Year 2007

VCE

Specialist Mathematics Solutions Trial Examination 1

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Question 1

$\dfrac{x}{e^{2y}} - 5y + 6x^2 + 3 = 0$ taking $\dfrac{d}{dx}$ of each term (implicit differentiation)

$\dfrac{d}{dx}\left(xe^{-2y}\right) - \dfrac{d}{dx}(5y) + \dfrac{d}{dx}\left(6x^2\right) + \dfrac{d}{dx}(3) = 0$ M1

product rule in the first term

$x\dfrac{d}{dy}\left(e^{-2y}\right)\dfrac{dy}{dx} + e^{-2y}\dfrac{d}{dx}(x) - \dfrac{d}{dy}(5y)\dfrac{dy}{dx} + \dfrac{d}{dx}\left(6x^2\right) + \dfrac{d}{dx}(3) = 0$

$-2xe^{-2y}\dfrac{dy}{dx} + e^{-2y} - 5\dfrac{dy}{dx} + 12x = 0$

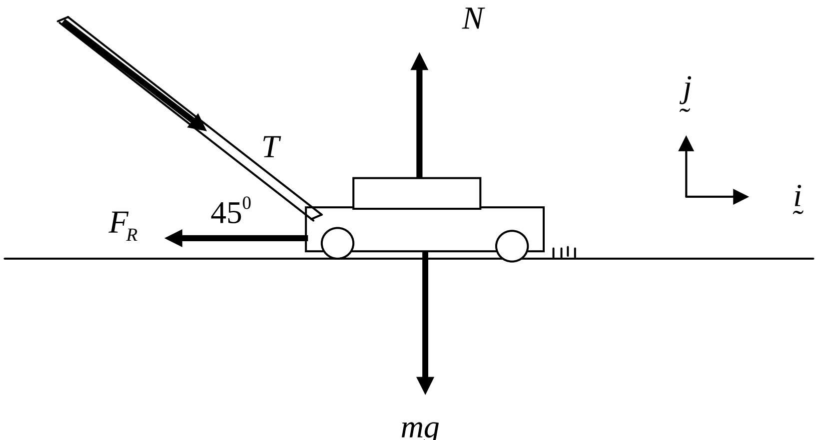
M1

$\dfrac{dy}{dx}\left(2xe^{-2y} + 5\right) = e^{-2y} + 12x$

$\dfrac{dy}{dx} = \dfrac{e^{-2y} + 12x}{2xe^{-2y} + 5}$ A1

Question 2

a.



for all the forces A1

b. now $T = 12\sqrt{2}$ $m = 10\,\text{kg}$ $g = 9.8$ $\mu = \dfrac{1}{7}$ $\theta = 45^0$

resolving horizontally to the lawn in the $\underset{\sim}{i}$ direction.

(1) $T\cos(\theta) - F_R = 0$

from (1) $F_R = T\cos(\theta) = 12\sqrt{2}\cos\left(45^0\right) = 12$ A1

resolving perpendicular to the lawn in the $\underset{\sim}{j}$ direction.

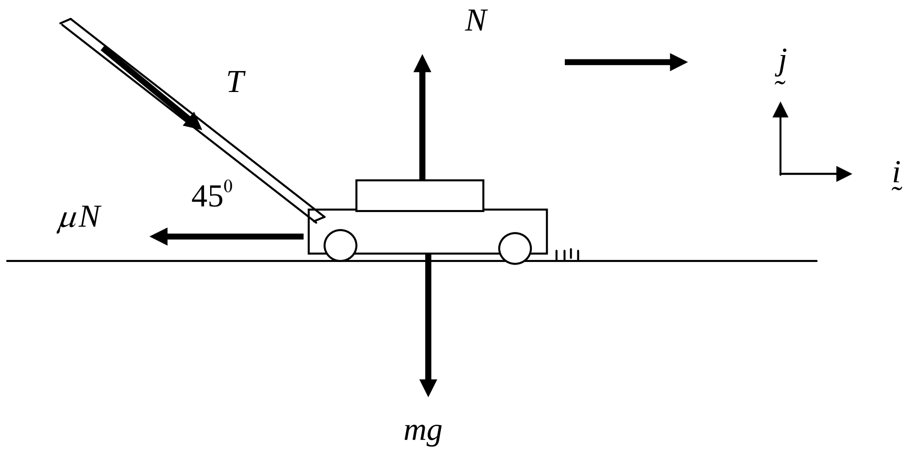
(2) $N - T\sin(\theta) - mg = 0$

from (2) $N = mg + T\sin(\theta) = 10 \times 9.8 + 12\sqrt{2}\cos\left(45^0\right) = 98 + 12 = 110$

and $\mu N = \dfrac{110}{7} = 15\dfrac{5}{7}$

since $\mu N > F_R$ it is **not** on the point of moving A1

c.



now $\quad T=\dfrac{49\sqrt{2}}{2} \quad m=10\,\text{kg} \quad g=9.8 \quad \mu=\dfrac{1}{7} \quad \theta=45^0 \quad a=?$

resolving horizontally to the lawn in the $\underset{\sim}{i}$ direction.

$(1) \quad T\cos(\theta)-\mu N=ma$ M1

resolving perpendicular to the lawn in the $\underset{\sim}{j}$ direction.

$(2) \quad N-T\sin(\theta)-mg=0$ from (2) $\quad N=T\sin(\theta)+mg$ into (1)

$ma=T\cos(\theta)-\mu\left(T\sin(\theta)+mg\right)$

$ma=T\left(\cos(\theta)-\mu\sin(\theta)\right)-\mu mg$ A1

$a=\dfrac{T}{m}\left(\cos(\theta)-\mu\sin(\theta)\right)-\mu g$

$a=\dfrac{49\sqrt{2}}{20}\left(\cos\left(45^0\right)-\dfrac{1}{7}\sin\left(45^0\right)\right)-\dfrac{9.8}{7}$

$a=\dfrac{49\sqrt{2}}{20}\left(\dfrac{1}{\sqrt{2}}-\dfrac{1}{7\sqrt{2}}\right)-1.4$

$a=\dfrac{49\sqrt{2}}{20}\times\dfrac{6}{7\sqrt{2}}-1.4$

$a=2.1-1.4$

$a=0.7\ \text{m/s}^2$ A1

Question 3

$\dfrac{dy}{dx}=\dfrac{3x-5}{\sqrt{9-4x^2}}$ integrating with respect to x

$y=\displaystyle\int \frac{3x-5}{\sqrt{9-4x^2}}\,dx$ separating into two integrals

$y=\displaystyle\int \frac{3x}{\sqrt{9-4x^2}}\,dx - \int \frac{5}{\sqrt{9-4x^2}}\,dx$ M1

in the first integral let $u=9-4x^2$, $\dfrac{du}{dx}=-8x$

in the second integral let $v=2x$, $\dfrac{dv}{dx}=2$

$y=-\dfrac{3}{8}\displaystyle\int u^{-\frac{1}{2}}\,du - \frac{5}{2}\int \frac{1}{\sqrt{9-v^2}}\,dv$ M1

$y=-\dfrac{3}{4}u^{\frac{1}{2}}-\dfrac{5}{2}\sin^{-1}\left(\dfrac{v}{3}\right)+C$

$y=-\dfrac{3}{4}\sqrt{9-4x^2}-\dfrac{5}{2}\sin^{-1}\left(\dfrac{2x}{3}\right)+C$ A2

to find C use $y(0)=0$

$0=-\dfrac{9}{4}-0+C \quad C=\dfrac{9}{4}$

$y=-\dfrac{3}{4}\sqrt{9-4x^2}-\dfrac{5}{2}\sin^{-1}\left(\dfrac{2x}{3}\right)+\dfrac{9}{4}$ A1

Note that a possible correct alternative answer is

$y=-\dfrac{3}{4}\sqrt{9-4x^2}+\dfrac{5}{2}\cos^{-1}\left(\dfrac{2x}{3}\right)+\dfrac{9}{4}-\dfrac{5\pi}{4}$

Question 4

a. let $P(z)=z^3+pz^2+qz+15=0$ since p and q are real

by the conjugate root theorem $P(1-2i)=P(1+2i)=0$

so $1+2i$ is also a root A1

b. let $\alpha=1+2i \quad \beta=1-2i$

now $\alpha+\beta=2$ and $\alpha\beta=1-4i^2=5$

so the quadratic z^2-2z+5 is a factor

$P(z)=z^3+pz^2+qz+15=\left(z^2-2z+5\right)(z+c)=0$ M1

now $5c=15$ so that $c=3$ and expanding $\left(z^2-2z+5\right)(z+3)$

coefficient of z^2 : $p=3-2=1$

coefficient of z : $q=5-6=-1$ A2

and all the roots are $z=1\pm 2i$ and $z=-3$

Question 5

a. **Method I** using addition theorems $\frac{\pi}{12}=15^0 \quad 15^0=45^0-30^0$

$$\tan\left(\frac{\pi}{12}\right)=\tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\frac{\tan\left(\frac{\pi}{4}\right)-\tan\left(\frac{\pi}{6}\right)}{1+\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}$$ M1

$$\tan\left(\frac{\pi}{12}\right)=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}\text{ x }\frac{\sqrt{3}-1}{\sqrt{3}-1}$$ M1

$$\tan\left(\frac{\pi}{12}\right)=\frac{3-2\sqrt{3}+1}{3-1}=\frac{4-2\sqrt{3}}{2}$$

$$\tan\left(\frac{\pi}{12}\right)=2-\sqrt{3}$$ A1

Alternative Method II using double angle formulae let $A=\frac{\pi}{12}$ $2A=\frac{\pi}{6}$

$$\tan(2A)=\frac{2\tan(A)}{1-\tan^2(A)}$$

$$\frac{1}{\sqrt{3}}=\frac{2\tan\left(\frac{\pi}{12}\right)}{1-\tan^2\left(\frac{\pi}{12}\right)}$$ M1

$$1-\tan^2\left(\frac{\pi}{12}\right)=2\sqrt{3}\tan\left(\frac{\pi}{12}\right)$$

$$\tan^2\left(\frac{\pi}{12}\right)+2\sqrt{3}\tan\left(\frac{\pi}{12}\right)-1=0$$

let $u=\tan\left(\frac{\pi}{12}\right)$ $u^2+2\sqrt{3}u-1=0$

$$\Delta=\left(2\sqrt{3}\right)^2+4=16$$

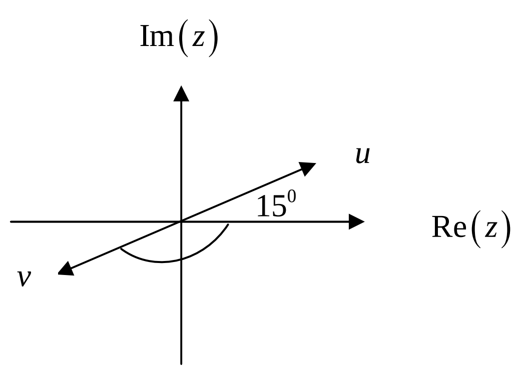
$u=\tan\left(\frac{\pi}{12}\right)=\frac{-2\sqrt{3}\pm\sqrt{16}}{2}$ but $\tan\left(\frac{\pi}{12}\right)>0$ take the positive M1

so that $\tan\left(\frac{\pi}{12}\right)=2-\sqrt{3}$ A1

b. let $u=1+\left(2-\sqrt{3}\right)i$ now $\text{Arg}(u)=\tan^{-1}\left(2-\sqrt{3}\right)=\frac{\pi}{12}$

let $v=-1+\left(\sqrt{3}-2\right)i=-u=i^2u$ v is a rotation of 180^0 from u, so

$\text{Arg}(v)=\text{Arg}\left(-1+\left(\sqrt{3}-2\right)i\right)=-\pi+\frac{\pi}{12}=-\frac{11\pi}{12}$ $\left(\text{or } -165^0\right)$ A1



Question 6

a. Let $y = \tan^{-1}\left(\sqrt{\frac{3}{x}}\right) = \tan^{-1}(u)$ where $u = \sqrt{\frac{3}{x}} = \sqrt{3}\,x^{-\frac{1}{2}}$

$\frac{dy}{du} = \frac{1}{1+u^2}$ $\frac{du}{dx} = -\frac{\sqrt{3}}{2}x^{-\frac{3}{2}} = \frac{-\sqrt{3}}{2\sqrt{x^3}}$ chain rule M1

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{-\sqrt{3}}{2\sqrt{x^3}\left(1+\frac{3}{x}\right)}$$

$\frac{dy}{dx} = \frac{-\sqrt{3}}{2\sqrt{x^3}\left(\frac{x+3}{x}\right)}$ since $x > 0$

$$\frac{dy}{dx} = \frac{-\sqrt{3}}{2\sqrt{x}(x+3)}$$

so shown $\frac{d}{dx}\left(\tan^{-1}\left(\sqrt{\frac{3}{x}}\right)\right) = \frac{-\sqrt{3}}{2\sqrt{x}(x+3)}$ for $x > 0$ A1

b. $\int_1^9 \frac{1}{\sqrt{x^3+6x^2+9x}}\,dx = \int_1^9 \frac{1}{\sqrt{x(x^2+6x+9)}}\,dx =$

$\int_1^9 \frac{1}{\sqrt{x(x+3)^2}}\,dx = \int_1^9 \frac{1}{\sqrt{x}(x+3)}\,dx$ since $x > 0$

$= -\frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{\frac{3}{x}}\right)\right]_1^9$ M1

$$= -\frac{2}{\sqrt{3}}\left(\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) - \tan^{-1}\left(\sqrt{3}\right)\right)$$

$$= -\frac{2\sqrt{3}}{3}\left(\frac{\pi}{6} - \frac{\pi}{3}\right)$$

$= \frac{\sqrt{3}\,\pi}{9}$ A1

Question 7

a. $\underset{\sim}{r}(t) = \frac{3}{2}\left(e^{2t} + e^{-2t}\right)\underset{\sim}{i} + \frac{5}{2}\left(e^{2t} - e^{-2t}\right)\underset{\sim}{j}$ vector equation,

the parametric equations are $x = \frac{3}{2}\left(e^{2t} + e^{-2t}\right)$ and $y = \frac{5}{2}\left(e^{2t} - e^{-2t}\right)$

now $x^2 = \frac{9}{4}\left(e^{4t} + 2 + e^{-4t}\right)$ and $y^2 = \frac{25}{4}\left(e^{4t} - 2 + e^{-4t}\right)$ A1

so that $\frac{4x^2}{9} = \left(e^{4t} + 2 + e^{-4t}\right)$ and $\frac{4y^2}{25} = \left(e^{4t} - 2 + e^{-4t}\right)$

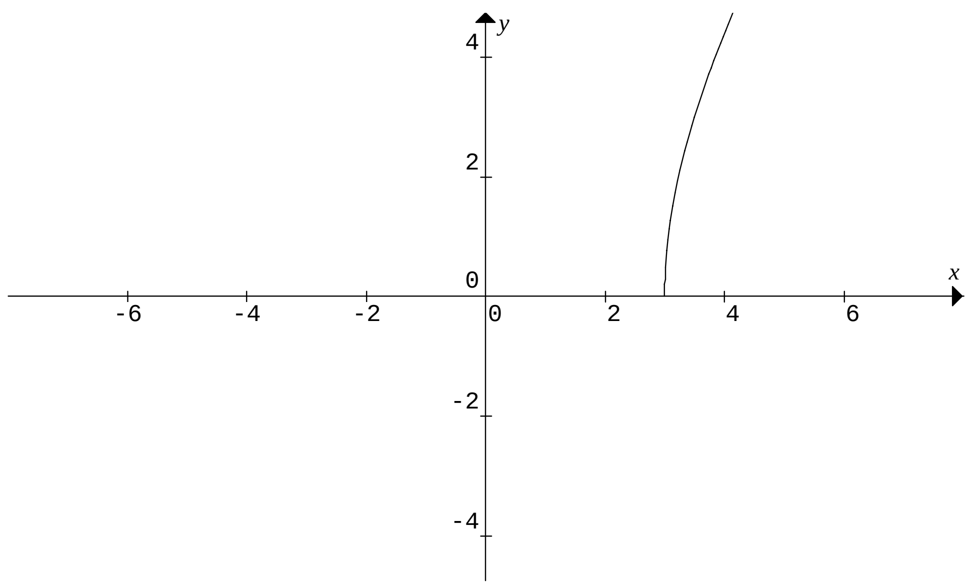
subtracting to eliminate t gives M1

$$\frac{4x^2}{9} - \frac{4y^2}{25} = 4$$

$$\frac{x^2}{9} - \frac{y^2}{25} = 1 \qquad a^2 = 9 \qquad b^2 = 25$$

since $a > 0$ and $b > 0$ $a = 3$ $b = 5$ A1

b. since $t \geq 0$ both $x \geq 0$ and $y \geq 0$, the graph is not the whole hyperbola only the upper right branch, touching the x-axis at $(3, 0)$ A1



Question 8

a. $y = 6\sin\left(\dfrac{\pi x}{2}\right)$

$V = \pi\displaystyle\int_0^b 36\sin^2\left(\dfrac{\pi x}{2}\right)dx \qquad 0 < b < 4$

$V = 18\pi\displaystyle\int_0^b \big(1-\cos(\pi x)\big)dx$ A1

$V = 18\pi\left[x - \dfrac{1}{\pi}\sin(\pi x)\right]_0^b$

M1

$V = 18\pi\left[b - \dfrac{1}{\pi}\sin(b\pi) - 0\right]$

$V = 18\big(b\pi - \sin(b\pi)\big)$ A1

b. if $V = 18\big(b\pi - \sin(b\pi)\big) = 9(7\pi + 2)$

then $b = \dfrac{7}{2}$ since $\sin\left(\dfrac{7\pi}{2}\right) = -1$ A1

Question 9

a. $y = \dfrac{4}{x^2 - 4x} = \dfrac{4}{x(x-4)}$

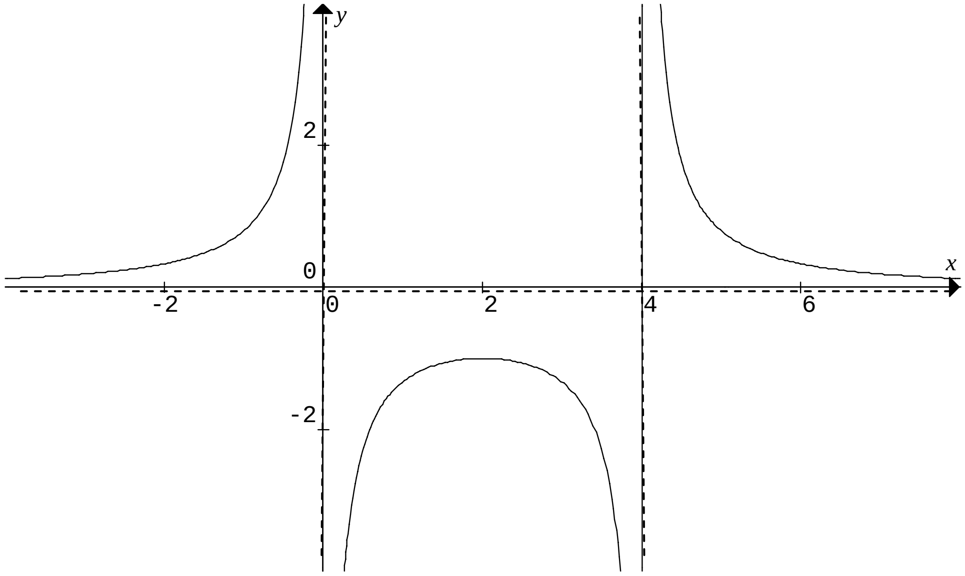
vertical asymptotes at $x = 0$ (the *y*-axis) and $x = 4$

horizontal asymptotes at $y = 0$ (the *x*-axis) A1

the turning point is when $2x - 4 = 0$ at $x = 2$ $y(2) = \dfrac{4}{4-8} = -1$

maximum turning point at $(2, -1)$

correct graph and turning point A1



b. the area $= \int_1^2 \frac{4}{x^2-4x}\,dx = \int_1^2 \frac{4}{x(x-4)}\,dx$

by partial fractions $\frac{4}{x^2-4x} = \frac{4}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$ adding the partial fractions

$= \frac{A(x-4)+Bx}{x(x-4)} = \frac{x(A+B)-4A}{x^2-4x}$ M1

$(1)\quad A+B=0$ and $(2)\quad -4A=4$ so that $A=-1$ and $B=-A \quad B=1$

the area $A = \int_1^2 \left(\frac{1}{x-4} - \frac{1}{x}\right)dx$ but the area is under the x-axis, so $A<0$

$A = \int_1^2 \left(\frac{1}{x} - \frac{1}{x-4}\right)dx$ now $A>0$ M1

$= \left[\log_e(|x|) - \log_e(|x-4|)\right]_1^2$

$= \left[\log_e\left(\left|\frac{x}{x-4}\right|\right)\right]_1^2$ A1

$= \left(\log_e\left(\left|\frac{2}{-2}\right|\right) - \log_e\left(\left|\frac{1}{-3}\right|\right)\right)$

$= \log_e(3)$ A1

END OF SUGGESTED SOLUTIONS