{7T}{3}$$

$$T = \frac{12g}{7}$$

Section 2: Extended Answer

Question 1

a.



A1

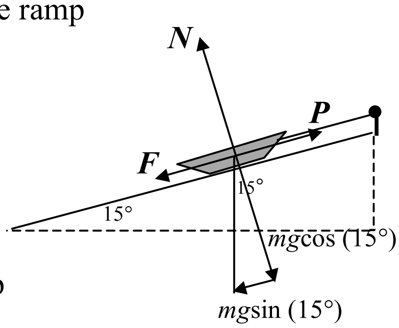
P = pulling force
F = frictional force
N = normal reaction
mg = weight force

b. Resolving forces perpendicular to the ramp

$N = mg\cos(15°) \quad \ldots(1)$

Friction $F = \mu N$

$\Rightarrow F = \mu mg\cos(15°)$



Boat is moving upwards
Resolving forces parallel to the ramp

$P - mg\sin(15°) - F = ma \quad \ldots(2)$ A1

$P - mg\sin(15°) - \mu mg\cos(15°) = ma$ M1

$P - 1000g\sin(15°) - 0.25\times 1000g\cos(15°) = 1000\times 0.1$

$P = 100 + 1000g\sin(15°) + 0.25\times 1000g\cos(15°)$

$P = 100 + 1000g\sin(15°) + 250g\cos(15°)$

$P = 100 + 2536.4 + 2366.5$

$P = 5003$ newtons A1

c. Boat slides down the ramp under the force of gravity.

$mg\sin(15°) - F = ma$ M1

$mg\sin(15°) - \mu mg\cos(15°) = ma$

$a = g\sin(15°) - 0.25g\cos(15°)$ A1

$a = 0.17$ m/s^2 A1

d. $u = 0, \quad s = 6, \quad a = 0.17$

$v^2 = u^2 + 2as$

$v^2 = 0^2 + 2\times 0.17\times 6$ M1

$v = \sqrt{2.04}$

$v = 1.43$ m/s A1

e. $v = u + at$

$1.43 = 0 + 0.17t$

$t = \dfrac{1.43}{0.17}$

$t = 8.4$ seconds A1

Question 2

a. $\underset{\sim}{v} = 20\cos(\theta)\underset{\sim}{i} + (20\sin(\theta) - 9.8t)\underset{\sim}{j}$

When $t = 0$, $\underset{\sim}{v} = 20\cos(\theta)\underset{\sim}{i} + 20\sin(\theta)\underset{\sim}{j}$

$$\left|\underset{\sim}{v}\right| = \sqrt{(20\cos(\theta))^2 + (20\sin(\theta))^2}$$

$$\left|\underset{\sim}{v}\right| = 20\sqrt{(\cos^2(\theta) + \sin^2(\theta))}$$

$$\left|\underset{\sim}{v}\right| = 20 \text{ m/s}$$

The initial speed of the stone is 20m/s A1

b. $\underset{\sim}{r} = \int\left(20\cos(\theta)\underset{\sim}{i} + (20\sin(\theta) - 9.8t)\underset{\sim}{j}\right)dt$

$\underset{\sim}{r} = 20t\cos(\theta)\underset{\sim}{i} + (20t\sin(\theta) - 4.9t^2)\underset{\sim}{j} + C$ C is constant A1

When $t = 0$, $\underset{\sim}{r} = 0\underset{\sim}{i} + 50\underset{\sim}{j}$ $\Rightarrow C = 50\underset{\sim}{j}$

$\underset{\sim}{r} = 20t\cos(\theta)\underset{\sim}{i} + (20t\sin(\theta) - 4.9t^2)\underset{\sim}{j} + 50\underset{\sim}{j}$

$\underset{\sim}{r} = 20t\cos(\theta)\underset{\sim}{i} + (20t\sin(\theta) - 4.9t^2 + 50)\underset{\sim}{j}$ A1

c. Given $\theta = 60°$.

i. Stone reaches its greatest height when the $\underset{\sim}{j}$ component of the velocity is zero.

$20\sin(60°) - 9.8t = 0$

$t = \dfrac{20\sin 60°}{9.8}$

$t = 1.8$ seconds A1

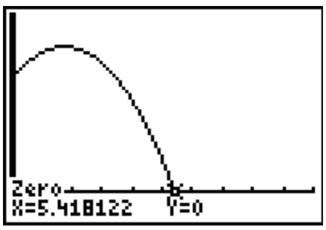
Greatest height is given by the $\underset{\sim}{j}$ component of $\underset{\sim}{r}$ when $t = 1.8$ seconds

$20 \times 1.8 \times \sin(60°) - 4.9(1.8)^2 + 50 = 65.3$ metres A1

ii. Stone is at ground level when $\underset{\sim}{j}$ component of $\underset{\sim}{r}$ is zero

$20t\sin(60°) - 4.9t^2 + 50 = 0$

Solve for t using calculator



$t = 5.42$ seconds M1

Distance from base of building is given by the $\underset{\sim}{i}$ component of $\underset{\sim}{r}$ when $t = 5.42$

$20 \times 5.42 \times \cos(60°) = 54.2$ metres A1

d. Landing place of stone $\underset{\sim}{r} = 25\underset{\sim}{i} + 30\underset{\sim}{j}$

Solve $20t\cos(\theta)\underset{\sim}{i} + \left(20t\sin(\theta) - 4.9t^2 + 50\right)\underset{\sim}{j} = 25\underset{\sim}{i} + 30\underset{\sim}{j}$ to find θ M1

Equating $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components

$20t\cos(\theta) = 25$ $\qquad$ $20t\sin(\theta) - 4.9t^2 + 50 = 30 \quad \ldots(2)$

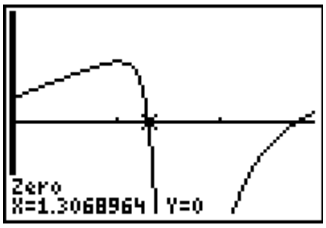
$t = \dfrac{5}{4\cos(\theta)} \quad \ldots(1)$

Substitute (1) into (2) to eliminate t from the equation

$$20\left(\frac{5}{4\cos(\theta)}\right)\sin(\theta) - 4.9\left(\frac{5}{4\cos(\theta)}\right)^2 + 50 = 30$$ A1

$$25\tan(\theta) - \frac{245}{32\cos^2(\theta)} + 20 = 0$$

Solve for θ using calculator



$\theta = 1.31$ radian.
The angle of projection is 74.9º. A1

Question 3

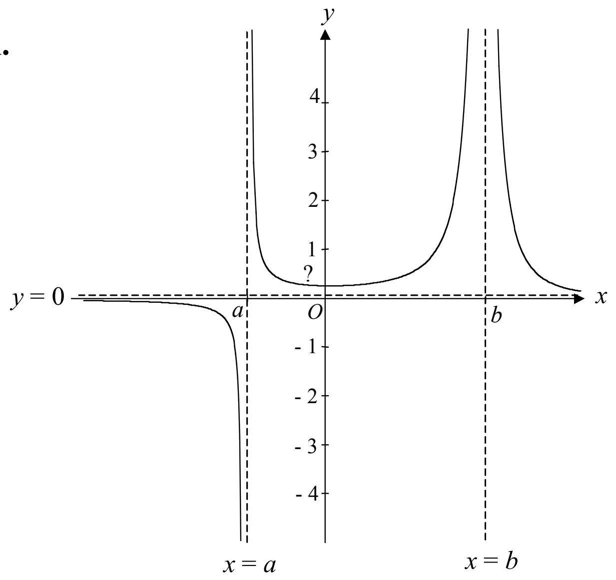
a. Curve is of the form $y = k(x-a)(x-b)^2$ where k is the dilation factor. M1

(0, 4) is a point on the curve $\Rightarrow$ $4 = k(0-a)(0-b)^2$

$$k = -\frac{4}{ab^2}$$

Equation of curve is $f(x) = -\frac{4}{ab^2}(x-a)(x-b)^2$ A1

b. **i.**



Shape and y-intercept A1
Asymptotes A1

ii. Domain: $R\setminus\{a, b\}$, Range: $R\setminus\{0\}$ A1

c. $f(x)$ is the derivative or gradient of $g(x)$

At $x = a$ and $x = b$ the gradient is zero ($f(x) = 0$)

$\therefore$ $g(x)$ has stationary points at $x = a$ and $x = b$

When $x < a$ the gradient is negative ($f(x) < 0$)

When $a < x < b$ the gradient is positive ($f(x) > 0$)

$\therefore$ $x = a$ is a minimum turning point on $g(x)$.

The y-coordinate of this point cannot be found accurately from graph of $f(x)$.

When $a < x < b$ the gradient is positive ($f(x) > 0$)

When $x < b$ the gradient is positive ($f(x) > 0$)

$\therefore$ $x = b$ is a stationary point of inflexion on $g(x)$.

The y-coordinate of this point cannot be found accurately from graph of $f(x)$.

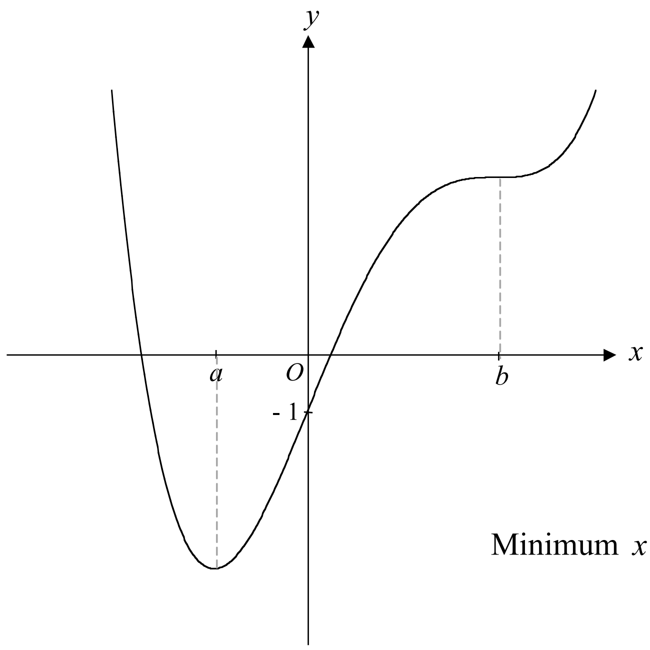
The graph of $f(x)$ shows a local maximum at the point (0, 4).

This means $x = 0$ is a non stationary point of inflexion on $g(x)$.

Given $g(0) = -1$, then the graph of $g(x)$ will pass through the point (0, -1).

This is the point of maximum gradient over the interval $a < x < b$.

$\therefore$ The gradient of $g(x)$ is 4 at point (0, -1). (It is not possible to show this a accurately on the scale that is given.)

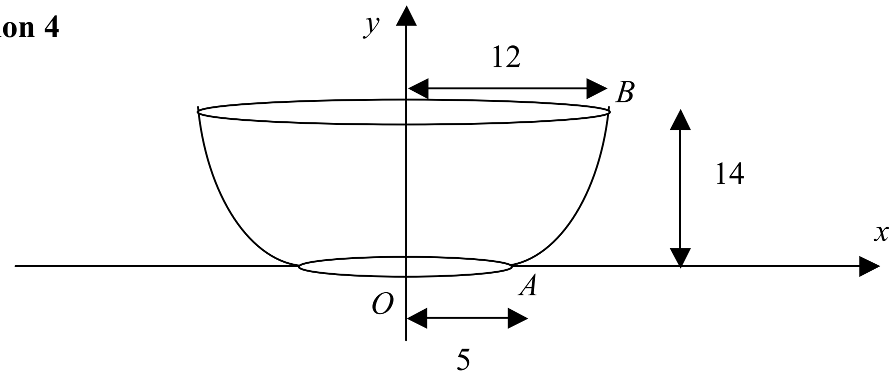


Shape A1

y-intercept A1

Minimum $x = a$, Inflexion $x = b$ A1

Question 4



a. $A(5,0)\quad B(12,14)$

Since the parabola is symmetrical about the y-axis $b=0$

Substituting the points A and B into $y = ax^2 + c$

$(1)\quad 0 = 25a + c$ M1

$(2)\quad 14 = 144a + c$

Subtracting gives $119a = 14$ so that $a = \frac{14}{119} = \frac{2}{17}$

and $c = -25a \quad c = -\frac{50}{17}$ so that $y = \frac{2}{17}\left(x^2 - 25\right)$ A1

b. $V = \pi \int_a^b x^2\, dy$ now $\frac{17y}{2} = x^2 - 25$ $\quad x^2 = \frac{17y}{2} + 25 = \frac{1}{2}(17y + 50)$

$$V = \pi \int_0^{14} \frac{1}{2}(17y + 50)\,dy$$ M1

$$V = \frac{\pi}{2}\left[\frac{17y^2}{2} + 50y\right]_0^{14}$$ A1

$$V = \frac{\pi}{2}\left[\frac{17(14)^2}{2} + 50 \text{x} 14 - 0\right] = 1183\pi$$

$V = 3716.5\ \text{cm}^3$ A1

c. $$V = \pi \int_0^{h} \frac{1}{2}(17y + 50)\,dy$$ M1

$$V = \frac{\pi}{2}\left[\frac{17y^2}{2} + 50y\right]_0^{h}$$

$$V = \frac{\pi}{2}\left[\frac{17h^2}{2} + 50h - 0\right]$$

$V = \frac{\pi h}{4}(17h + 100)$ for $0 \le h \le 14$ A1

d. Given $\dfrac{dV}{dt} = 2000\,\text{cm}^3/\text{min}$ we need to find $\dfrac{dh}{dt}$ when $h = 7$.

$$V = \frac{\pi h}{4}\left(17h + 100\right) = \frac{\pi}{4}\left(17h^2 + 100h\right)$$

$\dfrac{dV}{dh} = \dfrac{\pi}{4}\left(34h + 100\right) = \dfrac{\pi}{2}\left(17h + 50\right)$ and by the chain rule M1

$\dfrac{dh}{dt} = \dfrac{dh}{dV}\dfrac{dV}{dt} = \dfrac{4000}{\pi\left(17h + 50\right)}$ A1

$\left.\dfrac{dh}{dt}\right|_{h=7} = \dfrac{4000}{\pi\left(17\text{x}7 + 50\right)} = 7.53\,\text{cm/min}$ A1

e. $\dfrac{dV}{dt} = -28\sqrt{h}\ \text{cm}^3/\text{min}$

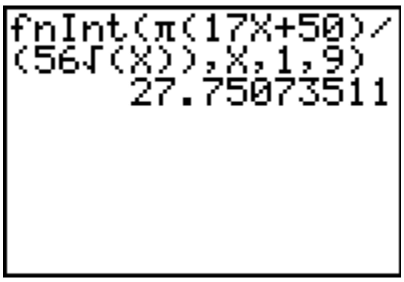
$\dfrac{dh}{dt} = \dfrac{dh}{dV}\dfrac{dV}{dt} = \dfrac{-28\sqrt{h}}{\dfrac{\pi}{2}\left(17h + 50\right)} = \dfrac{-56\sqrt{h}}{\pi\left(17h + 50\right)}$ inverting

$\dfrac{dt}{dh} = \dfrac{\pi\left(17h + 50\right)}{-56\sqrt{h}}$ integrating with respect to h M1

$t = \displaystyle\int_9^1 \frac{\pi\left(17h + 50\right)}{-56\sqrt{h}}\,dh = \int_1^9 \frac{\pi\left(17h + 50\right)}{56\sqrt{h}}\,dh$ A1

f. $= \displaystyle\int_1^9 \frac{\pi\left(17h + 50\right)}{56\sqrt{h}}\,dh = \frac{53\pi}{6} \approx 27.75$

The time for the height of the water to fall from 9 cm to 1 cm is 28 minutes



A1

Question 5

a. **i.** $P(-2,-1)$ $Q\left(-2-2\sqrt{3},5\right)$ $R\left(-2+2\sqrt{3},5\right)$ $C(-2,3)$

$\overrightarrow{OP} = -2\underset{\sim}{i} - \underset{\sim}{j}$ $\quad\overrightarrow{OQ} = \left(-2-2\sqrt{3}\right)\underset{\sim}{i} + 5\underset{\sim}{j}$

$\overrightarrow{OR} = \left(-2+2\sqrt{3}\right)\underset{\sim}{i} + 5\underset{\sim}{j}$ $\quad\overrightarrow{OC} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$

$\overrightarrow{CP} = \overrightarrow{CO} + \overrightarrow{OP}$ $\quad\overrightarrow{CQ} = \overrightarrow{CO} + \overrightarrow{OQ}$

$\overrightarrow{CP} = \left(2\underset{\sim}{i} - 3\underset{\sim}{j}\right) + \left(-2\underset{\sim}{i} - \underset{\sim}{j}\right)$ $\quad\overrightarrow{CQ} = \left(2\underset{\sim}{i} - 3\underset{\sim}{j}\right) + \left(-2-2\sqrt{3}\right)\underset{\sim}{i} + 5\underset{\sim}{j}$

$\overrightarrow{CP} = -4\underset{\sim}{j}$ $\quad\overrightarrow{CQ} = -2\sqrt{3}\,\underset{\sim}{i} + 2\underset{\sim}{j}$

$\left|\overrightarrow{CP}\right| = 4$ $\quad\left|\overrightarrow{CQ}\right| = \sqrt{12+4} = 4$ M1

$\overrightarrow{CP}.\overrightarrow{CQ} = -8$ M1

$$\cos(\theta) = \frac{\overrightarrow{CP}.\overrightarrow{CQ}}{\left|\overrightarrow{CP}\right|\left|\overrightarrow{CQ}\right|} = -\frac{8}{16} = -\frac{1}{2}$$

$\theta = 120^0$ A1

ii. $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$

$\overrightarrow{PQ} = \left(2\underset{\sim}{i} + \underset{\sim}{j}\right) + \left(-2-2\sqrt{3}\right)\underset{\sim}{i} + 5\underset{\sim}{j}$

$\overrightarrow{PQ} = -2\sqrt{3}\,\underset{\sim}{i} + 6\underset{\sim}{j}$

$\left|\overrightarrow{PQ}\right| = \sqrt{12+36}$

$\left|\overrightarrow{PQ}\right| = 4\sqrt{3}$

$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$

$\overrightarrow{PR} = \left(2\underset{\sim}{i} + \underset{\sim}{j}\right) + \left(-2+2\sqrt{3}\right)\underset{\sim}{i} + 5\underset{\sim}{j}$

$\overrightarrow{PR} = 2\sqrt{3}\,\underset{\sim}{i} + 6\underset{\sim}{j}$

$\left|\overrightarrow{PR}\right| = \sqrt{12+36} = 4\sqrt{3}$

$\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$

$\overrightarrow{QR} = \left(2+2\sqrt{3}\right)\underset{\sim}{i} - 5\underset{\sim}{j} + \left(-2+2\sqrt{3}\right)\underset{\sim}{i} + 5\underset{\sim}{j}$

$\overrightarrow{QR} = 4\sqrt{3}\,\underset{\sim}{i}$

$\left|\overrightarrow{QR}\right| = 4\sqrt{3}$ M1

since $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{PR}\right| = \left|\overrightarrow{QR}\right|$ PQR forms an equilateral triangle. A1

b. **i.** $S = \{ z : |z - c| = 4 \}$ where $c = -2 + 3i$ $\quad |z - (-2+3i)| = 4$

When $z = p = -2 - i$

$|(-2 - i) - (-2 + 3i)| = |-4i| = 4$ A1

When $z = q = -2 - 2\sqrt{3} + 5i$

$|(-2 - 2\sqrt{3} + 5i) - (-2 + 3i)| = |-2\sqrt{3} + 2i| = \sqrt{12+4} = \sqrt{16} = 4$ A1

When $z = r = -2 + 2\sqrt{3} + 5i$

$|(-2 + 2\sqrt{3} + 5i) - (-2 + 3i)| = |2\sqrt{3} + 2i| = \sqrt{12+4} = \sqrt{16} = 4$ A1

so that p, q and r all belong to S

ii. $|z - c| = 4 \quad c = -2 + 3i \qquad z = x + yi$

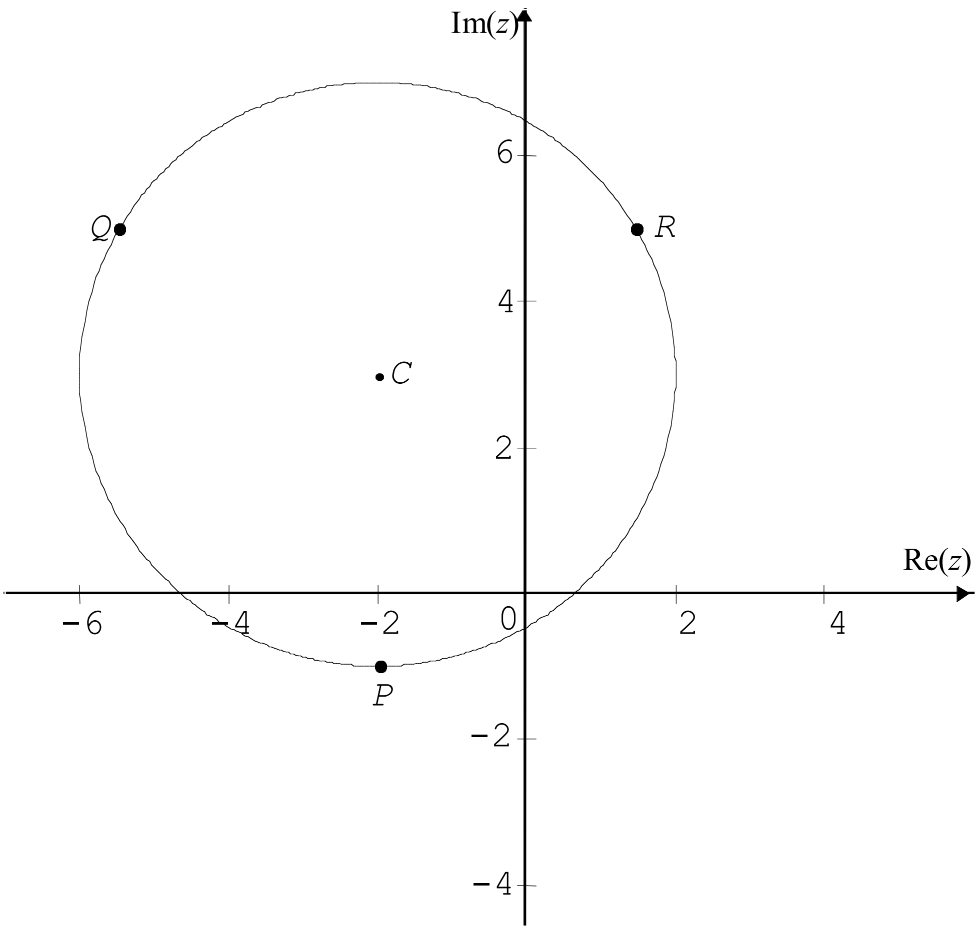
$|(x + yi) - (-2 + 3i)| = 4$

$|(x+2) + i(y-3)| = 4$

$\sqrt{(x+2)^2 + (y-3)^2} = 4$ squaring both sides

$(x+2)^2 + (y-3)^2 = 16$

S is a circle centre at $C(-2,3)$ radius 4 A1



For the graph circle and points. A1

c. **i.** $(z-c)^3 = 64i$ expanding

$z^3 - 3z^2c + 3zc^2 - c^3 = 64i$

$z^3 - 3z^2(-2+3i) + 3z(-2+3i)^2 - (-2+3i)^3 = 64i$ A1

$z^3 - 3z^2(-2+3i) + 3z(-5-12i) - (46+9i) = 64i$

$z^3 + z^2(6-9i) - z(15+36i) - (46+9i) = 64i$

$z^3 + (6-9i)z^2 - (15+36i)z - (46+73i) = 0$ M1

ii. $z^3 + (6-9i)z^2 - (15+36i)z - (46+73i) = 0$

$(z-c)^3 = 64i$

$(z-c)^3 = 64\text{cis}\left(\frac{\pi}{2} + 2k\pi\right)$ M1

$(z-c) = 64^{\frac{1}{3}}\text{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$

$z - (-2+3i) = 4\text{cis}\left(\frac{\pi}{6}\right) = 2\sqrt{3} + 2i$ when $k = 0$ A1

$z - (-2+3i) = 4\text{cis}\left(\frac{5\pi}{6}\right) = -2\sqrt{3} + 2i$ when $k = 1$ A1

$z - (-2+3i) = 4\text{cis}\left(-\frac{\pi}{2}\right) = -4i$ when $k = -1$ A1

so $z_R = -2 + 2\sqrt{3} + 5i$ $z_Q = -2 - 2\sqrt{3} + 5i$ $z_P = -2 - i$

These are the points P, Q and R. These points lie on the circle S of radius 4 and are equally spaced around the circle.
The points P , Q and R form an equilateral triangle. A1