THE HEFFERNAN GROUP

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SPECIALIST MATHS TRIAL EXAMINATION 1 SOLUTIONS 2007

Question 1

a. Let $y = \arctan\left(\sqrt{2x-1}\right),\ x > \frac{1}{2}$

$y = \arctan(u)$ where $u = \sqrt{2x-1}$

$\frac{dy}{du} = \frac{1}{1+u^2}$

$\frac{du}{dx} = \frac{1}{2}(2x-1)^{-\frac{1}{2}} \times 2 = \frac{1}{\sqrt{2x-1}}$

Now, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (chain rule) **(1 mark)**

$$= \frac{1}{1+u^2} \cdot \frac{1}{\sqrt{2x-1}}$$

$$= \frac{1}{1+2x-1} \cdot \frac{1}{\sqrt{2x-1}}$$

$$= \frac{1}{2x\sqrt{2x-1}} \text{ as required.}$$

(1 mark)

b. From **a.** $\frac{d}{dx}\left(\arctan\left(\sqrt{2x-1}\right)\right) = \frac{1}{2x\sqrt{2x-1}}$

So, $\int_1^2 \frac{d}{dx}\left(\arctan\left(\sqrt{2x-1}\right)\right)dx = \frac{1}{2}\int_1^2 \frac{1}{x\sqrt{2x-1}}\,dx$ **(1 mark)**

$$2\left[\arctan\left(\sqrt{2x-1}\right)\right]_1^2 = \int_1^2 \frac{1}{x\sqrt{2x-1}}\,dx$$

$$2\left\{\arctan\left(\sqrt{3}\right) - \arctan(1)\right\} = \int_1^2 \frac{1}{x\sqrt{2x-1}}\,dx$$

So $\int_1^2 \frac{1}{x\sqrt{2x-1}}\,dx = 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$$= \frac{\pi}{6}$$

(1 mark)

Question 2

a. $2x^2y+y^2-5x=3$

$$2x^2\frac{dy}{dx}+4xy+2y\frac{dy}{dx}-5=0$$ **(1 mark)**

$$\frac{dy}{dx}\left(2x^2+2y\right)=5-4xy$$

$$\frac{dy}{dx}=\frac{5-4xy}{2x^2+2y}$$ **(1 mark)**

b. When $y=0$,

$2x^2y+y^2-5x=3$

becomes $-5x=3$

$$x=-\frac{3}{5}$$ **(1 mark)**

So $\frac{dy}{dx}=\dfrac{5-4\times-\frac{3}{5}\times 0}{2\times\left(-\frac{3}{5}\right)^2+0}$

$$=5\div\frac{18}{25}$$

$$=\frac{125}{18}$$

$$=6\frac{17}{18}$$

(1 mark)

Question 3

$$\frac{dy}{dx}=(x-2)\sqrt{x-1}$$

$$\int\frac{dy}{dx}\,dx=\int(x-2)\sqrt{x-1}\,dx$$

$$y=\int(u-1)u^{\frac{1}{2}}\frac{du}{dx}\,dx$$

Let $u=x-1$

$\frac{du}{dx}=1$

and $u-1=x-2$

$$=\int\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du$$ **(1 mark)**

$$=\frac{2u^{\frac{5}{2}}}{5}-\frac{2u^{\frac{3}{2}}}{3}+c$$

$$=\frac{2(x-1)^{\frac{5}{2}}}{5}-\frac{2(x-1)^{\frac{3}{2}}}{3}+c$$ **(1 mark)**

When $x=1,\ y=0$

So $0=0-0+c$

$c=0$

So $y=\frac{2(x-1)^{\frac{5}{2}}}{5}-\frac{2(x-1)^{\frac{3}{2}}}{3}$

(1 mark)

Question 4

a. $z^2 - z + 2.5 = 0$ is a quadratic equation so we can use the quadratic formula.

$$z = \frac{1 \pm \sqrt{1 - 4 \times 1 \times 2.5}}{2}$$ **(1 mark)**

$$= \frac{1 \pm \sqrt{-9}}{2}$$

$$= \frac{1 \pm 3i}{2} \text{ since } \sqrt{-1} = i$$

(1 mark)

b. $z^3 + z^2 + 3z - 5 = 0$

The coefficients of all the terms are real so two of the three solutions form a conjugate pair and the third is a real solution.

Let $p(z) = z^3 + z^2 + 3z - 5$

$p(1) = 1 + 1 + 3 - 5 = 0$

$z - 1$ is a factor.

Method 1

$$\begin{array}{r|l} & z^2 + 2z + 5 \\ z - 1 & z^3 + z^2 + 3z - 5 \\ & z^3 - z^2 \\ & \quad 2z^2 + 3z \\ & \quad 2z^2 - 2z \\ & \qquad 5z - 5 \\ & \qquad 5z - 5 \end{array}$$

$p(z) = (z-1)(z^2 + 2z + 5)$ **(1 mark)**

Method 2

$$\begin{aligned} p(z) &= z^3 + z^2 + 3z - 5 \\ &= (z-1) \times \_\_ + (z-1) \times \_\_ + (z-1) \times \_\_ \\ &= (z-1)z^2 + (z-1) \times 2z + (z-1) \times 5 \quad \text{by inspection} \\ &= (z-1)(z^2 + 2z + 5) \end{aligned}$$ **(1 mark)**

$$\begin{aligned} \text{So } p(z) &= (z-1)(z^2 + 2z + 5) \\ &= (z-1)\left((z^2 + 2z + 1) - 1 + 5\right) \text{ completing the square} \\ &= (z-1)\left((z+1)^2 + 4\right) \\ &= (z-1)\left((z+1)^2 - 4i^2\right) \\ &= (z-1)(z + 1 - 2i)(z + 1 + 2i) \end{aligned}$$ **(1 mark)**

So for $p(z) = 0$,

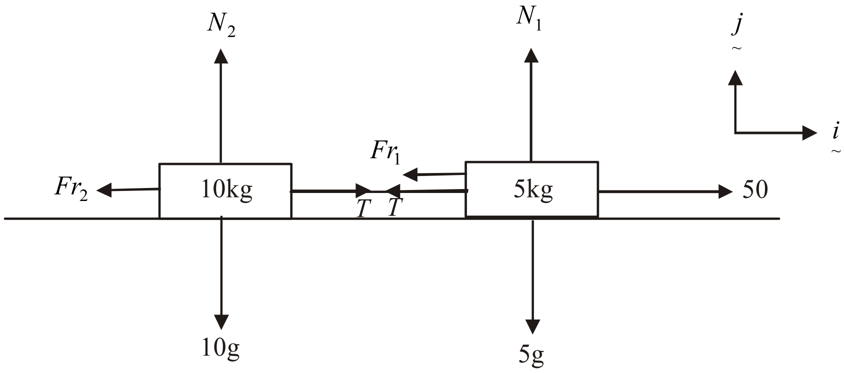
$z = 1, \quad -1 \pm 2i$

(1 mark)

Question 5

a. Method 1 – resolving around each of the containers

Show all the forces on the diagram.



Around the 5kg container

$$\underset{\sim}{R} = m\underset{\sim}{a}$$

$$(50 - Fr_1 - T)\underset{\sim}{i} + (N_1 - 5g)\underset{\sim}{j} = 5 \times 2\underset{\sim}{i}$$

So, $50 - Fr_1 - T = 10$ and $N_1 - 5g = 0$

$-\mu N_1 - T = -40$ $\quad N_1 = 5g$

$-5g\mu - T = -40$ **(1 mark)**

$T = 40 - 5g\mu \qquad -(1)$

Around the 10kg container

$$\underset{\sim}{R} = m\underset{\sim}{a}$$

$$(T - Fr_2)\underset{\sim}{i} + (N_2 - 10g)\underset{\sim}{j} = 10 \times 2\underset{\sim}{i}$$

$T - Fr_2 = 20$ and $N_2 - 10g = 0$

$T - \mu N_2 = 20$ $\quad N_2 = 10g$ **(1 mark)**

$T - 10g\mu = 20$

$T = 10g\mu + 20 \qquad -(2)$

From (1), $T = 40 - 5g\mu$

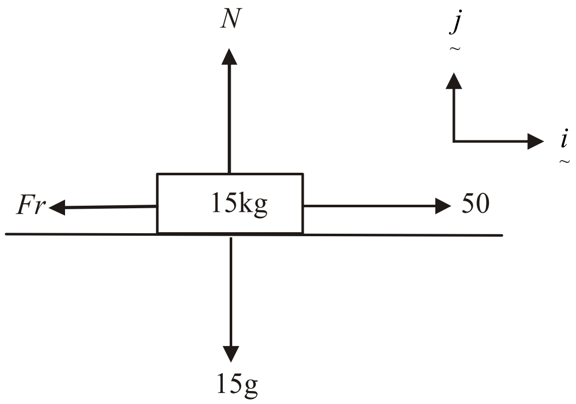
So $40 - 5g\mu = 10g\mu + 20$

$20 = 15g\mu$

$\mu = \dfrac{20}{15g}$

$\mu = \dfrac{4}{3g}$ as required. **(1 mark)**

Method 2 - Combining the containers into a single mass



(1 mark)

$$\underset{\sim}{R} = m\underset{\sim}{a}$$

$$(50 - Fr)\underset{\sim}{i} + (N - 15g)\underset{\sim}{j} = 15 \times 2\underset{\sim}{i}$$

So, $50 - Fr = 30$ and $N - 15g = 0$

$-\mu N = -20$ $\quad N = 15g$ **(1 mark)**

$15g\mu = 20$

$\mu = \dfrac{4}{3g}$ as required **(1 mark)**

b. Method 1 - following on from Method 1 in part **a.**

Substitute $\mu = \dfrac{4}{3g}$ into (1)

$$T = 40 - 5g \times \frac{4}{3g}$$
$$= 40 - \frac{20}{3}$$
$$= \frac{100}{3}\text{N}$$ **(1 mark)**

(Check in $(2)\ T = 10g\mu + 20$
$= 10g \times \dfrac{4}{3g} + 20$
$= \dfrac{40}{3} + 20$
$= \dfrac{100}{3}$N)

Method 2 - following on from Method 2 in part **a.**

Around the 10 kg container

$$\underset{\sim}{R} = m\underset{\sim}{a}$$

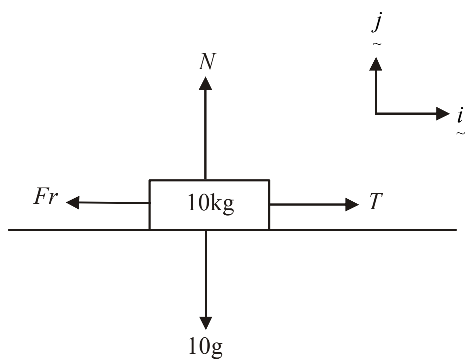
$$(T - Fr)\underset{\sim}{i} + (N - 10g)\underset{\sim}{j} = 10 \times 2\underset{\sim}{i}$$

$T - Fr = 20$ and $N - 10g = 0$

$T - \mu N = 20$ $\quad N = 10g$

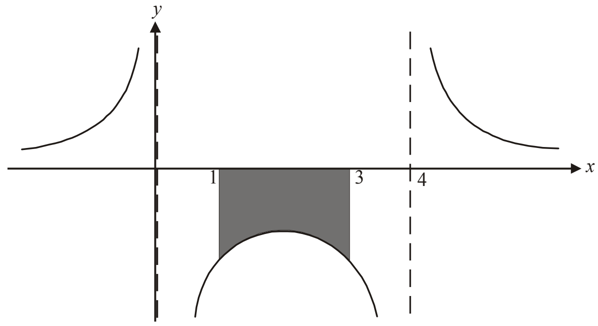
$$T - \frac{4 \times 10g}{3g} = 20$$

$$T = \frac{100}{3}\text{N}$$



(1 mark)

Question 6



Do a quick sketch.

$y = \dfrac{1}{x^2 - 4x}$

$= \dfrac{1}{x(x-4)}$

There are vertical asymptotes at $x = 0$ and $x = 4$.

The required area lies below the x-axis.

$$\text{Area} = -\int_1^3 \frac{1}{x^2 - 4x}\,dx$$ **(1 mark)**

Let $\dfrac{1}{x(x-4)} \equiv \dfrac{A}{x} + \dfrac{B}{x-4}$

$\equiv \dfrac{A(x-4) + Bx}{x(x-4)}$

True iff $1 \equiv A(x-4) + Bx$

Put $x = 4$, $1 = 4B$, $B = \dfrac{1}{4}$

Put $x = 0$, $1 = -4A$, $A = -\dfrac{1}{4}$

So $\dfrac{1}{x(x-4)} = \dfrac{-1}{4x} + \dfrac{1}{4(x-4)}$ **(1 mark)**

$$\text{Area required} = -\int_1^3 \left(-\frac{1}{4x} + \frac{1}{4(x-4)} \right) dx$$

$$= -\left[-\frac{1}{4}\log_e|x| + \frac{1}{4}\log_e|x-4| \right]_1^3$$

$$= -\frac{1}{4}\left[\log_e \frac{|x-4|}{|x|} \right]_1^3$$ **(1 mark)**

$$= -\frac{1}{4}\left\{ \log_e\left(\frac{1}{3}\right) - \log_e\left(\frac{3}{1}\right) \right\}$$

$$= -\frac{1}{4}\left(\log_e\left(\frac{1}{3} \div 3\right) \right)$$

$$= -\frac{1}{4}\log_e\left(\frac{1}{9}\right)$$ **(1 mark)**

$$= -\frac{1}{4}\log_e\left(9^{-1}\right)$$

$$= \frac{1}{4}\log_e(9)\text{ square units}$$

So $a = \dfrac{1}{4}$ and $b = 9$

(1 mark)

Question 7

The function is continuous for $x \in R$ and also $\frac{1}{\sqrt{4+x^2}} > 0$ for $x \in R$.

$$\text{Volume} = \pi\int_0^2 y^2\,dx$$

$$= \pi\int_0^2 \frac{1}{4+x^2}\,dx \qquad \textbf{(1 mark)}$$

$$= \frac{\pi}{2}\int_0^2 \frac{2}{4+x^2}\,dx \qquad \textbf{(1 mark)}$$

$$= \frac{\pi}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \frac{\pi}{2}\left(\tan^{-1}(1) - \tan^{-1}(0)\right)$$

$$= \frac{\pi}{2} \times \frac{\pi}{4} - 0$$

$$= \frac{\pi^2}{8} \text{ cubic units}$$

(1 mark)

Question 8

$$\int_0^3 \frac{4(x-1)}{\sqrt{9-x^2}}\,dx = \int_0^3 \frac{4x-4}{\sqrt{9-x^2}}\,dx$$

$$= \int_0^3 \frac{4x}{\sqrt{9-x^2}}\,dx - \int_0^3 \frac{4}{\sqrt{9-x^2}}\,dx \qquad \textbf{(1 mark)}$$

$$= \int_9^0 -2\frac{du}{dx}u^{-\frac{1}{2}}\,dx - 4\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$$

where $u = 9 - x^2$

$\frac{du}{dx} = -2x$

$x = 3, \quad u = 0$

$x = 0, \quad u = 9$

$$= -2\int_9^0 u^{-\frac{1}{2}}\,du - 4\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$= -2\left[2u^{\frac{1}{2}}\right]_9^0 - 4\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

(1 mark) **(1 mark)**

$$= -2(0-6) - 4\left(\sin^{-1}(1) - \sin^{-1}(0)\right)$$

$$= 12 - 4\left(\frac{\pi}{2} - 0\right)$$

$$= 12 - 2\pi$$

(1 mark)

Question 9

a. $\underset{\sim}{v}(t)=\left(2\sin^2(t)-1\right)\underset{\sim}{i}-\sin(2t)\underset{\sim}{j} \quad t\geq 0$

$\text{speed}=|\underset{\sim}{v}|$

$$=\sqrt{\left(2\sin^2(t)-1\right)^2+\left(-\sin(2t)\right)^2}$$

(1 mark)

$$=\sqrt{4\sin^4(t)-4\sin^2(t)+1+\left(-2\sin(t)\cos(t)\right)^2}$$
$$=\sqrt{4\sin^4(t)-4\sin^2(t)+1+4\sin^2(t)\left(1-\sin^2(t)\right)}$$
$$=\sqrt{4\sin^4(t)-4\sin^2(t)+1+4\sin^2(t)-4\sin^4(t)}$$
$$=\sqrt{1}$$
$$=1$$

so speed is constant.

(1 mark)

b. $\underset{\sim}{v}(t)=\left(2\sin^2(t)-1\right)\underset{\sim}{i}-\sin(2t)\underset{\sim}{j} \quad t\geq 0$

$\underset{\sim}{v}(t)=-\cos(2t)\underset{\sim}{i}-\sin(2t)\underset{\sim}{j}$

$\underset{\sim}{r}(t)=-\frac{1}{2}\sin(2t)\underset{\sim}{i}+\frac{1}{2}\cos(2t)\underset{\sim}{j}+\underset{\sim}{c}$

(1 mark)

Now, when $t=0$, $\underset{\sim}{r}=\frac{1}{2}\underset{\sim}{j}$

so, $\frac{1}{2}\underset{\sim}{j}=0\underset{\sim}{i}+\frac{1}{2}\underset{\sim}{j}+\underset{\sim}{c}$

$\underset{\sim}{c}=\underset{\sim}{0}$

$\underset{\sim}{r}(t)=-\frac{1}{2}\sin(2t)\underset{\sim}{i}+\frac{1}{2}\cos(2t)\underset{\sim}{j}$

as required. **(1 mark)**

c.

$x=-\frac{1}{2}\sin(2t) \qquad y=\frac{1}{2}\cos(2t)$

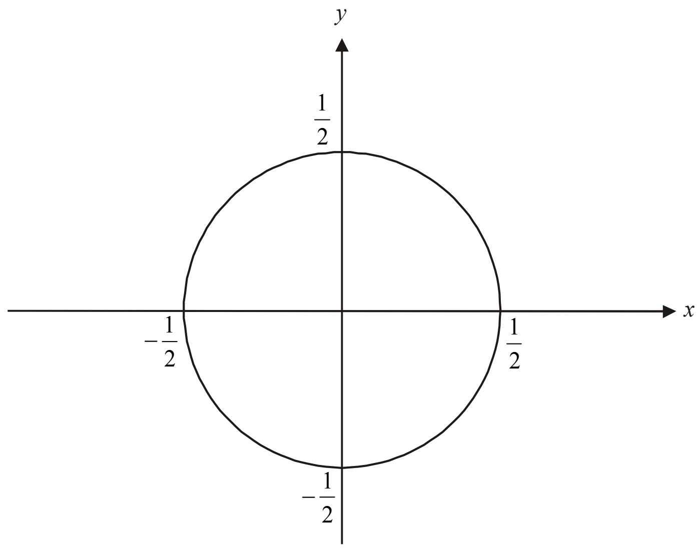
$x^2=\frac{1}{4}\sin^2(2t) \qquad y^2=\frac{1}{4}\cos^2(2t)$

$x^2+y^2=\frac{1}{4}\left(\sin^2(2t)+\cos^2(2t)\right)$

$x^2+y^2=\frac{1}{4}$

(1 mark)

d. **i.**



(1 mark)

ii. At $t = 0$,

$$\underset{\sim}{r} = 0\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j}$$

The particle starts at the point $\left(0, \frac{1}{2}\right)$. **(1 mark)**

At $t = \frac{\pi}{4}$

$$\underset{\sim}{r} = -\frac{1}{2}\underset{\sim}{i} + 0\underset{\sim}{j}$$

At $t = \frac{\pi}{4}$ seconds, the particle is at the point $\left(-\frac{1}{2}, 0\right)$.

So the particle moves around the circle indefinitely in an anticlockwise direction having started it's motion at the point $\left(0, \frac{1}{2}\right)$.

(1 mark)