The Mathematical Association of Victoria

2006 SPECIALIST MATHEMATICS

VCAA Sample Examination 1 and 2

Suggested Answers and Solutions

Examination 1

The examination will consist of short answer questions which are to be answered without the use of technology.

Examination 2

The examination will consist of two parts. Part 1 will be a multiple-choice section containing 22 questions and Part II will consist of extended answer questions, involving multi-stage solutions of increasing complexity.

The sample exams are found at:
http://www.vcaa.vic.edu.au/vce/studies/mathematics/specialist/pastexams/2006/2006specmath.pdf

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VCAA Sample Questions 2006 – Specialist Mathematics

Written Examination 1

Suggested answers and solutions

Question 1

a.
$$F_R = \mu N$$
$$= \frac{1}{7} \times 100 \times 9.8$$
$$= 140$$

The crate will move if $F > F_R$ ie. If $F > 140$

Since $F = 120$, the crate will not move.

b.
$$F - F_R = ma$$
$$190 - 140 = 100a$$
$$50 = 100a$$
$$a = 0.5\,\text{m/s}^2$$

Question 2

$\overrightarrow{BC} = \underset{\sim}{a}$ and $\overrightarrow{AC} = \underset{\sim}{b}$ since OABC is a parallelogram.

$\overrightarrow{OC} \bullet \overrightarrow{AB} = 0$ since $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{OC}$.

$\therefore \left(\underset{\sim}{a} + \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{b} - \underset{\sim}{a}\right) = 0$

$\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{a} = 0$

$\left|\underset{\sim}{b}\right|^2 - \left|\underset{\sim}{a}\right|^2 = 0$

$\left|\underset{\sim}{b}\right| = \left|\underset{\sim}{a}\right|$ so all sides have the same length.

Opposite sides are parallel, since it is a parallelogram.
Hence OABC is a rhombus.

Question 3

a.
$$f(x) = \arctan(x) + x\arctan(x)$$
$$f'(x) = \frac{1}{1+x^2} + x \times \frac{1}{1+x^2} + 1 \times \arctan(x)$$
$$= \frac{1}{1+x^2} + \frac{x}{1+x^2} + \arctan(x)$$
$$= \arctan(x) + \frac{1+x}{1+x^2}$$

b. Points of inflection occur when $f''(x) = 0$.
$$f''(x) = 0$$
$$\frac{1}{1+x^2} + \frac{(x^2+1) - 2x(x+1)}{(x^2+1)^2} = 0$$
$$x^2 + 1 + x^2 + 1 - 2x^2 - 2x = 0$$
$$2 - 2x = 0$$
$$x = 1$$
$$f(1) = \arctan(1) + \arctan(1)$$
$$= \frac{\pi}{4} + \frac{\pi}{4}$$
$$= \frac{\pi}{2}$$

Point of inflection at $\left(1, \frac{\pi}{2}\right)$

Question 4

a. Differentiating with respect to x:
$$\frac{d}{dx}(2y) - \frac{d}{dx}\left(xy^2\right) = \frac{d}{dx}(8)$$
$$2\frac{dy}{dx} - \left(\frac{d(x)}{dy}y^2 + x\frac{d\left(y^2\right)}{dx}\right) = 0$$
$$2\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} = 0$$
$$(2 - 2xy)\frac{dy}{dx} = y^2$$
$$\frac{dy}{dx} = \frac{y^2}{2(1-xy)}$$

b. When $y = 2$
$$4 - 4x = 8$$
$$x = -1$$
$$\therefore \frac{dy}{dx} = \frac{2^2}{2(1 - 2 \times -1 \times 2)}$$
$$= \frac{2}{5}$$

Question 5

a. Substituting $z = 3i$ we have

$$(3i)^3 - 2(3i)^2 + 9(3i) - 18$$
$$= -27i + 18 + 27i - 18$$
$$= 0$$

So $z = 3i$ is a solution.

b. As all coefficients are real and $z - 3i$ is a factor then $z + 3i$ is also a factor.

$$(z - 3i)(z + 3i)(z + a) = 0$$
$$(z^2 + 9)(z + a) = 0$$
$$\text{So} \quad 9a = -18$$
$$\therefore \quad a = -2$$

So the solutions are $-3i,\ 3i,\ 2$

Question 6

a.
$$\frac{d}{dx}\left(\cos^{-1}\left(\sqrt{3x}\right)\right) = \frac{-1}{\sqrt{1-\left(\sqrt{3x}\right)^2}} \times \frac{1}{2}(3x)^{-\frac{1}{2}} \times 3$$
$$= \frac{-1}{\sqrt{1-3x}} \times \frac{3}{2\sqrt{3x}}$$
$$= \frac{-1}{\sqrt{1-3x}} \times \frac{3}{2\sqrt{3}\sqrt{x}}$$
$$= \frac{-\sqrt{3}}{2\sqrt{1-3x}\sqrt{x}}$$
$$= \frac{-\sqrt{3}}{2\sqrt{x-3x^2}}$$

b.
$$\int \frac{-\sqrt{3}}{2\sqrt{x-3x^2}}dx = \cos^{-1}\left(\sqrt{3x}\right)$$

so
$$\frac{-\sqrt{3}}{2}\int \frac{1}{\sqrt{x-3x^2}}dx = \cos^{-1}\left(\sqrt{3x}\right)$$
$$\int \frac{1}{\sqrt{x-3x^2}}dx = \frac{-2}{\sqrt{3}}\cos^{-1}\left(\sqrt{3x}\right)$$

$$\int_{\frac{1}{6}}^{\frac{1}{4}} \frac{1}{\sqrt{x-3x^2}}dx = \left[\frac{-2}{\sqrt{3}}\cos^{-1}\left(\sqrt{3x}\right)\right]_{\frac{1}{6}}^{\frac{1}{4}}$$
$$= \frac{-2}{\sqrt{3}}\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$$
$$= \frac{-2}{\sqrt{3}}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)$$
$$= \frac{-2}{\sqrt{3}} \times \frac{-\pi}{12}$$
$$= \frac{\sqrt{3}\pi}{18}$$

Question 7

$$y = \int \sin^3(x)\cos^2(x)dx$$
$$= \int \sin(x)\sin^2(x)\cos^2(x)dx$$
$$= \int \sin(x)(1-\cos^2(x))\cos^2(x)dx$$

Let $u = \cos(x)$, so $\frac{du}{dx} = -\sin(x)$

$$dx = \frac{du}{\sin(x)}$$

Therefore, making the substitutions

$$y = \int \sin(x)\left(1-u^2\right)u^2 \frac{-du}{\sin(x)}$$
$$= \int -\left(u^2 - u^4\right)du$$
$$= \int \left(u^4 - u^2\right)du$$
$$= \frac{u^5}{5} - \frac{u^3}{3} + c$$
$$= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + c$$

$$y(0) = 0$$
$$\frac{\cos^5(0)}{5} - \frac{\cos^3(0)}{3} + c = 0$$
$$\frac{1}{5} - \frac{1}{3} + c = 0$$
$$c = \frac{2}{15}$$

$$\therefore y = \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + \frac{2}{15}$$

Question 8

$$V = \pi\int x^2 dy$$

$$y = \frac{16}{x^2+4} - 2$$

$$\therefore y + 2 = \frac{16}{x^2+4}$$

$$x^2 + 4 = \frac{16}{y+2}$$

$$x^2 = \frac{16}{y+2} - 4$$

$$V = \int_0^2 \left(\frac{16}{y+2} - 4\right) dy$$

$$= \pi\left[16\log_e(y+2) - 4y\right]_0^2$$

$$= \pi\left[\left(16\log_e(4) - 8\right) - \left(16\log_e(2) - 0\right)\right]$$

$$= \pi\left[16\log_e(4) - 16\log_e(2) - 8\right]$$

$$= \pi\left[16\log_e(2) - 8\right]$$

$$= 8\pi\left[2\log_e(2) - 1\right]$$

$$= 8\pi\left[\log_e(4) - 1\right] \text{ cubic units}$$

Question 9

a. Vertical asymptotes occur when $x^2 + 4x + 5 = 0$

$$\Delta = (4)^2 - 4(1)(5)$$

$$= 16 - 20$$

$$= -4$$

As $\Delta < 0$, $x^2 + 4x + 5$ has no real solutions

So there are no vertical asymptotes.

b.

$$\int \frac{3}{x^2+4x+5}dx = \int \frac{3}{(x+2)^2+1}dx$$

$$= 3\tan^{-1}(x+2) + c$$

$$\therefore A = 3,\ B = 2$$

c. Turning points occur when $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = \frac{-3(2x+4)}{(x^2+4x+5)^2}$$

When $\frac{dy}{dx} = 0$

$$2x + 4 = 0$$

$$x = -2$$

$$\text{Area} = \int_{-2}^{\sqrt{3}-2} \frac{3}{x^2+4x+5}dx$$

$$= \left[3\tan^{-1}(x+2)\right]_{-2}^{\sqrt{3}-2}$$

$$= 3\tan^{-1}\left(\sqrt{3}\right) - 3\tan^{-1}(0)$$

$$= \pi$$

d.

$$-\frac{\pi}{2} < \tan^{-1}(x+7) < \frac{\pi}{2}$$

$$-\frac{5\pi}{2} < 5\tan^{-1}(x+7) < \frac{5\pi}{2}$$

$$-\frac{5\pi}{2} + c < 5\tan^{-1}(x+7) + c < \frac{5\pi}{2} + c$$

For $5\tan^{-1}(x+7) + c > 0$ then $-\frac{5\pi}{2} + c > 0$, so

$$c > \frac{5\pi}{2}.$$

VCAA Sample Questions 2006 – Specialist Mathematics

Written Examination 2 Section 1

Suggested answers and solutions

1	E	2	C	3	E	4	D	5	B
6	A	7	C	8	A	9	D	10	B
11	E	12	D	13	C	14	E	15	B
16	D	17	A	18	B	19	A	20	E
21	E	22	C						

Question 1

$$y=\frac{-x^2+1}{2x}$$
$$=\frac{-x^2}{2x}+\frac{1}{2x}$$
$$=\frac{-x}{2}+\frac{1}{2x}$$

Vertical asymptote at $x=0$

As $x\to\pm\infty,\ \frac{1}{2x}\to 0,\ \therefore\ y\to\frac{-x}{2}$

Therefore $y=\frac{-x}{2}$ is also an asymptote

$\therefore$ E

Question 2

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Centre: $(1,-2)$

$\therefore\ h=1, k=-2$

Also, the lengths of the major and minor axes give $a=1, b=2$

$$(x-1)^2+\frac{(y+2)^2}{4}=1$$

$\therefore$ C

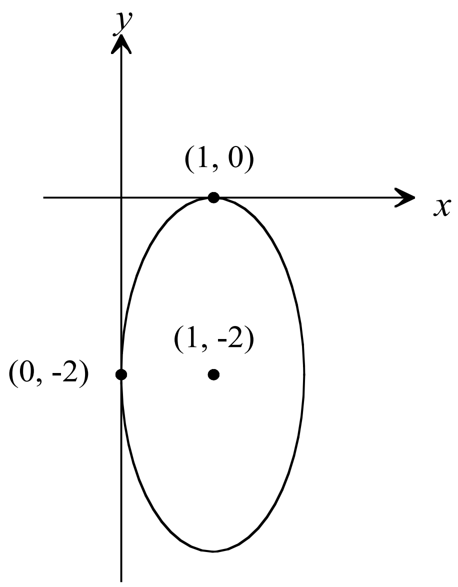


Diagram for Question 2

Question 3

A. $$\frac{\sin\left(\frac{\pi}{5}\right)}{\cos\left(\frac{\pi}{5}\right)}=\tan\left(\frac{\pi}{5}\right)$$

B. $$\frac{1}{\cot\left(\frac{\pi}{5}\right)}=\frac{1}{\left(\frac{1}{\tan\left(\frac{\pi}{5}\right)}\right)}$$
$$=\tan\left(\frac{\pi}{5}\right)$$

C $\cot\left(\frac{3\pi}{10}\right) = \cot\left(\frac{\pi}{2} - \frac{\pi}{5}\right)$

$$= \frac{\cos\left(\frac{\pi}{2} - \frac{\pi}{5}\right)}{\sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right)}$$

$$= \frac{\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{5}\right) + \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{5}\right)}$$

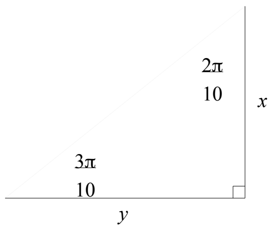
$$= \frac{\sin\left(\frac{\pi}{5}\right)}{\cos\left(\frac{\pi}{5}\right)}$$

$$= \tan\left(\frac{\pi}{5}\right)$$

Alternatively

$$\cot\left(\frac{3\pi}{10}\right) = \frac{y}{x}$$

$$\tan\left(\frac{2\pi}{10}\right) = \frac{y}{x}$$



$$\therefore \cot\left(\frac{3\pi}{10}\right) = \tan\left(\frac{2\pi}{10}\right)$$

$$= \tan\left(\frac{\pi}{5}\right)$$

D. $$\frac{2\tan\left(\frac{\pi}{10}\right)}{1 - \tan^2\left(\frac{\pi}{10}\right)} = \tan\left(2 \times \frac{\pi}{10}\right)$$

$$= \tan\left(\frac{\pi}{5}\right)$$

E. $$\frac{2\tan\left(\frac{2\pi}{5}\right)}{1 - \tan^2\left(\frac{2\pi}{5}\right)} = \tan\left(2 \times \frac{2\pi}{5}\right)$$

$$= \tan\left(\frac{4\pi}{5}\right)$$

$$= \tan\left(\pi - \frac{\pi}{5}\right)$$

$$= -\tan\left(\frac{\pi}{5}\right)$$

Therefore option E is incorrect.

Question 4

The period of the graph is π. Therefore,

$$\frac{2\pi}{a} = \pi$$
$$\therefore a = 2$$

The graph is translated by $\frac{\pi}{4}$ units in the positive x- direction. Therefore $b = \frac{\pi}{4}$

$\therefore$ D

Question 5

As the polynomial has real coefficients then the Complex Conjugate Rule applies. Hence, any complex roots must occur as a conjugate pair.

Hence, it is not possible to have an odd number of non-real roots. So, answer **B** must be false

Question 6

If $|w| = 1.5$ then $w = 1.5\text{cis}\theta$

Therefore,

$$w^{-1} = 1.5^{-1}\text{cis}(-\theta)$$
$$= \frac{2}{3}\text{cis}(-\theta)$$

So, P is the best representation

$\therefore$ A

Question 7

The shaded region corresponds to

$$\left\{z : \text{Arg}(z) > \frac{\pi}{3}\right\} \cap \left\{z : \text{Arg}(z) < \frac{\pi}{2}\right\}$$

$\therefore$ C

Question 8

$$\int_0^{\frac{\pi}{3}} \cos^2(x)\sin^3(x)dx = \int_0^{\frac{\pi}{3}} \cos^2(x)\sin^2(x)\sin(x)dx$$

$$= \int_0^{\frac{\pi}{3}} \cos^2(x)\left(1-\cos^2(x)\right)\sin(x)dx$$

Let $u = \cos(x)$

$$\frac{du}{dx} = -\sin(x)$$

$$dx = \frac{-du}{\sin(x)}$$

When $x = \frac{\pi}{3}$ $u = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$x = 0$ $u = \cos(0) = 1$

$$\int_0^{\frac{\pi}{3}} \cos^2(x)\left(1-\cos^2(x)\right)\sin(x)dx$$

$$= \int_1^{\frac{1}{2}} u^2\left(1-u^2\right)\sin(x)\frac{-du}{\sin(x)}$$

$$= -\int_1^{\frac{1}{2}} u^2\left(1-u^2\right)du$$

$$= \int_{\frac{1}{2}}^1 u^2\left(1-u^2\right)du$$

$\therefore$ A

Question 9

When $x = \frac{\pi}{2}$, $F'(x) = 0$

$0 < x < \frac{\pi}{2}$, $F'(x) > 0$

$\frac{\pi}{2} < x < \pi$, $F'(x) < 0$

Therefore a local maximum occurs at $x = \frac{\pi}{2}$

When $x = \pi$, $F'(x) = 0$

$\frac{\pi}{2} < x < \pi$, $F'(x) < 0$

$\pi < x < \frac{3\pi}{2}$, $F'(x) < 0$

Therefore a stationary point of inflexion occurs at $x = \pi$

When $x = \frac{3\pi}{2}$ $F'(x) = 0$

$\pi < x < \frac{3\pi}{2}$, $F'(x) < 0$

$\frac{3\pi}{2} < x < 2\pi$, $F'(x) > 0$

Therefore a local minimum occurs at $x = \frac{3\pi}{2}$

$\therefore$ D

Question 10

$x^2 + y^2 = 9$

Differentiating both sides with respect to x

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

When $x = 1$, $y = \sqrt{8}$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{8}} = \frac{-1}{2\sqrt{2}}$$

$\therefore$ B

Question 11

$$\int_a^b \frac{1}{x}dx = \left[\log_e|x|\right]_a^b$$

$$= \log_e|b| - \log_e|a|$$

$\therefore$ E

Question 12

$y = \sqrt{x^2 - 9}$

$\therefore y^2 = x^2 - 9$

$\therefore x^2 = y^2 + 9$

$$V = \pi\int_0^4 5^2\, dy - \pi\int_0^4 \left(y^2 + 9\right)dy$$
$$= \pi\int_0^4 \left(25 - \left(y^2 + 9\right)\right)dy$$
$$= \pi\int_0^4 \left(16 - y^2\right)dy$$

$\therefore$ D

Question 13

A $\quad |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2$ by Pythagoras' Theorem
Therefore: **true**

B

$\underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{a}$

$\therefore \underset{\sim}{a} - \underset{\sim}{c} = \underset{\sim}{b}$

So

$$\underset{\sim}{b} \cdot \left(\underset{\sim}{a} - \underset{\sim}{c}\right) = \underset{\sim}{b} \cdot \underset{\sim}{b}$$
$$= |\underset{\sim}{b}|^2$$

Therefore: **true**

C

$\underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{a}$

$\therefore \underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{c}$

So

$$\underset{\sim}{b} \cdot \left(\underset{\sim}{a} - \underset{\sim}{b}\right) = \underset{\sim}{b} \cdot \underset{\sim}{c}$$
$$= |\underset{\sim}{b}||\underset{\sim}{c}|\cos\left(90°\right)$$
$$= 0$$
$$\neq |\underset{\sim}{b}||\underset{\sim}{c}|$$

Therefore: **false**

D $\quad \underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\left(\theta\right)$ by dot product rule
Therefore: **true**

E

$$\underset{\sim}{a} \cdot \underset{\sim}{c} = |\underset{\sim}{a}||\underset{\sim}{c}|\cos\left(90° - \theta\right)$$
$$= |\underset{\sim}{a}||\underset{\sim}{c}|\sin\left(\theta\right)$$

Therefore: **true**

Hence the correct answer is **C**

Question 14

$\underset{\sim}{r}\left(t\right) = 2\sin\left(t\right)\underset{\sim}{i} + \cos\left(t\right)\underset{\sim}{j}$

$x = 2\sin\left(t\right) \qquad y = \cos\left(t\right)$

$\dfrac{x}{2} = \sin\left(t\right)$

$\sin^2\left(t\right) + \cos^2\left(t\right) = 1$

$\therefore \left(\dfrac{x}{2}\right)^2 + y^2 = 1$

$\dfrac{x^2}{4} + y^2 = 1$

Now $\quad 0 \le t \le \pi$

When $\; t = 0,\ x = 0$

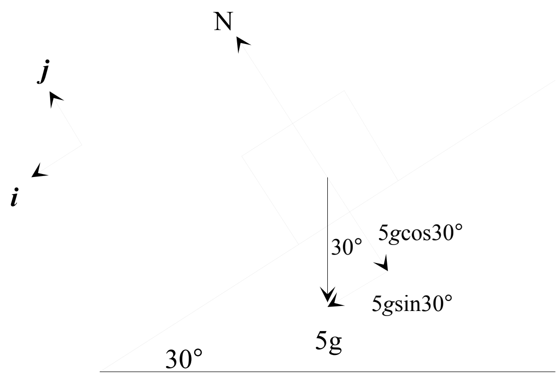
When $\; t = \dfrac{\pi}{2},\ x = 2$

When $\; t = \pi,\ x = 0$

$\therefore 0 \le x \le 2$

$\therefore$ E

Question 15



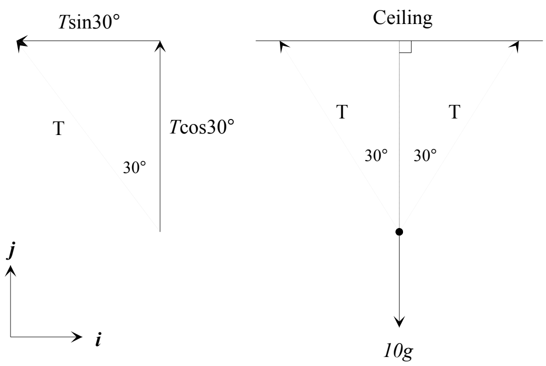
$5g\sin 30\mathbf{i} + (N - 5g\cos 30)\mathbf{j} = 5a\mathbf{i}$

$5g\sin 30 = 5a$

$a = \frac{g}{2}$

$\therefore$ B

Question 16



$\left(T\sin 30 - T\sin 30\right)\mathbf{i} + \left(T\cos 30° + T\cos 30° - 10g\right) = 0$

$2T\cos 30 - 10g = 0$

$\frac{2T\sqrt{3}}{2} = 10g$

$T = \frac{10g}{\sqrt{3}}$

$= \frac{10g\sqrt{3}}{3}$

$\therefore$ D

Question 17

$\sum \underset{\sim}{F} = 5\underset{\sim}{a}$

$\left(2\underset{\sim}{i} + \underset{\sim}{j}\right) + \left(\underset{\sim}{i} + 10\underset{\sim}{j}\right) + \left(3\underset{\sim}{i} - 3\underset{\sim}{j}\right) = 5\underset{\sim}{a}$

$6\underset{\sim}{i} + 8\underset{\sim}{j} = 5\underset{\sim}{a}$

$\left|\underset{\sim}{a}\right| = \frac{\sqrt{6^2 + 8^2}}{5}$

$= 2$

$\therefore$ A

Question 18

$u = -21, t = 10, a = 9.8$

$s = ut + \frac{1}{2}at^2$

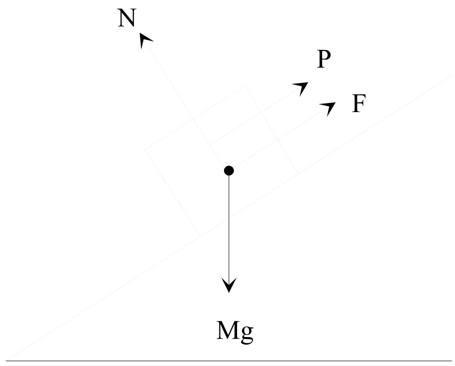
$s = -21 \times 10 + \frac{1}{2} \times 9.8 \times 10^2$

$= 280$

$\therefore h = 280\,\text{m}$

$\therefore$ B

Question 19



As the body is on the point of sliding down the plane, the friction force acts up the plane.

$\therefore$ A

Question 20

A. $\frac{dy}{dx} = \frac{-1}{x^2}$

B. $\frac{dy}{dx} = \frac{-1}{x}$

C. $\frac{dy}{dx} = -e^{-x}$

D. $\frac{dy}{dx} = \frac{-2}{x^3}$

E. $\frac{dy}{dx} = \frac{1}{x}$

As the slope field has the form of a hyperbola in the form $y = \frac{a}{x}$, for $a > 0$, this corresponds to answer **E**

Question 21

$$\frac{dy}{dt} = -k\left(y-4\right), \qquad t=0,\ y=20$$

$\therefore$ E

Question 22

$$\frac{dy}{dx} = \cos\left(\frac{x}{2}\right)$$

$$\begin{aligned} y_1 &= y_0 + hf'\left(x_0\right) \\ &= 2 + 0.2\cos\left(0\right) \\ &= 2.2 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + hf'\left(x_1\right) \\ &= 2.2 + 0.2f'\left(0.2\right) \\ &= 2.2 + 0.2\cos\left(\frac{0.2}{2}\right) \\ &= 2.2 + 0.2\cos\left(0.1\right) \end{aligned}$$

$\therefore$ C

VCAA Sample Questions 2006 – Specialist Mathematics

Written Examination 2 Section 2

Suggested answers and solutions

Question 1

a.

$$\underset{\sim}{r} \cdot \underset{\sim}{s}$$
$$= (2-2\cos(t))(\sin(t)) + (1+\sin(t))(2\cos(t))$$
$$= 2\sin(t) - 2\cos(t)\sin(t) + 2\cos(t) + 2\sin(t)\cos(t)$$
$$= 2\sin(t) + 2\cos(t)$$

The particles will be at right angles to each other when $\underset{\sim}{r} \cdot \underset{\sim}{s} = 0$

Hence,

$$2\sin(t) + 2\cos(t) = 0$$
$$2\sin(t) = -2\cos(t)$$
$$\tan(t) = -1$$

$$\therefore t = \frac{3\pi}{4}$$

b.

$$x = 2 - 2\cos(t) \qquad y = 1 + \sin(t)$$
$$\cos(t) = \frac{2-x}{2} \qquad \sin(t) = 1 - y$$

$$\cos^2(t) + \sin^2(t) = 1$$
$$\left(\frac{2-x}{2}\right)^2 + (1-y)^2 = 1$$
$$\frac{(x-2)^2}{4} + (y-1)^2 = 1$$

Now $t \geq 0 \therefore x \geq 0$

$$\therefore \frac{(x-2)^2}{4} + (y-1)^2 = 1, \; x \geq 0$$

c. Velocity of particle R:

$$\dot{\underset{\sim}{r}} = 2\sin(t)\underset{\sim}{i} + \cos(t)\underset{\sim}{j}$$

Speed of particle R:

$$|\dot{\underset{\sim}{r}}| = \sqrt{(2\sin(t))^2 + (\cos(t))^2}$$
$$= \sqrt{4\sin^2(t) + \cos^2(t)}$$

Velocity of particle S:

$$\dot{\underset{\sim}{s}} = \cos(t)\underset{\sim}{i} - 2\sin(t)\underset{\sim}{j}$$

Speed of particle S:

$$|\dot{\underset{\sim}{s}}| = \sqrt{(\cos(t))^2 + (2\sin(t))^2}$$
$$= \sqrt{\cos^2(t) + 4\sin^2(t)}$$

Therefore particles R and S move at the same speed at any given time

Velocities will be equal when

$$2\sin(t)\underset{\sim}{i} + \cos(t)\underset{\sim}{j} = \cos(t)\underset{\sim}{i} - 2\sin(t)\underset{\sim}{j}$$

Therefore:

$$2\sin(t) = \cos(t) \quad \text{and} \quad \cos(t) = -2\sin(t)$$
$$\tan(t) = \frac{1}{2} \qquad \tan(t) = \frac{-1}{2}$$

As $\tan(t)$ cannot equal both $\frac{1}{2}$ and $\frac{-1}{2}$ simultaneously, the particles can never have the same velocity.

Question 2

a. i.

$u = 0,\ s = 400,\ t = 8$

$s = ut + \frac{1}{2}at^2$

$400 = 0 \times 8 + \frac{1}{2} \times a \times 64$

$400 = 32a$

$a = 12.5 \text{ m/s}^2$

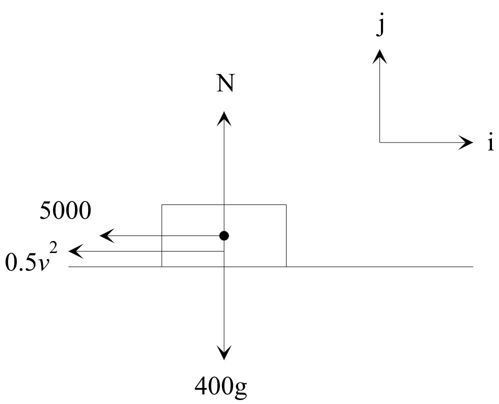
ii.

$v = u + at$

$v = 0 + 12.5 \times 8$

$= 100 \text{ m/s}$

b. i.



$\sum F = ma$

$\left(-5000 - 0.5v^2\right)\underset{\sim}{i} + \left(N - 400g\right)\underset{\sim}{j} = 400a\underset{\sim}{i}$

$\therefore\ 400a = -5000 - 0.5v^2$

$a = \dfrac{-5000 - 0.5v^2}{400}$

ii.

$a = v\dfrac{dv}{dx}$

$v\dfrac{dv}{dx} = \dfrac{-5000 - 0.5v^2}{400}$

$= \dfrac{-\left(10000 + v^2\right)}{800}$

$\dfrac{dv}{dx} = -\dfrac{\left(10^4 + v^2\right)}{800v}$

iii.

$\dfrac{dv}{dx} = -\dfrac{\left(10^4 + v^2\right)}{800v}$

$\dfrac{dx}{dv} = -\dfrac{800v}{\left(10^4 + v^2\right)}$

When $x = 0,\ v = 100$ (from a. ii.)

$x = \displaystyle\int_{100}^{0} -\dfrac{800v}{\left(10^4 + v^2\right)}dv$

Let $u = 10^4 + v^2$

When $v = 100 \quad u = 2 \times 10^4$

$v = 0 \quad u = 10^4$

$\dfrac{du}{dv} = 2v$

$dv = \dfrac{du}{2v}$

$x = \displaystyle\int_{2\times10^4}^{10^4} -\dfrac{800v}{u} \times \dfrac{du}{2v}$

$= \displaystyle\int_{2\times10^4}^{10^4} -\dfrac{400}{u}du$

$= \displaystyle\int_{10^4}^{2\times10^4} \dfrac{400}{u}du$

$= 400\left[\log_e u\right]_{10^4}^{2\times10^4}$

$= 400\left[\log_e\left(2 \times 10^4\right) - \log_e\left(10^4\right)\right]$

$= 400\log_e(2)$

$= 277 \text{ m}$

c.

$$a = -\frac{\left(10^4 + v^2\right)}{800}$$

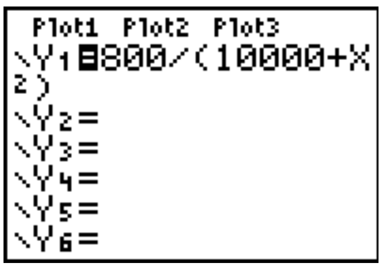
$$\frac{dv}{dT} = -\frac{\left(10^4 + v^2\right)}{800}$$

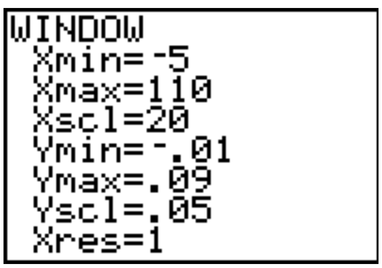
$$\frac{dT}{dv} = -\frac{800}{\left(10^4 + v^2\right)}$$

$$T = -800\int_{100}^{0} \frac{1}{10^4 + v^2}\,dv$$

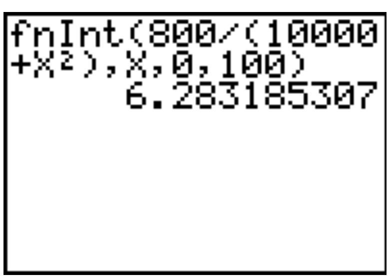
$$= 800\int_{0}^{100} \frac{1}{10^4 + v^2}\,dv$$

$$= 6.28\,\text{s}$$



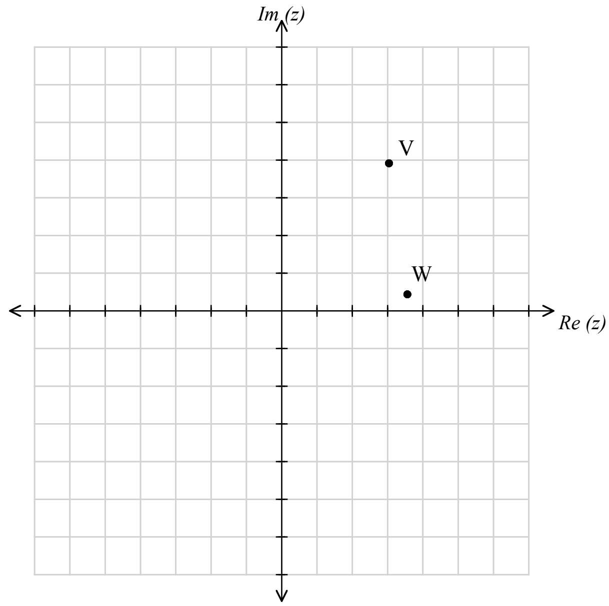


Alternatively:



Question 3

a.



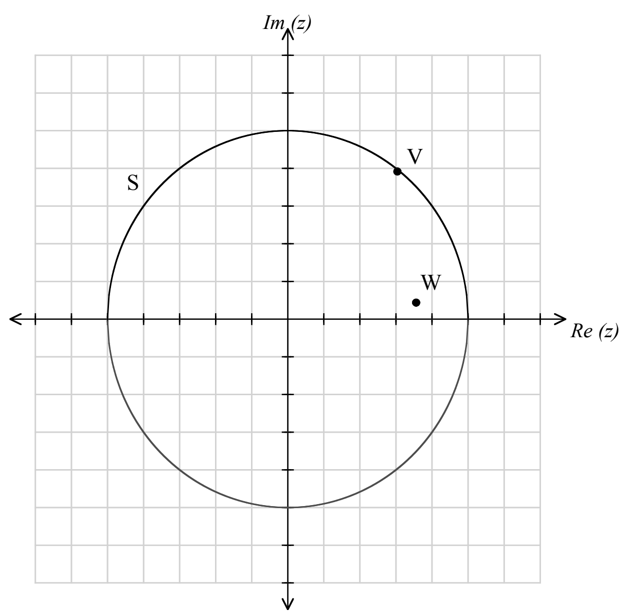
b.

i.

$$|v| = \sqrt{6^2 + 8^2}$$
$$= \sqrt{100}$$
$$= 10$$

$$\therefore v \in S$$

ii.



c. $u + i\overline{w} = \overline{w}$

$u = \overline{w} + i\overline{w}$

$= 7 - i - 7i - 1$

$u = 6 - 8i$

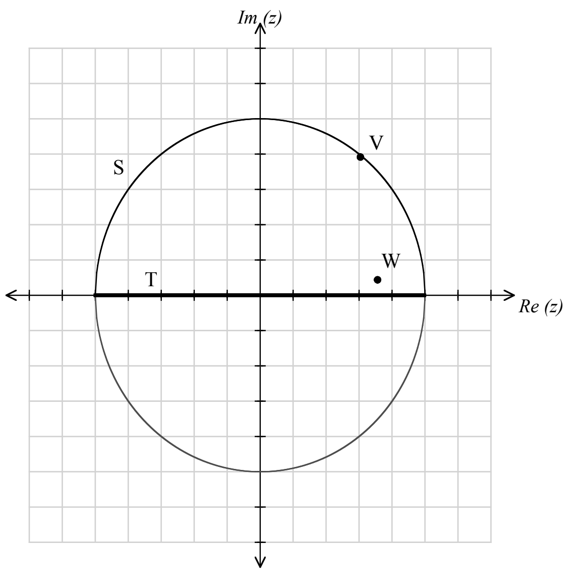
d. $|z-u| = |z-v|$

$|x+iy-6+8i| = |x+iy-6-8i|$

$\sqrt{(x-6)^2+(y+8)^2} = \sqrt{(x-6)^2+(y-8)^2}$

$(y+8)^8 = (y-8)^2$

$\therefore y = 0$



e. $\overrightarrow{OV} = 6\underset{\sim}{i} + 8\underset{\sim}{j}$

$\overrightarrow{OW} = 7\underset{\sim}{i} + \underset{\sim}{j}$

$\therefore \overrightarrow{WO} = -7\underset{\sim}{i} - \underset{\sim}{j}$

$\overrightarrow{WV} = \overrightarrow{WO} + \overrightarrow{OV}$

$= -\underset{\sim}{i} + 7\underset{\sim}{j}$

$\overrightarrow{WO} \cdot \overrightarrow{WV} = |\overrightarrow{WO}||\overrightarrow{WV}|\cos\theta$ where θ is $\angle OWV$

$(-7)(-1) + (-1)(7) = \sqrt{49+1} \times \sqrt{49+1}\cos\theta$

$0 = 2\sqrt{50}\cos\theta$

$\cos\theta = 0$

$\therefore \theta = 90°$

$\therefore \angle OWV$ is a right angle.

Question 4

a. $x \ge 0, \quad 1 - x^2 > 0$

$-1 < x < 1$

$\therefore x \in [0,1)$

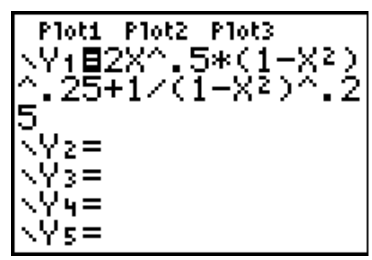
b. $D = 2 \times f(0.5)$

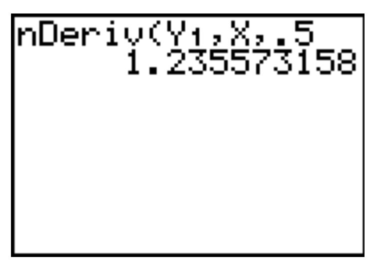
$= 2 \times 2(0.5)^{\frac{1}{2}}(1-0.5^2)^{\frac{1}{4}} + \dfrac{1}{(1-0.5^2)^{\frac{1}{4}}}$

$= 3.71$

c. $\tan\theta = f'(0.5)$ where θ is the angle $f(x)$ makes with the x-axis.

Note that the angle the platform makes with the surface is $\dfrac{\pi}{2} - \theta$.





$f'(0.5) = 1.23557$

$\tan\theta = 1.23557$

$\theta = 51.015°$

So, the angle the platform makes with the surface is $39°$.

d. i.

$$\left(f(x)\right)^2 = \left(2x^{\frac{1}{2}}\left(1-x^2\right)^{\frac{1}{4}} + \frac{1}{\left(1-x^2\right)^{\frac{1}{4}}}\right)^2$$

The middle term is given by

$$2 \times 2x^{\frac{1}{2}}\left(1-x^2\right)^{\frac{1}{4}} \times \frac{1}{\left(1-x^2\right)^{\frac{1}{4}}}$$

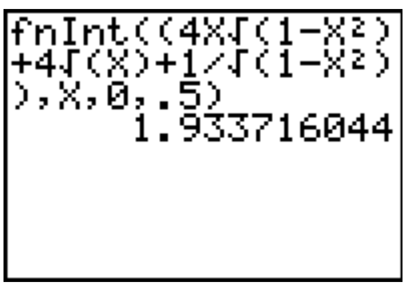
$$= 4x^{\frac{1}{2}}$$

$$= 4\sqrt{x}$$

ii. $V = \pi\int y^2 dx$

$$= \pi\int_0^{0.5}\left(4x\sqrt{1-x^2} + 4\sqrt{x} + \frac{1}{\sqrt{1-x^2}}\right)dx$$

iii. The volume of the platform is 1.9π m^3



Therefore 7 packages are needed.

Alternative solution:

$$V_x = \pi\int y^2 \ dx$$

$$= \pi\int_0^{\frac{1}{2}}\left(4x\ \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}} + 4\sqrt{x}\right)\ dx$$

$$= \pi\ \left[-\frac{4}{3}\left(1-x^2\right)^{\frac{3}{2}} + \text{Sin}^{-1}(x) + \frac{8}{3}\ x^{\frac{3}{2}}\right]_0^{\frac{1}{2}}$$

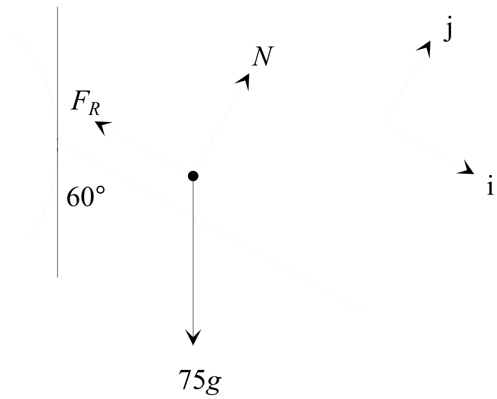
$$= \pi\ \left[\frac{4}{3} + \frac{\pi}{6} + \frac{2\sqrt{2}}{3} - \frac{\sqrt{3}}{2}\right]$$

$$= 6.075 = 6.1 \text{ m}^3$$

so they need 7 packages.

Question 5

a.



b. $\sum F = ma$

$$\left(75g\sin 30° - F_R\right)\underset{\sim}{i} + \left(N - 75g\cos 30°\right)\underset{\sim}{j} = 75a\underset{\sim}{i}$$

$$75g\sin 30° - F_R = 75a \qquad (1)$$

$$N = 75g\cos 30° \qquad (2)$$

$$F_R = \mu N$$

$$= \frac{1}{5} \times 75g \times \frac{\sqrt{3}}{2}$$

Sub in (1)

$$75a = 75g \times \frac{1}{2} - \frac{1}{5} \times 75g \times \frac{\sqrt{3}}{2}$$

$$a = \frac{g}{2} - \frac{1}{5} \times g \times \frac{\sqrt{3}}{2}$$

$$a = \frac{g}{2}\left(1 - \frac{\sqrt{3}}{5}\right)$$

c. i. $u = 0,\ a = \frac{g}{2}\left(1 - \frac{\sqrt{3}}{5}\right),\ s = 6$

$$s = ut + \frac{1}{2}at^2$$

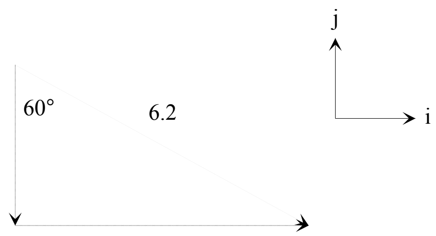
$$6 = \frac{1}{2} \times \frac{g}{2}\left(1 - \frac{\sqrt{3}}{5}\right)t^2$$

$$t = 1.94 \text{ s}$$

ii.

$$v^2 = u^2 + 2as$$
$$= 0 + 2 \times \frac{g}{2}\left(1 - \frac{\sqrt{3}}{5}\right) \times 6$$
$$v^2 = 38.431$$
$$v = 6.2 \text{ m/s}$$

d.



$$\underset{\sim}{v}(t) = 6.2\sin 60°\underset{\sim}{i} - 6.2\cos 60°\underset{\sim}{j} - gt\underset{\sim}{j}$$
$$= 3.1\sqrt{3}\underset{\sim}{i} - (3.1 + gt)\underset{\sim}{j}$$
$$\underset{\sim}{r}(t) = 3.1\sqrt{3}t\underset{\sim}{i} - \left(3.1t + \frac{gt^2}{2}\right)\underset{\sim}{j} + c$$

Jay hits the ground when

$$3.1t + \frac{gt^2}{2} = 2$$
$$4.9t^2 + 3.1t - 2 = 0$$
$$t = 0.3966 \text{ s}$$

Therefore the horizontal distance is given by

$$3.1\sqrt{3} \times 0.3966$$
$$= 2.1 \text{ m}$$

e.

$$\underset{\sim}{v}(t) = 3.1\sqrt{3}\underset{\sim}{i} - (3.1 + gt)\underset{\sim}{j}$$
$$\underset{\sim}{v}(0.3966) = 3.1\sqrt{3}\underset{\sim}{i} - (3.1 + 9.8 \times 0.3966)\underset{\sim}{j}$$
$$= 5.369\underset{\sim}{i} - 6.987\underset{\sim}{j}$$
$$|\underset{\sim}{v}| = 8.812$$
$$p = mv$$
$$p = 75 \times 8.812$$
$$= 661 \text{ kg m/s}$$