THE HEFFERNAN GROUP

P.O. Box 1180
Surrey Hills North VIC 3127
ABN 47 122 161 282
Phone 9836 5021
Fax 9836 5025

SPECIALIST MATHS TRIAL EXAMINATION 2 SOLUTIONS 2007

Section 1 – Multiple-choice answers

1.	**D**	**7.**	**A**	**13.**	**E**	**19.**	**A**
2.	**B**	**8.**	**E**	**14.**	**B**	**20.**	**D**
3.	**E**	**9.**	**B**	**15.**	**B**	**21.**	**D**
4.	**B**	**10.**	**E**	**16.**	**C**	**22.**	**B**
5.	**C**	**11.**	**A**	**17.**	**E**		
6.	**C**	**12.**	**D**	**18.**	**C**		

Section 1- Multiple-choice solutions

Question 1

The hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ has asymptotes given by $y-k=\pm\frac{b}{a}(x-h)$.
Now $y=\pm 4(x-2)$, so (h,k) is $(2,0)$ so options A and B can be eliminated.
Also, $\frac{b}{a}=4$. Only option D offers this since $\frac{\sqrt{64}}{\sqrt{4}}=\frac{8}{2}=4$.
The answer is D.

Question 2

The graph of $y=f(x)$ has one vertical asymptote if the equation $x^2+px+q=0$ has one solution.
The quadratic equation $x^2+px+q=0$ has one solution if

$$p^2-4\times 1\times q=0$$
$$p^2=4q$$

The answer is B.

Question 3

For the inverse circular function $y = \text{arsin}(x)$, $d = [-1, 1]$ and $r = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

For the inverse circular function $f(x) = \frac{\pi}{2} + \text{arsin}\left(\frac{x}{a}\right), d_f = [-a, a]$

since $1 \le \frac{x}{a} \le 1$

$-a \le x \le a$

and $r_f = \left[-\frac{\pi}{2} + \frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi}{2}\right]$

$= [0, \pi]$

The answer is E.

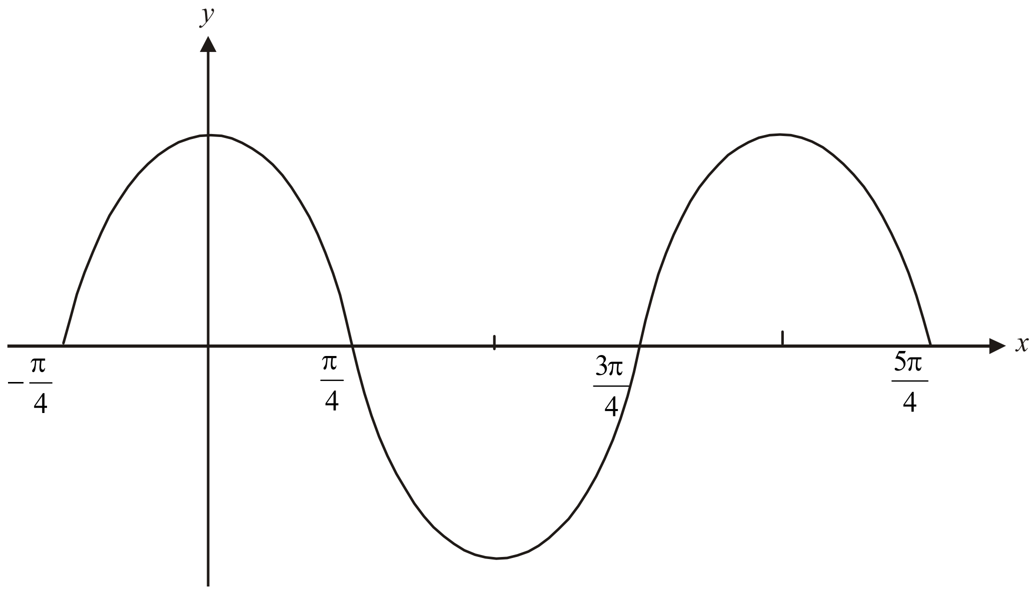
Question 4

$y = \text{cosec}(a(x-b))$

$= \frac{1}{\sin(a(x-b))}$

Asymptotes occur at $x = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$ and $\frac{5\pi}{4}$.

Consider the graph of $y = \sin(a(x-b))$ at which $y = 0$ at $x = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$ and $\frac{5\pi}{4}$.



period $= \pi = \frac{2\pi}{a}$ so $a = 2$ For this graph above, $y = \sin 2\left(x + \frac{\pi}{4}\right)$.

Also, for the graph of $y = \text{cosec}(a(x-b))$,

$y > 0$ for $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$ and $y < 0$ for $x \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$. This is true also for the graph of $y = \sin(a(x-b))$ where $b = -\frac{\pi}{4}$ but not for $b = \frac{\pi}{4}$. So $a = 2$ and $b = -\frac{\pi}{4}$.

The answer is B.

Question 5

$v = a + ai \qquad |v| = \sqrt{a^2 + a^2} \qquad \arg(v) = \tan^{-1}\left(\frac{a}{a}\right)$ (first quadrant)

$\qquad = \sqrt{2a^2} \qquad\qquad = \frac{\pi}{4}$

$\qquad = \sqrt{2}a$ since a is a positive constant.

$$\frac{v}{w} = \frac{\sqrt{2}a\text{cis}\left(\frac{\pi}{4}\right)}{\text{cis}\left(\frac{\pi}{3}\right)}$$

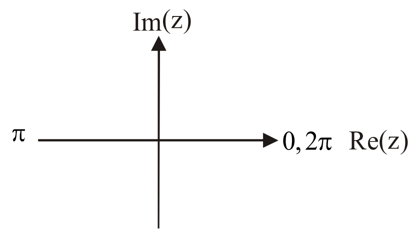
$$= \sqrt{2}a\text{cis}\left(-\frac{\pi}{12}\right)$$

The answer is C.

Question 6

The solutions to the equation $z^{12} = a$ are equally spaced around a circle, $\frac{2\pi}{12} = \frac{\pi}{6}$ apart.

Since one solution is $a^{\frac{1}{12}}\text{cis}\left(\frac{7\pi}{6}\right)$, the others that will have an imaginary part that is less than zero; that is, that have an argument that is greater than π and less than 2π are $a^{\frac{1}{12}}\text{cis}\left(\frac{8\pi}{6}\right)$, $a^{\frac{1}{12}}\text{cis}\left(\frac{9\pi}{6}\right)$, $a^{\frac{1}{12}}\text{cis}\left(\frac{10\pi}{6}\right)$ and $a^{\frac{1}{12}}\text{cis}\left(\frac{11\pi}{6}\right)$.

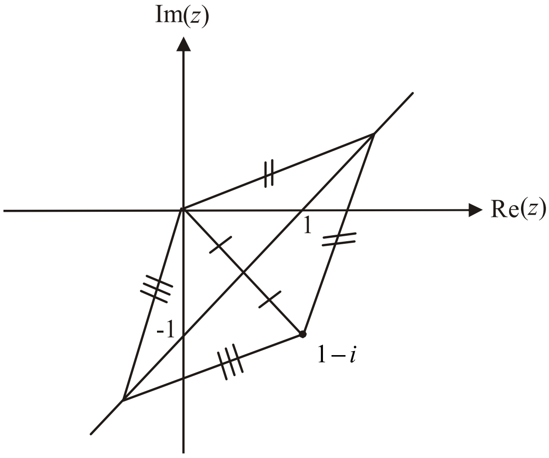


So there are five in total.

Note that the solutions $a^{\frac{1}{12}}cis\left(\frac{6\pi}{6}\right)$ and $a^{\frac{1}{12}}\text{cis}(0)$ have an imaginary part equal to zero.

The answer is C.

Question 7

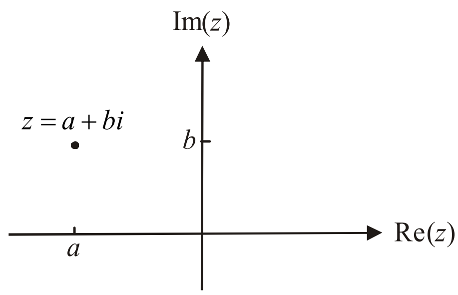


Let z be any point lying on the line.
The distance between any point z on the line and the origin is given by $|z|$. This is the same as the distance between z and the point $1-i$; that is, $|z-(1-i)|$

$$=|z-1+i|.$$

So the required equation of the line is $|z|=|z-1+i|$.
The answer is A.

Question 8



Since z lies in the second quadrant, $a<0$ and $b>0$.
Multiplying z by i rotates z by 90° in an anticlockwise direction which means that zi lies in the third quadrant.

$$|z|=\sqrt{a^2+b^2}$$
$$=\sqrt{2a^2} \text{ if } |a|=b$$
$$=\sqrt{2}\,|a|$$
$$\text{Arg}(z)=\tan^{-1}(-1) \text{ if } |a|=b$$
$$=\frac{3\pi}{4} \text{ since } z \text{ is in the second quadrant}$$

So $\text{Arg}(z)\neq\dfrac{\pi}{4}$

The answer is E.

Question 9

$x^2 - 3y^3 = 1$

When $x = 2$, $\quad 2^2 - 3y^3 = 1$

$-3y^3 = -3$

$y = 1$

$$x^2 - 3y^3 = 1$$

$$2x - 3 \times 3y^2 \times \frac{dy}{dx} = 0 \qquad \text{(implicit differentiation)}$$

$$-9y^2 \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{-9y^2}$$

$$= \frac{2x}{9y^2}$$

When $x = 2$, $y = 1$ and $\frac{dy}{dx} = \frac{4}{9}$.

The answer is B.

Question 10

$$\int_0^{\frac{\pi}{6}} \sin^3(x)\cos^4(x)dx$$

$$= \int_0^{\frac{\pi}{6}} \sin^2(x)\cos^4(x)\sin(x)dx$$

$$= \int_0^{\frac{\pi}{6}} \left(1 - \cos^2(x)\right)\cos^4(x)\sin(x)dx$$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$

$x = \frac{\pi}{6}$, $u = \frac{\sqrt{3}}{2}$

$x = 0$, $\quad u = 1$

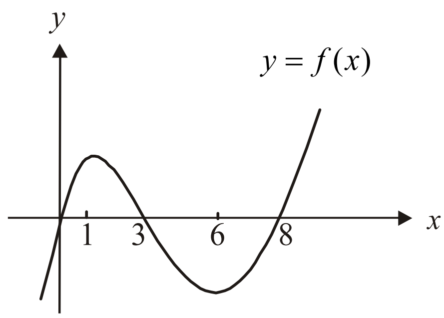
$$= \int_1^{\frac{\sqrt{3}}{2}} \left(1 - u^2\right)u^4 \times -\frac{du}{dx}dx$$

$$= \int_{\frac{\sqrt{3}}{2}}^1 \left(u^4 - u^6\right)du$$

The answer is E.

Question 11

Do a quick sketch of $y = f(x)$.



The graph of $y = f(x)$ is the gradient function of the graph of $y = F(x)$.
At $x = 0,\ 3$ and 8, the gradient of $y = F(x)$ is zero, so there must be a turning point or a stationary point of inflection at $x = 0, x = 3$ and $x = 8$.
Only options A and D show this.
The gradient of the graph of $y = F(x)$ is positive for $0 < x < 3$ and $x > 8$ and negative for $x < 0$ and $3 < x < 8$.
Only option A shows this.
The answer is A.

Question 12

From the slope field when $x = 0,\ \frac{dy}{dx} = 0$. This eliminates options A, B, C and E.
When $x = 0,\ \frac{dy}{dx}$ is defined; that is, values of $\frac{dy}{dx}$ are indicated by the lines. Conversely when $y = 0$, no such lines are apparent because $\frac{dy}{dx}$ is not defined for $y = 0$. Option D shows this.
Also for a given value of *y*; for example $y = 1$, as *x* increases from zero, the gradient becomes steeper which is indicated by a larger negative number. For another given value of *y*; for example $y = -1$, as *x* increases from zero, the gradient becomes steeper which is indicated by a larger positive number and so on.
The answer is D.

Question 13

The two particles meet iff

$$\begin{aligned} 4 &= t + 1 \quad \text{AND} \quad 2t - 6 = 8 \\ t &= 3 \qquad\qquad\qquad t = 7 \end{aligned}$$

Since the particles are in the same spot at different times, they never meet.
The answer is E.

Question 14

$\underset{\sim}{x}(t)=\frac{1}{t}\underset{\sim}{i}+\sqrt{t}\,\underset{\sim}{j}+e^{2t}\,\underset{\sim}{k}$

$\dot{\underset{\sim}{x}}(t)=-1t^{-2}\,\underset{\sim}{i}+\frac{1}{2}t^{-\frac{1}{2}}\,\underset{\sim}{j}+2e^{2t}\,\underset{\sim}{k}$

$\dot{\underset{\sim}{x}}(1)=-\underset{\sim}{i}+\frac{1}{2}\underset{\sim}{j}+2e^{2}\,\underset{\sim}{k}$

The direction of motion at $t=1$ is given by $\dot{\underset{\sim}{x}}(1)$.

The answer is B.

Question 15

Sketch the graph of $y=x^5+2x^3+x+1$ and find the enclosed area.
This area occurs between $x=-0\cdot5698403$ and $x=0$.
The required area is closest to 0.34905 square units.
The answer is B.

Question 16

$\frac{dy}{dx}=\sqrt{x-9},\qquad x_0=9,\qquad y_0=0,\ h=0\cdot1$

$x_{n+1}=x_n+h,\qquad y_{n+1}=y_n+hf(x_n)$

So, $x_1=9+0\cdot1$
$=9\cdot1$

$y_1=0+0\cdot1\times\sqrt{9-9}$
$=0$

$x_2=9\cdot1+0\cdot1$
$=9\cdot2$

$y_2=0+0\cdot1\times\sqrt{9\cdot1-9}$
$=0\cdot1\times\sqrt{0\cdot1}$
$=0\cdot0316...$

When $x=9\cdot2$, the approximation for y is 0.0316…
The answer is C.

Question 17

$\underset{\sim}{a}$ and $\underset{\sim}{b}$ are at right angles because the triangle formed by $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ has it's vertices lying on a semicircle with one side forming the straight edge. So option A is correct. Option B is also correct since

$$\underset{\sim}{a} \bullet \underset{\sim}{a} = |\underset{\sim}{a}||\underset{\sim}{a}|\cos 0^\circ$$

$$= |\underset{\sim}{a}|^2$$

Similarly $\underset{\sim}{b} \bullet \underset{\sim}{b} = |\underset{\sim}{b}|^2$ and $\underset{\sim}{c} \bullet \underset{\sim}{c} = |\underset{\sim}{c}|^2$.

So because of Pythagoras theorem options B and C are correct.

$$\underset{\sim}{a} \bullet (\underset{\sim}{c} + \underset{\sim}{b}) = \underset{\sim}{a} \bullet \underset{\sim}{a}$$

$$= |\underset{\sim}{a}|^2$$

So option D is correct.
Option E is incorrect. The angle θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{c}$ not between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
The answer is E.

Question 18

$$\left|6\underset{\sim}{i} + 2\underset{\sim}{j} + 2\sqrt{6}\,\underset{\sim}{k}\right|$$
$$= \sqrt{36+4+24}$$
$$= \sqrt{64}$$
$$= 8$$

A vector with a magnitude of 4 that is parallel to the vector $6\underset{\sim}{i} + 2\underset{\sim}{j} + 2\sqrt{6}\,\underset{\sim}{k}$ is

$$4 \times \frac{1}{8}\left(6\underset{\sim}{i} + 2\underset{\sim}{j} + 2\sqrt{6}\,\underset{\sim}{k}\right)$$
$$= 3\underset{\sim}{i} + \underset{\sim}{j} + \sqrt{6}\,\underset{\sim}{k}$$

The answer is C.

Question 19

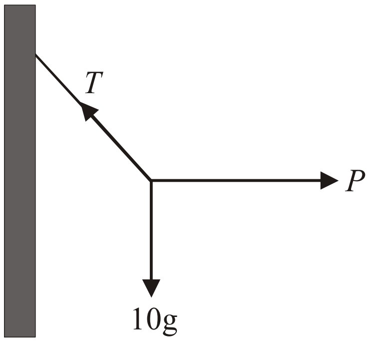
The change in temperature of the cake with respect to time is given by

$$\frac{dT}{dt} = -k(T-22),\ T(0) = 180$$

The answer is A.

Question 20

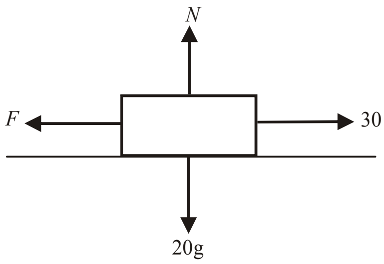
The correct force diagram is given by



The answer is D.

Question 21

Draw a force diagram.



Now $\mu N = 0\cdot 2 \times 20g$

$= 4g$

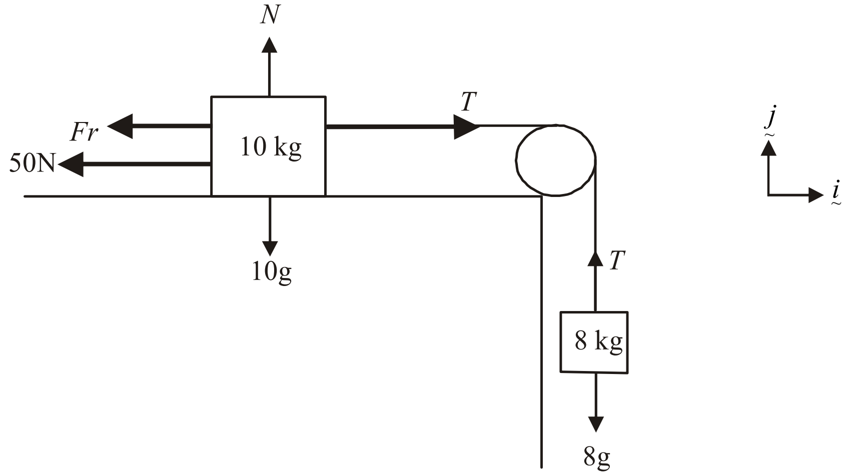
Since $F = 30$

$F < \mu N$

i.e. $F < 4g$

So the box is not at the point of sliding along the surface.
The box remains at rest because $F < 4g$.
The answer is D.

Question 22



Around the 8kg mass:

$$\underset{\sim}{R} = m\underset{\sim}{a}$$

$(T - 8g)\underset{\sim}{j} = -8a\underset{\sim}{j}$ assuming that the 8kg mass accelerates downwards

$$T - 8g = -8a$$

$$T = 8g - 8a \qquad \text{(A)}$$

Around the 10kg mass:

$$\underset{\sim}{R} = m\underset{\sim}{a}$$

$(T - Fr - 50)\underset{\sim}{i} + (N - 10g)\underset{\sim}{j} = 10a\underset{\sim}{i}$ assuming that the 10 kg mass accelerates to the right

$$T - \mu N - 50 = 10a \qquad N = 10g$$

$$T - 0\cdot 1 \times 10g - 50 = 10a$$

$$T = 10a + g + 50 \qquad \text{(B)}$$

Substitute (A) into (B):

$$8g - 8a = 10a + g + 50$$

$$-18a = -7g + 50$$

$$a = \frac{7g - 50}{18}$$

Since $a > 0$, the 10kg mass does move to the right and the 8kg mass does move downwards.

The answer is B.

SECTION 2

Question 1

a.

$$\underset{\sim}{a} = \overrightarrow{OA} = m\underset{\sim}{i} + 2\underset{\sim}{j}$$

$$\underset{\sim}{b} = \overrightarrow{OB} = \underset{\sim}{i} + 6\underset{\sim}{j}$$

$$\underset{\sim}{c} = \overrightarrow{OC} = n\underset{\sim}{i} + 6\underset{\sim}{j}$$

$$\underset{\sim}{d} = \overrightarrow{OD} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$$

Now, $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$$= -m\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{i} + 6\underset{\sim}{j}$$

$$= (1 - m)\underset{\sim}{i} + 4\underset{\sim}{j}$$

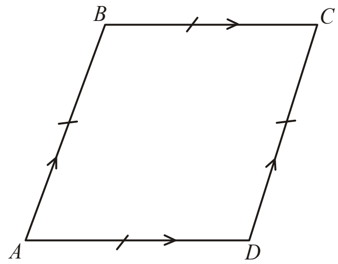
$$\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD}$$

$$= -m\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{i} + 2\underset{\sim}{j}$$

$$= (3 - m)\underset{\sim}{i}$$

(1 mark)

b. If $ABCD$ is a rhombus then $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AD}\right|$ and $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{BC} = \overrightarrow{AD}$.



Now, $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AD}\right|$

$$\sqrt{(1-m)^2 + 4^2} = \sqrt{(3-m)^2}$$

$$1 - 2m + m^2 + 16 = 9 - 6m + m^2$$

$$4m = -8$$

$$m = -2$$

(1 mark)

If $\overrightarrow{AB} = \overrightarrow{DC}$, then $(1-m)\underset{\sim}{i} + 4\underset{\sim}{j} = (n-3)\underset{\sim}{i} + 4\underset{\sim}{j}$

$$1 - m = n - 3$$

$$n + m = 4$$

(1 mark)

Substituting $m = -2$

into $n + m = 4$

gives $n - 2 = 4$

$n = 6$

(1 mark)

So $m = -2$ and $n = 6$

c. <u>To Show</u> $\overrightarrow{AC} \bullet \overrightarrow{BD} = 0$

(1 mark)

Now, $\underset{\sim}{a} = -2\underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{b} = \underset{\sim}{i} + 6\underset{\sim}{j}$

$\underset{\sim}{c} = 6\underset{\sim}{i} + 6\underset{\sim}{j}$

$\underset{\sim}{d} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

$$LS = \left(\overrightarrow{AO} + \overrightarrow{OC}\right) \bullet \left(\overrightarrow{BO} + \overrightarrow{OD}\right)$$

$$= \left(2\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{i} + 6\underset{\sim}{j}\right) \bullet \left(-\underset{\sim}{i} - 6\underset{\sim}{j} + 3\underset{\sim}{i} + 2\underset{\sim}{j}\right)$$

$$= \left(8\underset{\sim}{i} + 4\underset{\sim}{j}\right) \bullet \left(2\underset{\sim}{i} - 4\underset{\sim}{j}\right)$$

$$= 16 - 16$$

$$= 0$$

$$= RS$$

(1 mark)

Have shown.

Since $\overrightarrow{AC} \bullet \overrightarrow{BD} = 0$, then $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{BD}$.

d.

$$\overrightarrow{AE} = \left(\overrightarrow{AB} \bullet \hat{\overrightarrow{AD}}\right)\hat{\overrightarrow{AD}}$$ **(1 mark)**

$$= \left(\left(3\underset{\sim}{i} + 4\underset{\sim}{j}\right) \bullet \frac{1}{5}\left(5\underset{\sim}{i}\right)\right)\frac{1}{5}\left(5\underset{\sim}{i}\right)$$

$$= 3 \times \frac{1}{5}\left(5\underset{\sim}{i}\right)$$

$$= 3\underset{\sim}{i}$$

(1 mark)

e.

$$\overrightarrow{AB} \bullet \overrightarrow{AD} = \left|\overrightarrow{AB}\right|\left|\overrightarrow{AD}\right|\cos\theta$$

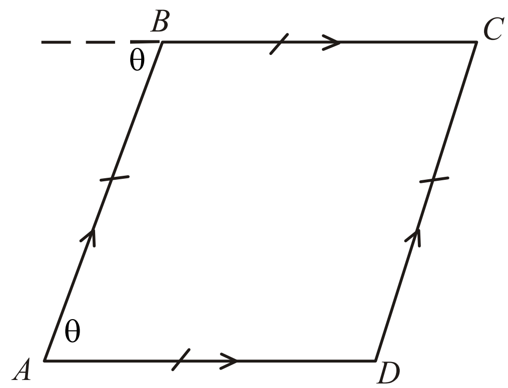
$$\left(3\underset{\sim}{i} + 4\underset{\sim}{j}\right) \bullet \left(5\underset{\sim}{i}\right) = \sqrt{9+16}\sqrt{5^2}\cos\left(\theta^\circ\right)$$ **(1 mark)**

$$15 = 25\cos\left(\theta^\circ\right)$$

$$\cos\left(\theta^\circ\right) = \frac{3}{5}$$ **(1 mark)**

f.

In the rhombus *ABCD,*
θ° is the angle between $\overrightarrow{AB}$ and $\overrightarrow{AD}$
so the angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ will be $(180 - \theta)^\circ$ because of alternate angles in parallel lines. **(1 mark)**



Now,

$\cos(180 - \theta)^\circ$

$= -\cos(\theta^\circ)$ (because $\theta < 90^\circ$ and $(180^\circ - \theta)$ is a second quadrant angle)

$= -\frac{3}{5}$ **(1 mark)**

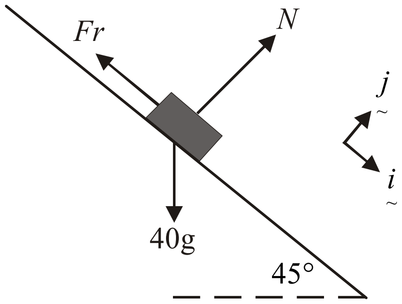
So the angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ is $\cos^{-1}\left(-\frac{3}{5}\right) = 126^\circ 52'$ to the nearest minute.

(1 mark)

Total 13 marks

Question 2

a.



(1 mark)

b.

$$R = m\underset{\sim}{a}$$

$$\left(40g\sin(45^\circ) - Fr\right)\underset{\sim}{i} + \left(N - 40g\cos 45^\circ\right)\underset{\sim}{j} = 40a\,\underset{\sim}{i}$$ **(1 mark)**

So, $\frac{40g}{\sqrt{2}} - Fr = 40a$ and $N - \frac{40g}{\sqrt{2}} = 0$

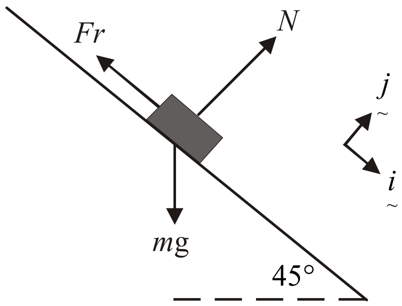
$20\sqrt{2}g - 0\cdot1\times 20\sqrt{2}g = 40a$ $\qquad N = 20\sqrt{2}g$ **(1 mark)**

$$a = \frac{18\sqrt{2}g}{40}$$

$$= \frac{9\sqrt{2}g}{20}\text{ m/s}^2$$ **(1 mark)**

as required.

c. Let the mass of the person on the slide be *m* kg.



$$\underset{\sim}{R} = m\underset{\sim}{a}$$

$$\left(mg\sin(45^\circ) - Fr\right)\underset{\sim}{i} + \left(N - mg\cos 45^\circ\right)\underset{\sim}{j} = ma\,\underset{\sim}{i}$$

$\frac{mg}{\sqrt{2}} - 0\cdot1\times\frac{mg}{\sqrt{2}} = ma$ $\qquad N = \frac{mg}{\sqrt{2}}$

$$m\left(\frac{g - 0\cdot1g}{\sqrt{2}}\right) = ma$$ **(1 mark)**

$$a = \frac{0\cdot9\sqrt{2}g}{2}$$

$$= \frac{9\sqrt{2}g}{20}\quad\text{as required}$$ **(1 mark)**

d. Since Rhiannon's acceleration down the slide is constant, and we know that $u = 0$, $a = \frac{9\sqrt{2}g}{20}$ and $s = 15$, we can use the formula

$$s = ut + \frac{1}{2}at^2$$ **(1 mark)**

$$15 = 0 + \frac{1}{2} \times \frac{9\sqrt{2}g}{20}t^2$$

$$t = \pm\sqrt{4 \cdot 8102...}$$

$$= 2.1932... \quad \text{since } t \ge 0$$

So Rhiannon is on the slide for 2.19 seconds (correct to 2 decimal places)

(1 mark)

e. Now, a is constant i.e. $a = \frac{9\sqrt{2}g}{20}$, $u = 0$ and $t = 2 \cdot 1932...$ (from part **d.**). Note that the "unrounded" value of 2.1932… from part **d.** should be carried through in these calculations.

We have,

$$v = u + at$$

$$v = 0 + \frac{9\sqrt{2}g}{20} \times 2 \cdot 1932...$$

$$= 13 \cdot 6784...$$

Rhiannon's speed at the end of the slide is 13.68 m/s (correct to 2 decimal places).

(1 mark)

f. Note that the "unrounded" value of 13.6784… from part **e.** should be carried through in these calculations.

Method 1

The only force acting on Rhiannon when she leaves the end of the slide is the gravitational force.

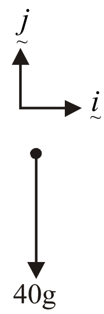
$$\underset{\sim}{R} = m\,\underset{\sim}{a}$$

becomes $-40g\,\underset{\sim}{j} = 40\,\underset{\sim}{a}$

$$\underset{\sim}{a} = -g\,\underset{\sim}{j}$$

$$\frac{d\underset{\sim}{v}}{dt} = -g\,\underset{\sim}{j}$$

$$\underset{\sim}{v} = -gt\,\underset{\sim}{j} + \underset{\sim}{c} \qquad -(1)$$



When $t = 0$, which is when Rhiannon leaves the slide,

$$\underset{\sim}{v} = 13 \cdot 6784\cos(45^\circ)\underset{\sim}{i} - 13 \cdot 6784\sin(45^\circ)\underset{\sim}{j}\ .$$

$$= \frac{13 \cdot 6784}{\sqrt{2}}\underset{\sim}{i} - \frac{13 \cdot 6784}{\sqrt{2}}\underset{\sim}{j}$$

In (1) $\frac{13 \cdot 6784}{\sqrt{2}}\underset{\sim}{i} - \frac{13 \cdot 6784}{\sqrt{2}}\underset{\sim}{j} = \underset{\sim}{c}$

So $\underset{\sim}{v} = \frac{13 \cdot 6784}{\sqrt{2}}\underset{\sim}{i} - \left(\frac{13 \cdot 6784}{\sqrt{2}} + gt\right)\underset{\sim}{j}$ **(1 mark)**

Note: the above working can be shortened.
It is shown here for explanatory purposes.

Now $\underset{\sim}{r} = \frac{13 \cdot 6784}{\sqrt{2}} t \underset{\sim}{i} - \left(\frac{13 \cdot 6784}{\sqrt{2}} t + \frac{gt^2}{2} \right) \underset{\sim}{j} + \underset{\sim}{c}_1$

When $t = 0, \underset{\sim}{r} = 0 \underset{\sim}{i} + 0 \underset{\sim}{j}$, taking the end of the slide as the origin of motion.

So $\underset{\sim}{c}_1 = \underset{\sim}{0}$

So $\underset{\sim}{r} = \frac{13 \cdot 6784}{\sqrt{2}} t \underset{\sim}{i} - \left(\frac{13 \cdot 6784}{\sqrt{2}} t + \frac{gt^2}{2} \right) \underset{\sim}{j}$

(1 mark)

Rhiannon lands when the $\underset{\sim}{j}$ component of $\underset{\sim}{r}$ equals – 3 (3 metres below the origin).

So $\frac{13 \cdot 6784}{\sqrt{2}} t + \frac{gt^2}{2} = 3$

$$t = 0.2725... \quad (t > 0)$$ **(1 mark)**

Rhiannon is in free fall for 0.27 secs (correct to 2 decimal places).

Method 2

Rhianon's velocity at the end of the slide is 13.6784…m/s at an angle of $45°$ downwards.
Taking the downwards direction as positive and considering the vertical component we have $u = \frac{13.6784}{\sqrt{2}}, a = 9.8$ and $s = 3$.

$$s = ut + \frac{1}{2}at^2$$ **(1 mark)**

$$3 = \frac{13.6784}{\sqrt{2}} t + \frac{9.8}{2} t^2$$

$$4.9t^2 + 9.6721t - 3 = 0$$ **(1 mark)**

$$t = \frac{-9.6721 \pm \sqrt{9.6721^2 - 4 \times 4.9 \times -3}}{9.8}$$

$$= 0.2725... \text{ since t} > 0$$

Rhiannon is in free fall for 0.27 secs (correct to 2 decimal places).

(1 mark)

g. Let the angle at which Rhiannon enters the water be θ.
Note again that the "unrounded" value of $t = 0.2725...$ together with the "unrounded" value of 13.6784… from part **e.** should be carried through in these calculations.

Method 1
The angle θ is the angle between $\underset{\sim}{v}$ at $t = 0\cdot 2725...$ secs (i.e. when she hits the water) and the vector $\underset{\sim}{i}$.

So, $\underset{\sim}{v} = \frac{13\cdot 6784}{\sqrt{2}}\underset{\sim}{i} - \left(\frac{13\cdot 6784}{\sqrt{2}} + 0\cdot 2725g\right)\underset{\sim}{j}$ **(1 mark)**

$\underset{\sim}{v} \bullet \underset{\sim}{i} = |\underset{\sim}{v}||\underset{\sim}{i}|\cos(\theta)$

$$\frac{13\cdot 6784}{\sqrt{2}} = \sqrt{\frac{13\cdot 6784^2}{2} + \left(\frac{13\cdot 6784}{\sqrt{2}} + 0\cdot 2725g\right)^2} \times 1 \times \cos(\theta)$$

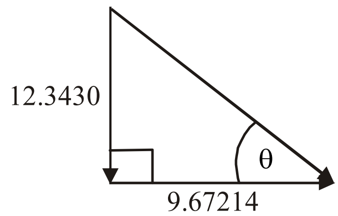
$\cos(\theta) = 0\cdot 6167...$

$\theta = 51^\circ 55'$ to the nearest minute

(1 mark)

Method 2
Calculate the vertical and horizontal velocities at the point of entry.
vertically:

$$\begin{aligned} v &= u + at \\ &= 9.67214 + 9.8 \times 0.27254 \\ &= 12.3430 \text{(down)} \end{aligned}$$



horizontally:

$v = 9.67214$ (constant)

(1 mark)

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{12.3430}{9.67214}\right) \\ &= 51^\circ 55' \text{ to the nearest minute} \end{aligned}$$

(1 mark)

Total 14 marks

Question 3

a. **i.**

$$\dot{y}(t)=\int\frac{-t}{\left(a^2-t^2\right)^{\frac{3}{2}}}dt,\ \text{t}>0$$ **(1 mark)**

where $u=a^2-t^2$

$$\frac{du}{dt}=-2t$$

$$=\int\frac{1}{2}\frac{du}{dt}\cdot u^{-\frac{3}{2}}dt$$

$$=\frac{1}{2}\int u^{-\frac{3}{2}}du$$

(1 mark)

ii.

$$\dot{y}(t)=\frac{1}{2}\times u^{-\frac{1}{2}}\times-2+c$$

$$=\frac{-1}{\sqrt{u}}+c$$

$$=\frac{-1}{\sqrt{a^2-t^2}}+c$$

Since $\dot{y}(1)=\dfrac{-1}{\sqrt{a^2-1}}$

$$-\frac{1}{\sqrt{a^2-1}}=\frac{-1}{\sqrt{a^2-1}}+c$$

$$c=0$$

(1 mark)

$$\dot{y}(t)=\frac{-1}{\sqrt{a^2-t^2}}\ \text{as required}$$

b.

$$y(t)=\int\frac{-1}{\sqrt{a^2-t^2}}dt,\quad t<a$$

$$=\arccos\left(\frac{t}{a}\right)+c$$

Now, $y\left(\dfrac{a}{2}\right)=\dfrac{\pi}{3}$

so $$\frac{\pi}{3}=\arccos\left(\frac{1}{2}\right)+c$$

$$\frac{\pi}{3}=\frac{\pi}{3}+c$$

$$c=0$$

$$y(t)=\arccos\left(\frac{t}{a}\right)$$ **(1 mark)**

c.

$$x = -a\log_e(t)$$

$$\frac{dx}{dt} = \frac{-a}{t}$$

$$\frac{d^2x}{dt^2} = \frac{a}{t^2}$$ **(1 mark)**

Now, $t^2\frac{d^2x}{dt^2} = a(1-t) - t^2\frac{dx}{dt}$

$$LS = t^2 \times \frac{a}{t^2}$$
$$= a$$
$$RS = a(1-t) - t^2\frac{dx}{dt}$$
$$= a - at - t^2 \times \frac{-a}{t}$$
$$= a - at + at$$
$$= a$$
$$= LS$$

Have verified.

(1 mark)

d. From part **b.**, $y(t) = \text{arcos}\left(\frac{t}{a}\right)$.

From part **c.**, $\frac{d^2x}{dt^2} = \frac{a}{t^2} = \ddot{x}(t)$.

So a possible expression for $x(t) = -a\log_e(t) + c$. **(1 mark)**

Given the condition $x(1) = 0$, we have

$$0 = -a\log_e(1) + c$$
$$0 = 0 + c$$
$$c = 0$$

So $-a\log_e(t)$ is the actual expression for $x(t)$.

So, $\underset{\sim}{r} = -a\log_e(t)\underset{\sim}{i} + \text{arcos}\left(\frac{t}{a}\right)\underset{\sim}{j}, \quad t > 0$

(1 mark)

e.

$$x = -a\log_e(t) \qquad y = \arccos\left(\frac{t}{a}\right)$$

$$\frac{-x}{a} = \log_e(t) \qquad \frac{t}{a} = \cos(y)$$

$$e^{-\frac{x}{a}} = t \qquad t = a\cos(y)$$

So,

$$e^{-\frac{x}{a}} = a\cos(y)$$

$$\frac{1}{ae^{\frac{x}{a}}} = \cos(y)$$

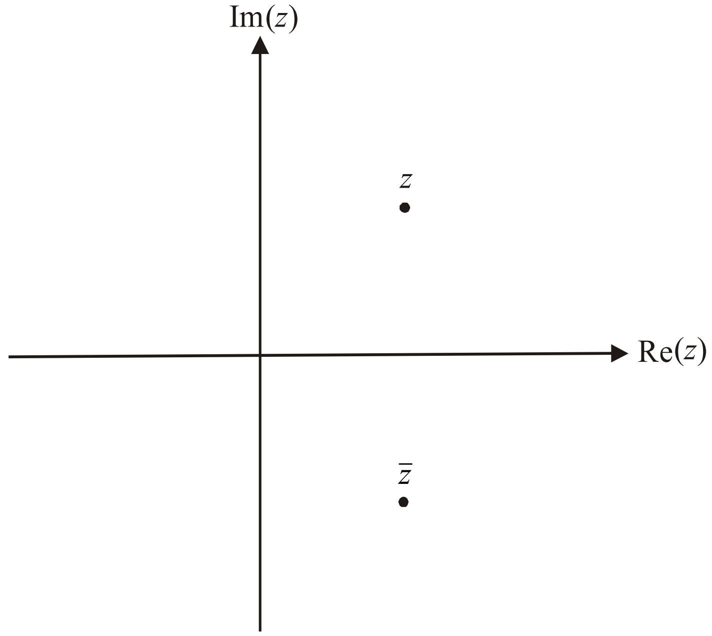
$$y = \arccos\left(\frac{1}{ae^{\frac{x}{a}}}\right)$$

(1 mark)

Total 9 marks

Question 4

a. **i.**



(1 mark)

ii. $\text{Arg}(z)=\theta$ so $\text{Arg}(\bar{z})=-\theta$

(1 mark)

iii. $z=r\text{cis}(\theta)$ so $\bar{z}=r\text{cis}(-\theta)$

(1 mark)

iv. To Show $(\bar{z})^n=\overline{(z^n)}$

$$\begin{aligned}LS&=(\bar{z})^n\\&=(r\text{cis}(-\theta))^n\\&=r^n\text{cis}(-n\theta)\ (\text{De Moivre})\end{aligned}$$

(1 mark)

$$\begin{aligned}RS&=\overline{(z^n)}\\&=\overline{(r\text{cis}(\theta))^n}\\&=\overline{r^n\text{cis}(n\theta)}\quad(\text{De Moivre})\\&=r^n\text{cis}(-n\theta)\\&=LS\end{aligned}$$

(1 mark)

Have shown.

b. To Show : $z^n + (\bar{z})^n = 2r^n \cos(n\theta)$

$$\begin{aligned} LS &= z^n + (\bar{z})^n \\ &= r^n cis(n\theta) + r^n cis(-n\theta) \\ &= r^n(\cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)) \\ &= r^n(\cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)) \\ &= r^n \times 2\cos(n\theta) \\ &= 2r^n \cos(n\theta) \\ &= RS \end{aligned}$$

(1 mark)

since $\cos(-\theta) = \cos(\theta)$
and $\sin(-\theta) = -\sin(\theta)$

Have shown.

(1 mark)

c. Let $(1+i)^{12} + (1-i)^{12}$

$= z^n + (\bar{z})^n$ where $z = 1+i$ from part **b.** – because of "hence" this must be shown

$= 2r^n \cos(n\theta)$ **(1 mark)**

$= 2(\sqrt{2})^{12} \cos\left(12 \times \frac{\pi}{4}\right)$

$= 128\cos(3\pi)$

$= 128\cos(\pi)$

$= -128$

Have shown.

since if $z = 1+i$

$r = |z| = \sqrt{2}$

$\theta = \tan^{-1}(1)$

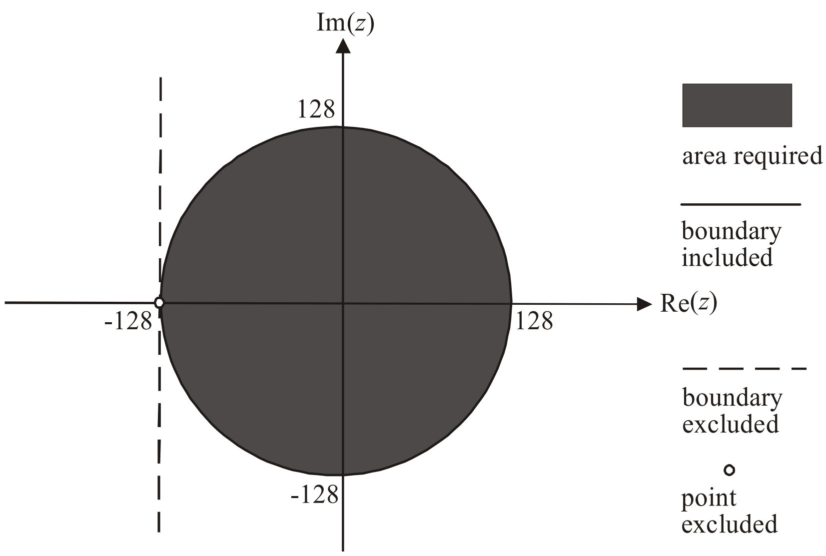
$= \frac{\pi}{4}$

Also, $n = 12$

(1 mark)

d. From part c. $(1+i)^{12} + (1-i)^{12} = -128$

So we require $\{z : \text{Re}(z) > -128\} \cap \{z : |z| \le 128\}$



Note that the complex number $z = -128$ is <u>not</u> included.

(1 mark) correct area
(1 mark) correct boundary
Total 11 marks

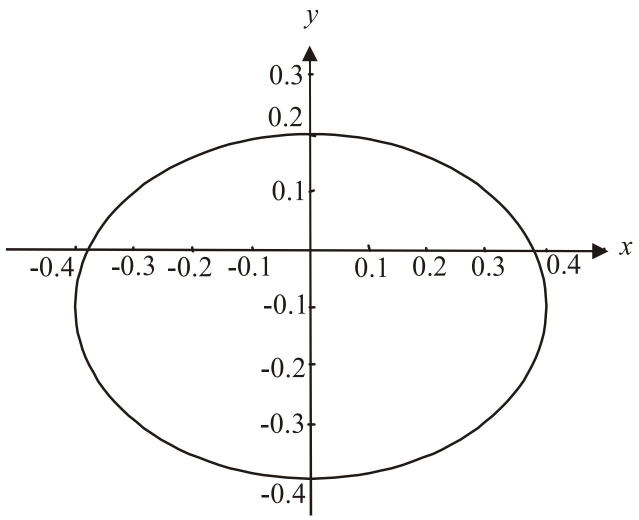
Question 5

a.

$$16(y+0\cdot 1)^2+9x^2=1\cdot 44$$
$$\frac{(y+0\cdot 1)^2}{9}+\frac{x^2}{16}=0\cdot 01$$
$$\frac{100(y+0\cdot 1)^2}{9}+\frac{100x^2}{16}=1$$
$$\frac{x^2}{0\cdot 16}+\frac{(y+0\cdot 1)^2}{0\cdot 09}=1$$

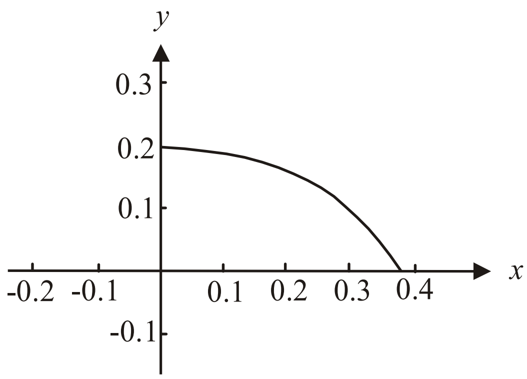
We have an ellipse with centre at $(0,-0\cdot 1)$, semi-major axis length of $0\cdot 4$ units and semi-minor axis length of $0\cdot 3$ units.

(1 mark)



(1 mark)

b.



$$\text{volume required}=\pi\int_0^{0\cdot 2}x^2\,dy$$
$$=\frac{\pi}{9}\int_0^{0\cdot 2}\left(1\cdot 44-16(y+0\cdot 1)^2\right)dy$$

(1 mark) - correct terminals
(1 mark) – correct integrand

c. Using a graphics or CAS calculator and the definite integral above,
$\text{volume} = 0\cdot 0521\text{m}^3$ (correct to 4 decimal places) **(1 mark)**

d. **i.**

$$\frac{dS}{dt} = \text{rate of inflow of salt} - \text{rate of outflow of salt}$$ **(1 mark)**

$$= \frac{dS_{\text{inflow}}}{dl}\cdot\frac{dl}{dt} - \frac{dS_{\text{outflow}}}{dl}\cdot\frac{dl}{dt} \quad \text{where } l \text{ represents litres}$$

$$= 35\times 0\cdot 5 - \frac{S}{50}\times 0\cdot 5$$

$$= 17\cdot 5 - \frac{S}{100}$$

$$= \frac{1750 - S}{100}$$

(1 mark)

ii.

$$\frac{dt}{dS} = \frac{100}{1750 - S}$$

$$t = \int \frac{100}{1750 - S}\, dS$$

$$= -100\log_e(1750 - S) + c$$ **(1 mark)**

Now, when $t = 0$, $S = 0$ because initially the base contained no sea water.

$$0 = -100\log_e(1750) + c$$

$$c = 100\log_e(1750)$$

$$t = 100\log_e\left(\frac{1750}{1750 - S}\right)$$

(1 mark)

iii. Method 1

The concentration of salt in the base will approach the concentration of the sea water being added which is 35g/l. **(1 mark)**

$\text{Total mass} = \frac{35g}{l}\times 50l = 1750g$. **(1 mark)**

Method 2

$$t = 100\log_e\left(\frac{1750}{1750 - S}\right)$$

$$\frac{t}{100} = \log_e\left(\frac{1750}{1750 - S}\right)$$

$$e^{\frac{t}{100}} = \frac{1750}{1750 - S}$$

$$1750 - S = 1750e^{\frac{-t}{100}}$$

$$S = 1750\left(1 - e^{\frac{-t}{100}}\right)$$ **(1 mark)**

As $t \to \infty, e^{\frac{-t}{100}} \to 0^+$ so $S \to 1750^-$. So in the long term the amount of salt in the base would approach 1750g. **(1 mark)**

Total 11 marks