Year 2007

VCE

Specialist Mathematics Solutions Trial Examination 2

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SECTION 1

ANSWERS

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SECTION 1

Question 1 **Answer D**

The quadratic in the denominator $x^2+2bx+9$ has a discriminant of
$\Delta=(2b)^2-4 \times 1 \times 9=4b^2-36=4(b^2-9)$ so

If $\Delta<0$ $|b|<3$ the quadratic has no real solutions, and hence $f(x)$ has no vertical asymptotes, option **A.** is true.

If $\Delta>0$ $|b|>3$ the quadratic has two real solutions, and hence $f(x)$ has two vertical asymptotes, option **B.** is true.

The *x*-axis is a horizontal asymptote, option **C.** is true.

however option **D.** is false, when $2x+2b=0$ $x=-b$, the point $\left(-b, \frac{1}{9-b^2}\right)$ is a maximum turning point.

When $x=0$ $y=\frac{1}{9}$ as the *y*-intercept, option **E.** is true.

Question 2 **Answer D**

Find the intercepts of the two asymptotes, $3x+11=-3x-7$ $\Rightarrow 6x=-18$
So that when $x=-3$ $y=2$, the centre is $(-3,2)=(h,k)$, $h=-3$ $k=2$,
now the distance from the centre to one of the vertices horizontally, that is from $(-3,2)$ to $(-1,2)$ is 2 units, so $a=2$, the asymptotes have gradients $\pm 3=\frac{b}{a}$
so that $b=6$.

Question 3 **Answer C**

$$\text{cosec}(x)=\frac{1}{\sin(x)}=\frac{\sqrt{5}}{2}$$

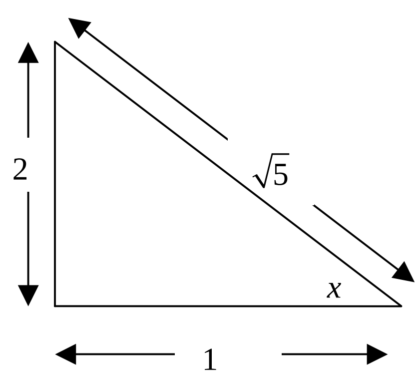
$$\sin(x)=\frac{2}{\sqrt{5}}$$

Since $\frac{\pi}{2}<x<\pi$ x is in the 2nd quadrant

$$\tan(x)<0 \quad \Rightarrow \quad \tan(x)=-2$$

$$\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}=\frac{-4}{1-4}=\frac{4}{3}$$

$$\cot(2x)=\frac{3}{4}$$



Question 4 **Answer E**

$\underset{\sim}{r}(t) = \cos^2(2t)\,\underset{\sim}{i} + \cos(4t)\,\underset{\sim}{j}$

$x = \cos^2(2t)$ and $y = \cos(4t) = 2\cos^2(2t) - 1$

$y = 2x - 1$ is the Cartesian equation, which is a straight line.

Question 5 **Answer C**

$y = \dfrac{ax^4 + b}{x^2} = ax^2 + bx^{-2}$

$\dfrac{dy}{dx} = 2ax - 2bx^{-3}$ for turning points $\dfrac{dy}{dx} = 0$

$\Rightarrow 2ax = \dfrac{2b}{x^3}$ $\quad x^4 = \dfrac{b}{a}$ $\quad x^2 = \pm\sqrt{\dfrac{b}{a}}$ however there are two turning points, so there are solutions for $x^2 = \sqrt{\dfrac{b}{a}}$ so a and b must both be positive, or both be negative, the product $ab > 0$ so $a < 0$ and $b < 0$ or $a > 0$ and $b > 0$ is the only possibility listed.

Question 6 **Answer C**

$i\,\text{cis}(-\theta)$

$= i\left(\cos(-\theta) + i\sin(-\theta)\right)$ since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$

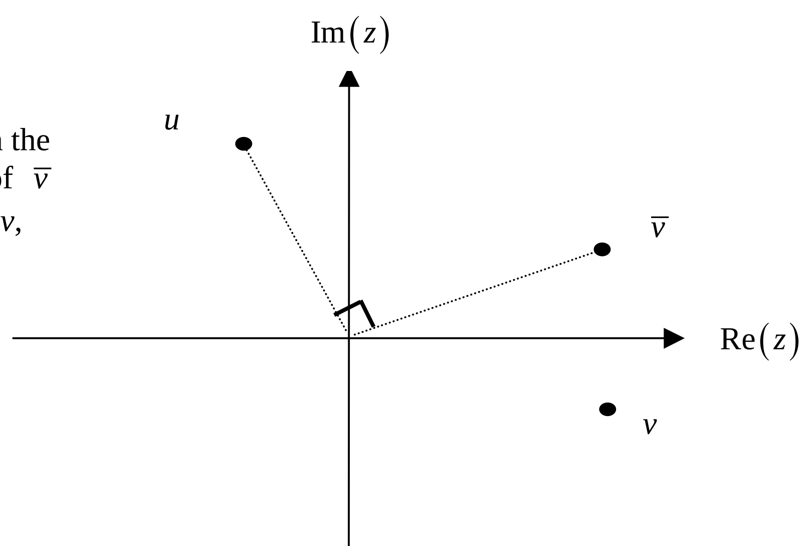
$= i\left(\cos(\theta) - i\sin(\theta)\right)$

$= -i^2\sin(\theta) + i\cos(\theta)$

$= \sin(\theta) + i\cos(\theta)$

Question 7 **Answer E**

$\overline{v}$ is the reflection of v in the real axis, u is a rotation of $\overline{v}$ 90^0 anti-clockwise from v, hence $u = i\overline{v}$



Question 8 **Answer B**

$\text{Arg}(a+bi)=\tan^{-1}\left(\dfrac{b}{a}\right)$, is only true, where the $\tan^{-1}$ function is defined, that is $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ or in the 1st and 4th quadrants, so $a>0$ and $b\in R$

Question 9 **Answer E**

Let $z=x+yi$ $\quad \overline{z}=x-yi$, checking each alternative,

A. $i(z+\overline{z})=\overline{z}-z \qquad \Rightarrow 2ix=-2iy \qquad \Rightarrow y=-x$

B. $|z+1|=|z-i| \quad \Rightarrow \sqrt{(x+1)^2+y^2}=\sqrt{x^2+(y-1)^2}$

$x^2+2x+1+y^2=x^2+y^2-2y+1 \qquad \Rightarrow y=-x$

C. $|z-1|=|z+i| \quad \Rightarrow \sqrt{(x-1)^2+y^2}=\sqrt{x^2+(y+1)^2}$

$x^2-2x+1+y^2=x^2+y^2+2y+1 \qquad \Rightarrow y=-x$

D. $\text{Re}(z)+\text{Im}(z)=0 \qquad \Rightarrow y=-x$

E. $\{z:\text{Arg}(z)=-\dfrac{\pi}{4}\}\cup\{z:\text{Arg}(z)=\dfrac{3\pi}{4}\}$ are two rays from the origin, making angles of $-\dfrac{\pi}{4}$ and $\dfrac{3\pi}{4}$ however the origin is **not** included, it is not the full line $y=-x$

Question 10 **Answer E**

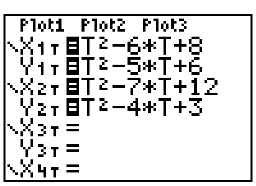
If we look at the parametric graphs, we see that the paths cross twice,
the paths are not parabolic and since,

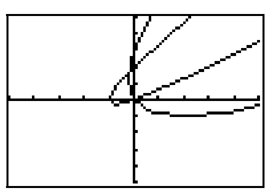
$\underset{\sim}{p}=(t^2-6t+8)\underset{\sim}{i}+(t^2-5t+6)\underset{\sim}{j} \qquad \underset{\sim}{q}=(t^2-7t+12)\underset{\sim}{i}+(t^2-4t+3)\underset{\sim}{j}$

$\underset{\sim}{p}=(t-4)(t-2)\underset{\sim}{i}+(t-3)(t-2)\underset{\sim}{j} \qquad \underset{\sim}{q}=(t-4)(t-3)\underset{\sim}{i}+(t-3)(t-1)\underset{\sim}{j}$

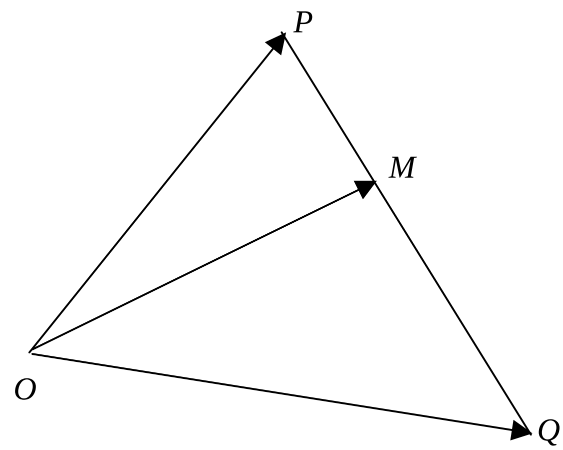
$\underset{\sim}{p}(2)\neq\underset{\sim}{q}(2)$ and $\underset{\sim}{p}(3)\neq\underset{\sim}{q}(3)$

P and *Q* are never in the same position.





Question 11 **Answer D**



$$\overrightarrow{PM} = \frac{3}{7}\overrightarrow{PQ}$$

$\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OQ} = \underset{\sim}{q}$

$$\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}$$

$$\overrightarrow{OM} = \overrightarrow{OP} + \frac{3}{7}\overrightarrow{PQ}$$

$$\overrightarrow{OM} = \overrightarrow{OP} + \frac{3}{7}\left(\overrightarrow{PO} + \overrightarrow{OQ}\right)$$

$$\overrightarrow{OM} = \overrightarrow{OP} + \frac{3}{7}\left(\overrightarrow{OQ} - \overrightarrow{OP}\right)$$

$$\overrightarrow{OM} = \frac{1}{7}\left(4\underset{\sim}{p} + 3\underset{\sim}{q}\right)$$

Question 12 **Answer A**

Let $\underset{\sim}{a} = 5\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}$ $\quad |\underset{\sim}{a}| = \sqrt{25+16+9} = \sqrt{50} = 5\sqrt{2}$

now a vector of magnitude 10 in the opposite direction to $\underset{\sim}{a}$ is

$$-10\hat{\underset{\sim}{a}} = -\frac{10}{5\sqrt{2}}\left(5\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}\right)$$

$$= \sqrt{2}\left(-5\underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k}\right)$$

Question 13 **Answer A**

Let $\underset{\sim}{s} = -3\underset{\sim}{i} + 12\underset{\sim}{j} - 4\underset{\sim}{k}$ $\quad |\underset{\sim}{s}| = \sqrt{9+144+16} = \sqrt{169} = 13$ so that $\hat{\underset{\sim}{s}} = \frac{1}{13}\left(-3\underset{\sim}{i} + 12\underset{\sim}{j} - 4\underset{\sim}{k}\right)$

The scalar resolute of the vector $\underset{\sim}{r}$ in the direction of $\underset{\sim}{s}$ is -2, so that $\underset{\sim}{r}.\hat{\underset{\sim}{s}} = -2$

The vector resolute of $\underset{\sim}{r}$ perpendicular to $\underset{\sim}{s}$ is $\underset{\sim}{r} - \left(\underset{\sim}{r}.\hat{\underset{\sim}{s}}\right)\hat{\underset{\sim}{s}} = \frac{1}{13}\left(20\underset{\sim}{i} - 2\underset{\sim}{j} - 21\underset{\sim}{k}\right)$

$$\underset{\sim}{r} + \frac{2}{13}\left(-3\underset{\sim}{i} + 12\underset{\sim}{j} - 4\underset{\sim}{k}\right) = \frac{1}{13}\left(20\underset{\sim}{i} - 2\underset{\sim}{j} - 21\underset{\sim}{k}\right)$$

$$\underset{\sim}{r} = \frac{1}{13}\left(20\underset{\sim}{i} - 2\underset{\sim}{j} - 21\underset{\sim}{k}\right) - \frac{2}{13}\left(-3\underset{\sim}{i} + 12\underset{\sim}{j} - 4\underset{\sim}{k}\right) = \frac{1}{13}\left(26\underset{\sim}{i} - 26\underset{\sim}{j} - 13\underset{\sim}{k}\right)$$

$$\underset{\sim}{r} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$$

Question 14 **Answer A**

$$\int_0^2 \frac{x^3}{\sqrt{3x^2+4}}\,dx$$

let $u = 3x^2 + 4$ $\quad \dfrac{du}{dx} = 6x \quad x\,dx = \dfrac{1}{6}du \qquad x^2 = \dfrac{u-4}{3}$

change terminals, when $x = 0$ $\quad u = 4$ and when $x = 2$ $\quad u = 16$

$$\int_0^2 \frac{x^2}{\sqrt{3x^2+4}}\;x.dx = \frac{1}{18}\int_4^{16} \frac{u-4}{\sqrt{u}}\,du$$

Question 15 **Answer C**

Let $y_1 = 1$ and $y_2 = 2e^{-x^2}$, to find the x-value where $y_1 = y_2$

$2e^{-x^2} = 1 \qquad e^{-x^2} = \dfrac{1}{2} \qquad e^{x^2} = 2$

$x^2 = \log_e(2) \qquad x = \sqrt{\log_e(2)}$

Now $y_2^2 = 4e^{-2x^2}$ so the volume required is

$V_x = \pi \int_a^b \left(y_2^2 - y_1^2\right)dx$ where y_1 and y_2 are the inner and outer radii respectively,

$$V = \pi \int_0^{\sqrt{\log_e(2)}} \left(4e^{-2x^2} - 1\right)dx$$

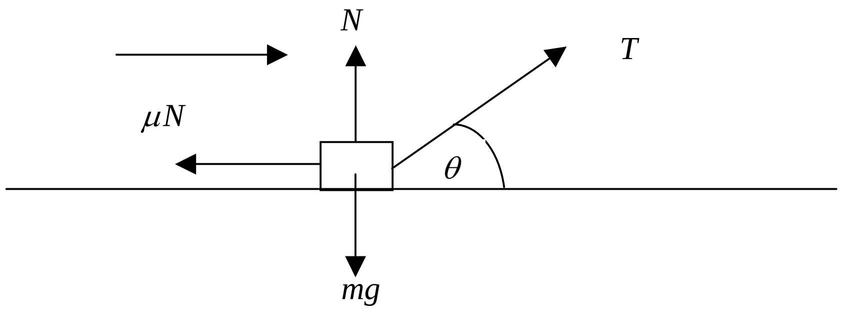
Question 16 **Answer B**

$v^2 = 9x$ for $x > 0$,
differentiating implicitly with respect to x, gives

$2v\dfrac{dv}{dx} = 9$

so that $a = v\dfrac{dv}{dx} = \dfrac{9}{2} = 4.5$

Question 17 **Answer C**



resolving parallel to the plane (1) $T\cos(\theta)-\mu N = ma$

resolving perpendicular to the plane (2) $T\sin(\theta)+N-mg=0$

to find a we need to eliminate N

from (2) $N = mg - T\sin(\theta)$ substituting into (1) gives

$$T\cos(\theta)-\mu\big(mg-T\sin(\theta)\big)=ma$$

$$ma = T\cos(\theta)-\mu mg+\mu T\sin(\theta)$$

$$ma = T\big(\cos(\theta)+\mu\sin(\theta)\big)-\mu mg$$

$$a=\frac{T}{m}\big[\cos(\theta)+\mu\sin(\theta)\big]-\mu g$$

now when $T = 30\,N$ $m = 10\,\text{kg}$ $\mu = 0.2$ $g = 9{,}8$ $a = ?$

$a = 3\big[\cos(\theta)+0.2\sin(\theta)\big]-1.96$, checking each alternative

A. $\theta = 0$ $\Rightarrow a = 1.04\,\text{m/s}^2$

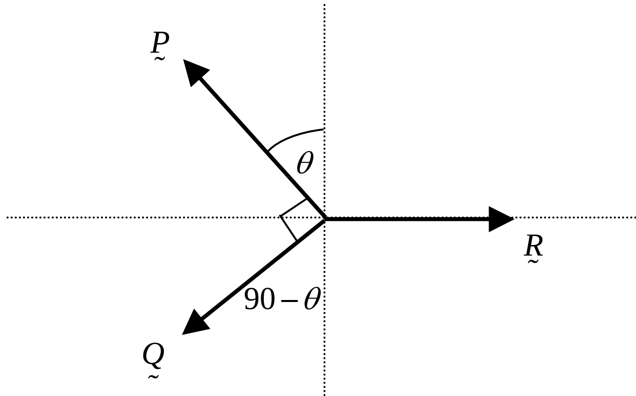
B. $\theta = 5$ $\Rightarrow a = 1.08\,\text{m/s}^2$

C. $\theta = 10$ $\Rightarrow a = 1.1\,\text{m/s}^2$

D. $\theta = 15$ $\Rightarrow a = 1.09\,\text{m/s}^2$

E. $\theta = 20$ $\Rightarrow a = 1.06\,\text{m/s}^2$

Question 18 **Answer D**



A. resolving vertically

$$P\cos(\theta)-Q\cos(90-\theta)=0$$
$$P\cos(\theta)=Q\sin(\theta)$$
$$\frac{P}{\sin(\theta)}=\frac{Q}{\cos(\theta)}$$
$$P\operatorname{cosec}(\theta)=Q\sec(\theta) \text{ is true}$$

B. $R^2=P^2+Q^2$ is true

C. resolving horizontally

$$R-P\sin(\theta)-Q\sin(90-\theta)=0$$
$$R=P\sin(\theta)+Q\cos(\theta) \quad \text{is true}$$

D. $P\cos(\theta)=Q\sin(\theta) \quad \Rightarrow \quad \dfrac{P}{Q}=\dfrac{\sin(\theta)}{\cos(\theta)}=\tan(\theta)$

$\cot(\theta)=\dfrac{Q}{P}$ **D.** is false

E. $\underset{\sim}{P}+\underset{\sim}{Q}+\underset{\sim}{R}=\underset{\sim}{0}$ is true

Question 19 **Answer A**

Using implicit differentiation $x^2-6xy-16y^2=0$.

$$\frac{d}{dx}\left(x^2\right)-\frac{d}{dx}(6xy)-\frac{d}{dx}\left(16y^2\right)=0 \quad \text{using the product rule in the middle term}$$
$$\frac{d}{dx}\left(x^2\right)-6x\frac{d}{dx}(y)-y\frac{d}{dx}(6x)-\frac{d}{dx}\left(16y^2\right)=0$$
$$\frac{d}{dx}\left(x^2\right)-6x\frac{d}{dy}(y)\frac{dy}{dx}-y\frac{d}{dx}(6x)-\frac{d}{dy}\left(16y^2\right)\frac{dy}{dx}=0$$
$$2x-6x\frac{dy}{dx}-6y-32y\frac{dy}{dx}=0$$
$$2x-6y=32y\frac{dy}{dx}+6x\frac{dy}{dx}=\frac{dy}{dx}(32y+6x)$$
$$\frac{dy}{dx}=\frac{x-3y}{16y+3x}$$

A. is false, all the options are true.

B. is true $\left.\dfrac{dy}{dx}\right|_{(2,-1)}=-\dfrac{1}{2} \quad m_N=2$

C. is true $\left.\dfrac{dy}{dx}\right|_{\left(2,\frac{1}{4}\right)}=\dfrac{1}{8}$

D. is true $\left.\dfrac{dy}{dx}\right|_{(-8,4)}=-\dfrac{1}{2}$

D. is true $\left.\dfrac{dy}{dx}\right|_{(-8,-1)}=\dfrac{1}{8} \quad m_N=-8$

Question 20 **Answer B**

Initially no x is present, $x(0)=0$, after a time of t, equal parts of x combine, leaving $(a-x)$ and $(b-x)$ of a and b respectively, since $k>0$ and initial the reaction rate is fastest, and slowing down as time goes on, the solution is **B.**

Question 21 **Answer B**

Consider the mass m_2 moving downwards, resolving downwards,

$(1)\qquad m_2 g - T = m_2 a$

Consider the mass m_1 moving upwards, resolving upwards,

$(2)\qquad T - m_1 g = m_1 a$

to solve for a add the two equations to eliminate T

$m_2 g - m_1 g = m_2 a + m_1 a$

so that $\quad (m_2 - m_1)g = (m_1 + m_2)a$

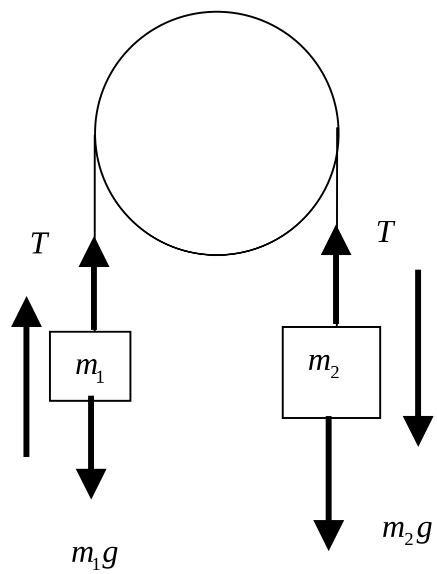
$a = \dfrac{(m_2 - m_1)g}{m_1 + m_2} = \dfrac{g}{5}$ and

$\dfrac{m_2 - m_1}{m_1 + m_2} = \dfrac{1}{5}$

$\dfrac{\frac{m_2}{m_1} - 1}{1 + \frac{m_2}{m_1}} = \dfrac{1}{5}$ let $\alpha = \dfrac{m_2}{m_1}$

$\dfrac{\alpha - 1}{\alpha + 1} = \dfrac{1}{5} \qquad 5(\alpha - 1) = 5\alpha - 5 = \alpha + 1 \qquad 4\alpha = 6$

$\alpha = \dfrac{3}{2}$



Question 22 **Answer D**

All the slopes (dashes are positive slopes) at $x=0 \quad t=0$ the slope is 2,

The solution curves are of the form $x = C - e^{-2t}$, differentiating gives

$\dfrac{dx}{dt} = 2e^{-2t}$ as the differential equation

END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a. $f(x) = 50\sin^{-1}\left(\dfrac{x-10}{10}\right) = y = 50\sin^{-1}\left(\dfrac{u}{10}\right)$ where $u = x - 10$

$$\frac{dy}{du} = \frac{50}{\sqrt{100-u^2}} \qquad \frac{du}{dx} = 1$$

M1

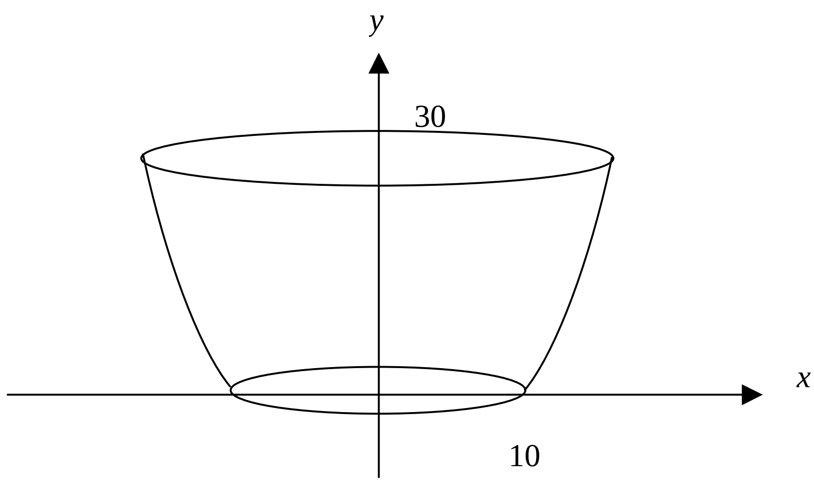
$$f'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{50}{\sqrt{100-u^2}} = \frac{50}{\sqrt{100-(x-10)^2}}$$

$$f'(x) = \frac{50}{\sqrt{100-\left(x^2-20x+100\right)}} = \frac{50}{\sqrt{20x-x^2}}$$

$$f'(x) = \frac{50}{\sqrt{x(20-x)}} \quad \text{for} \quad 0 < x < 20$$

so $a = 50 \quad b = 20$ A1

b.

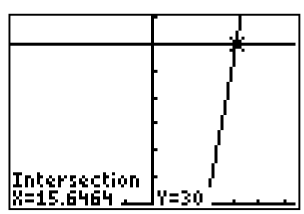


A1

c.

solving $50\sin^{-1}\left(\dfrac{x-10}{10}\right) = 30$

on a graphics calculator



$(15.6464, 30)$

so the diameter is $31.2928\,\text{cm}$

A1

Alternatively $\sin^{-1}\left(\dfrac{x-10}{10}\right)=\dfrac{3}{5}$ $\qquad \dfrac{x-10}{10}=\sin\left(\dfrac{3}{5}\right)$

$x=10+10\sin\left(\dfrac{3}{5}\right)\approx 15.6464$ so the diameter is $31.2928\,\text{cm}$

d. **i.** $V_y=\pi\int_a^b x^2\,dy$

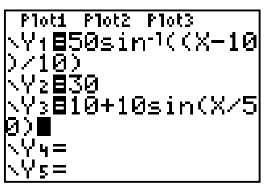
$y=50\sin^{-1}\left(\dfrac{x-10}{10}\right) \qquad \dfrac{y}{50}=\sin^{-1}\left(\dfrac{x-10}{10}\right)$

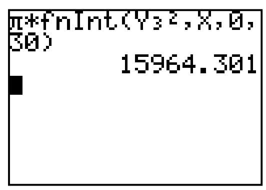
$\sin\left(\dfrac{y}{50}\right)=\dfrac{x-10}{10}$ M1

$x=10+10\sin\left(\dfrac{y}{50}\right)$

$V=\pi\displaystyle\int_0^{30}\left(10+10\sin\left(\frac{y}{50}\right)\right)^2 dy$ A1

ii. using a graphics calculator $V=15{,}964.301\text{ cm}^3$ A1





e. when the bowl is filled to a height of h its volume is

$V=\pi\displaystyle\int_0^{h}\left(10+10\sin\left(\frac{y}{50}\right)\right)^2 dy$ so that

$\dfrac{dV}{dh}=\pi\left(10+10\sin\left(\dfrac{h}{50}\right)\right)^2$ and given $\dfrac{dV}{dt}=-10\,\text{cm}^3/\text{sec}$ A1

$\dfrac{dh}{dt}=\dfrac{dh}{dV}\cdot\dfrac{dV}{dt}=\dfrac{-10}{\pi\left(10+10\sin\left(\dfrac{h}{50}\right)\right)^2}$ when $h=25$ M1

$\dfrac{dh}{dt}=\dfrac{-10}{\pi\left(10+10\sin\left(\dfrac{1}{2}\right)\right)^2}$

$\dfrac{dh}{dt}=-0.015\,\text{cm/sec}$

the water level is falling at $0.015\,\text{cm/sec}$ A1

Question 2

$\overrightarrow{OA} = \underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$ $\overrightarrow{OB} = \underset{\sim}{b} = 5\underset{\sim}{i} + y\underset{\sim}{j} - 3\underset{\sim}{k}$

$\overrightarrow{OC} = \underset{\sim}{c} = 3\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$ $\overrightarrow{OD} = \underset{\sim}{d} = -3\underset{\sim}{i} + 4\underset{\sim}{j} + 6\underset{\sim}{k}$

a. If $\left|\overrightarrow{AB}\right| = 13$ $y = ?$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(5\underset{\sim}{i} + y\underset{\sim}{j} - 3\underset{\sim}{k}\right) - \left(2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}\right)$ M1

$\overrightarrow{AB} = 3\underset{\sim}{i} + (y-3)\underset{\sim}{j} - 4\underset{\sim}{k}$

$\left|\overrightarrow{AB}\right| = \sqrt{9 + (y-3)^2 + 16} = \sqrt{25 + (y-3)^2} = 13$

$25 + (y-3)^2 = 169$

$(y-3)^2 = 144$

$y - 3 = \pm 12$

$y = 15$ or $y = -9$ both answers are acceptable A1

b. If $\overrightarrow{AB}$ makes an angle of 135^0 with the *z*-axis, $y = ?$

$\cos\left(135^0\right) = \dfrac{-4}{\sqrt{25 + (y-3)^2}} = -\dfrac{\sqrt{2}}{2}$ M1

$8 = \sqrt{2\left(25 + (y-3)^2\right)}$

$64 = 2\left(25 + (y-3)^2\right)$

$32 = 25 + (y-3)^2$

$(y-3)^2 = 7$

$y - 3 = \pm\sqrt{7}$

$y = 3 \pm \sqrt{7}$ both answers are acceptable A1

c. If $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CD}$ $y = ?$ $\overrightarrow{AB}.\overrightarrow{CD} = 0$

$\overrightarrow{CD} = \overrightarrow{CO} + \overrightarrow{OD} = \overrightarrow{OD} - \overrightarrow{OC} = \left(-3\underset{\sim}{i} + 4\underset{\sim}{j} + 6\underset{\sim}{k}\right) - \left(3\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}\right)$ M1

$\overrightarrow{CD} = -6\underset{\sim}{i} + 5\underset{\sim}{j} + 8\underset{\sim}{k}$

$\overrightarrow{AB}.\overrightarrow{CD} = -18 + 5(y-3) - 32 = 0$

$5(y-3) = 50$

$y - 3 = 10$

$y = 13$ A1

d. If $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ then $\overrightarrow{AB} = \lambda \overrightarrow{CD}$ M1

$\overrightarrow{AB} = -\frac{1}{2}\overrightarrow{CD}$ from the $\underset{\sim}{i}$ and $\underset{\sim}{k}$ components,

it must also be true for the $\underset{\sim}{j}$ so that $y - 3 = \frac{-5}{2}$

$y = \frac{1}{2}$ A1

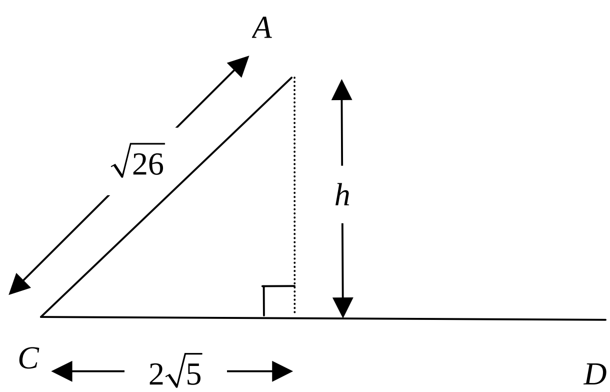
e. $\overrightarrow{CA} = \overrightarrow{CO} + \overrightarrow{OA} = \overrightarrow{OA} - \overrightarrow{OC} = \left(2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}\right) - \left(3\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

$\overrightarrow{CA} = -\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$ $\quad\left|\overrightarrow{CA}\right| = \sqrt{1+16+9} = \sqrt{26}$

$\overrightarrow{CD} = -6\underset{\sim}{i} + 5\underset{\sim}{j} + 8\underset{\sim}{k}$ $\quad\left|\overrightarrow{CD}\right| = \sqrt{36+25+64} = \sqrt{125} = 5\sqrt{5}$ M1

$\overrightarrow{CA}.\overrightarrow{CD} = 6 + 20 + 24 = 50$

$\cos(\angle DCA) = \frac{\overrightarrow{CA}.\overrightarrow{CD}}{\left|\overrightarrow{CA}\right|\left|\overrightarrow{CD}\right|} = \frac{50}{5\sqrt{5}\sqrt{26}} = \frac{2\sqrt{5}}{\sqrt{26}}$ or $\frac{\sqrt{130}}{13}$ A1



$h = \sqrt{\left(\sqrt{26}\right)^2 - \left(2\sqrt{5}\right)^2}$

$h = \sqrt{26 - 20}$

$h = \sqrt{6}$ A1

Question 3

a. $u = 22.5 \quad v = 17.5 \quad s = 8$

using constant acceleration formulae M1

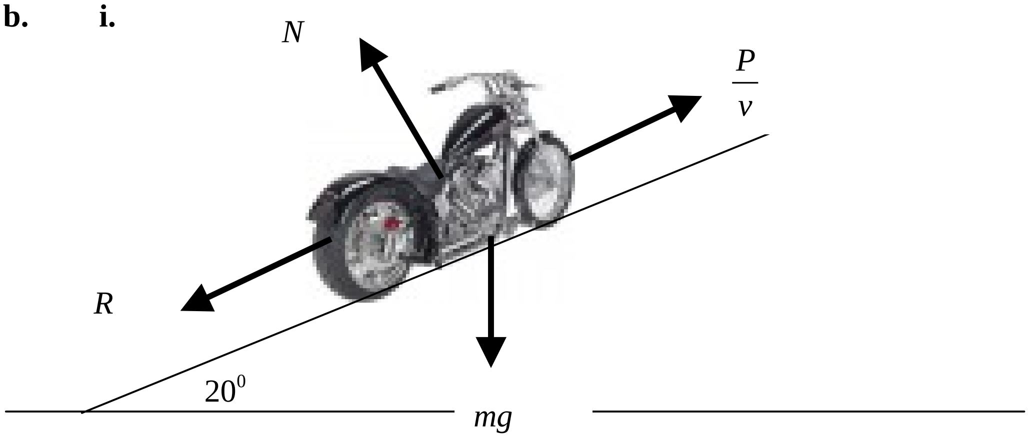
$s = \left(\frac{u+v}{2}\right)t \qquad 8 = \left(\frac{17.5 + 22.5}{2}\right)t$

$t = 0.4$ sec A1

$v^2 = u^2 + 2as \quad 17.5^2 = 22.5^2 + 16a$

$a = \frac{-200}{16} = -12.5\,\text{m/s}^2$ A1

b. **i.**



for forces on the diagram A1

ii. now $P = 120,000\,\text{W}$ $R = 16v^2$ $m = 500\,\text{kg}$ $\theta = 20^0$

resolving up and parallel to the slope

$$\frac{P}{v} - R - mg\sin(\theta) = ma \qquad \text{A1}$$

$$500a = \frac{120,000}{v} - 16v^2 - 500g\sin\left(20^0\right)$$

$$a = \frac{240}{v} - \frac{4v^2}{125} - g\sin\left(20^0\right) \qquad \text{A1}$$

iii. Use $a = v\dfrac{dv}{dx}$

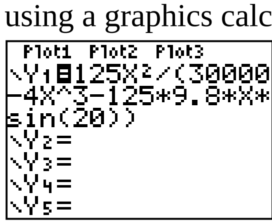
$$v\frac{dv}{dx} = \frac{30,000 - 4v^3 - 125g\,v\sin\left(20^0\right)}{125v} \qquad \text{M1}$$

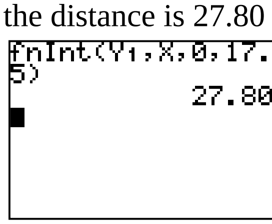
$$dx = \frac{125v^2}{30,000 - 4v^3 - 125g\,v\sin\left(20^0\right)}dv \qquad \text{A1}$$

the distance travelled from rest to 17.5 m/s, is the definite integral

$$x = \int_0^{17.5} \frac{125v^2}{30,000 - 4v^3 - 125g\,v\sin\left(20^0\right)}\,dv \qquad \text{A1}$$

iv. using a graphics calculator the distance is 27.80 m A1





Question 4

a. Given that $\cos\left(\frac{\pi}{5}\right)=\frac{\sqrt{5}+1}{4}$ and $\sin\left(\frac{\pi}{5}\right)=\frac{\sqrt{2\left(5-\sqrt{5}\right)}}{4}$

$\cos(2A)=\cos^2(A)-\sin^2(A)$ M1

$$\cos\left(\frac{2\pi}{5}\right)=\cos^2\left(\frac{\pi}{5}\right)-\sin^2\left(\frac{\pi}{5}\right)$$

$$\cos\left(\frac{2\pi}{5}\right)=\left(\frac{\sqrt{5}+1}{4}\right)^2-\left(\frac{\sqrt{2\left(5-\sqrt{5}\right)}}{4}\right)^2$$

$$\cos\left(\frac{2\pi}{5}\right)=\frac{5+2\sqrt{5}+1}{16}-\frac{2\left(5-\sqrt{5}\right)}{16}$$

$$\cos\left(\frac{2\pi}{5}\right)=\frac{4\sqrt{5}-4}{16}=\frac{4\left(\sqrt{5}-1\right)}{16}$$

$\cos\left(\frac{2\pi}{5}\right)=\frac{\sqrt{5}-1}{4}$ A1

b. using $\sin^2(A)=1-\cos^2(A)$

$$\sin^2\left(\frac{2\pi}{5}\right)=1-\cos^2\left(\frac{2\pi}{5}\right)$$

M1

$$\sin^2\left(\frac{2\pi}{5}\right)=1-\left(\frac{\sqrt{5}-1}{4}\right)^2=1-\frac{\left(5-2\sqrt{5}+1\right)}{16}$$

$$\sin^2\left(\frac{2\pi}{5}\right)=\frac{16-\left(6-2\sqrt{5}\right)}{16}$$

$$\sin^2\left(\frac{2\pi}{5}\right)=\frac{10+2\sqrt{5}}{16}$$

$\sin^2\left(\frac{2\pi}{5}\right)=\frac{2\left(5+\sqrt{5}\right)}{16}$ since $\sin\left(\frac{2\pi}{5}\right)>0$ take the positive only

$\sin\left(\frac{2\pi}{5}\right)=\frac{\sqrt{2\left(\sqrt{5}+5\right)}}{4}$ A1

c. $$4\left(\sqrt{\frac{\sqrt{5}-1}{4}+\frac{\sqrt{2\left(5+\sqrt{5}\right)}}{4}\,i}\right)^{21}$$

$$=4\left(\sqrt{\cos\left(\frac{2\pi}{5}\right)+i\sin\left(\frac{2\pi}{5}\right)}\right)^{21}$$ M1

$$=4\left(\text{cis}\left(\frac{2\pi}{5}\right)\right)^{\frac{21}{2}}$$

$$=4\text{cis}\left(\frac{21\pi}{5}\right)$$

M1

$$=4\text{cis}\left(\frac{21\pi}{5}-4\pi\right)$$

$$=4\text{cis}\left(\frac{\pi}{5}\right)$$

$$=\sqrt{5}+1+i\sqrt{2\left(5-\sqrt{5}\right)}$$ A1

d. $$\left(\left(\sqrt{5}+1\right)+\left(\sqrt{2\left(5-\sqrt{5}\right)}\right)i\right)^{n}$$

$$=\left(4\cos\left(\frac{\pi}{5}\right)+4i\sin\left(\frac{\pi}{5}\right)\right)^{n}$$

M1

$$=4^{n}\left(\text{cis}\left(\frac{\pi}{5}\right)\right)^{n}=4^{n}\text{cis}\left(\frac{n\pi}{5}\right)$$

$$=4^{n}\left(\cos\left(\frac{n\pi}{5}\right)+i\sin\left(\frac{n\pi}{5}\right)\right)$$

is a real number, so that the imaginary part must be zero

$$\sin\left(\frac{n\pi}{5}\right)=0$$

$$\frac{n\pi}{5}=k\pi$$

$n=5k$ where $k\in J$ A1

e. $z^5 = -32 = 32\text{cis}(\pi + 2k\pi)$

$z = 2\text{cis}\left(\dfrac{\pi}{5} + \dfrac{2k\pi}{5}\right)$ M1

$k = 0 \qquad z = 2\,\text{cis}\left(\dfrac{\pi}{5}\right) = \dfrac{1}{2}\left[\left(\sqrt{5}+1\right) + \left(2\left(5-\sqrt{5}\right)\right)i\right]$

$k = 1 \qquad z = 2\,\text{cis}\left(\dfrac{3\pi}{5}\right)$ A1

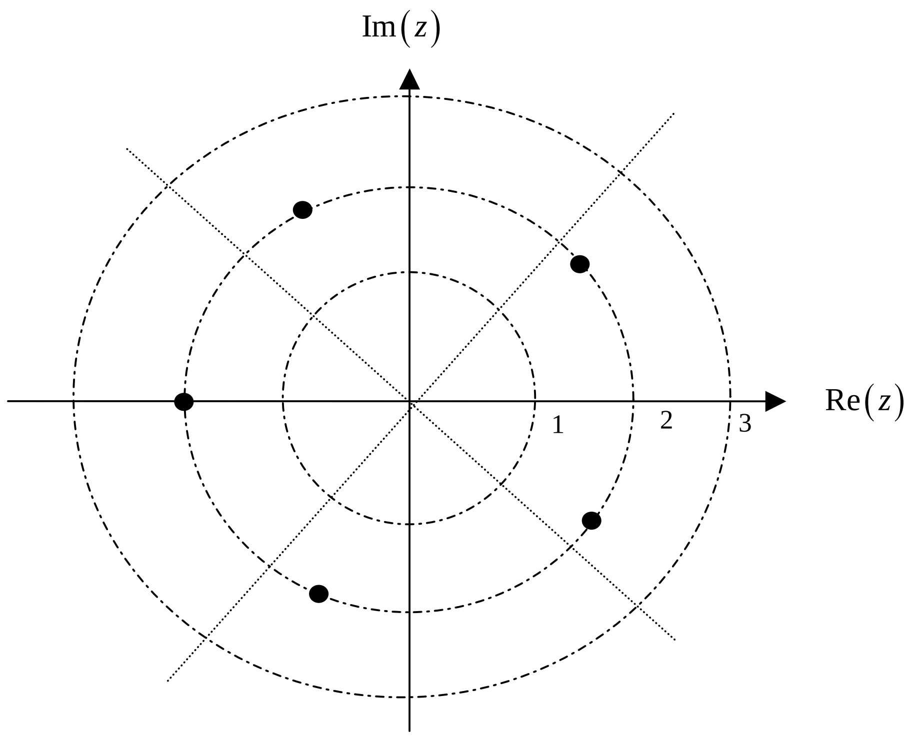
$k = 2 \qquad z = 2\,\text{cis}(\pi) = -2$

$k = -1 \qquad z = 2\,\text{cis}\left(-\dfrac{\pi}{5}\right) = \dfrac{1}{2}\left[\left(\sqrt{5}+1\right) - \left(2\left(5-\sqrt{5}\right)\right)i\right]$

$k = -2 \qquad z = 2\,\text{cis}\left(-\dfrac{3\pi}{5}\right)$

there are 5 roots, all the roots are equally spaced by $\dfrac{\pi}{5}$ or 36^0 around a circle of radius two, there is one real root and two pairs of complex conjugates. A1

For the five roots on the diagram below A1



Question 5

a. $\dfrac{dy}{dt}=10-gt-0.2y$

$\dfrac{dy}{dt}+0.2y=10-gt \quad y(0)=1.5$

$k=0.2 \quad b=10 \quad y_0=1.5$ A1

b. $\dfrac{dy}{dt}=f(y,t)=10-gt-0.2y \qquad y(0)=1.5$

Euler's method $\quad y_0=1.5 \quad t_0=0 \quad h=0.2$

$y_1=y_0+hf(y_0,t_0)=1.5+0.2(10-9.8\times 0-0.2\times 1.5)=3.44$ M1

$y_2=y_1+hf(y_1,t_1)=3.44+0.2(10-9.8\times 0.2-0.2\times 3.44)$

$y_2=4.9104$ A1

c. differentiating $y(t)=295-293.5e^{-\frac{t}{5}}-49t$ with respect to t

$\dfrac{dy}{dt}=0.2\times 293.5e^{-\frac{t}{5}}-49=58.7e^{-\frac{t}{5}}-49$ A1

substituting into LHS

$\dfrac{dy}{dt}+0.2y=58.7e^{-\frac{t}{5}}-49+0.2\left(295-293.5e^{-\frac{t}{5}}-49t\right)$ M1

$=-49+59-49t\times 0.2=10-9.8t=RHS$ shown

also it satisfies the initial conditions $y(0)=295-293.5-0=1.5$

d. solving $y(t)=295-293.5e^{-\frac{t}{5}}-49t=0$

on a graphics calculator gives $t=T=2.02676$

$y(2.02676)=295-293.5e^{-\frac{2.02676}{5}}-49\times 2.02676\approx 0$ shown A1

e. the horizontal component of velocity $\dot{x}=10$ so $x=10t$

horizontal distance travelled

$x(2.0267)=10\times 2.02676=20.268\,\text{m}$ A1

f. for maximum height $\frac{dy}{dt} = 58.7e^{-\frac{t}{5}} - 49 = 0$

$58.7e^{-\frac{t}{5}} = 49$

$e^{-\frac{t}{5}} = \frac{49}{58.7}$ M1

$e^{\frac{t}{5}} = \frac{587}{490}$

$t = 5\log_e\left(\frac{587}{490}\right) = 0.9031$

The time to reach maximum height is 0.903 sec A1

For maximum height

$y(0.9031) = 295 - 293.5e^{-\frac{0.9031}{5}} - 49 \times 0.9031$

The maximum height reached is $5.748\,\text{m}$ A1

horizontal distance travelled at this time

$x(0.9031) = 10 \times 0.9031 = 9.031\,\text{m}$ A1

g. $\dot{x} = 10$ always,

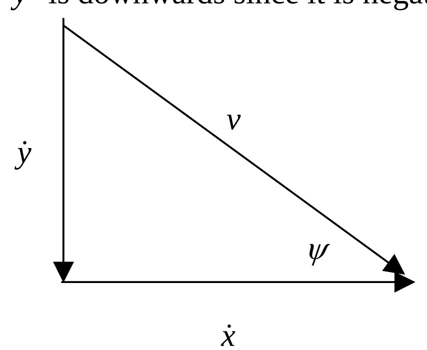
when it hits the ground $\dot{y} = 10 - 9.8 \times 2.0267 = -9.862$

the speed $v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{10^2 + 9.862^2}$

speed $v = 14.045$ m/s A1

h. the angle at which it hits the ground is ψ

$\dot{y}$ is downwards since it is negative

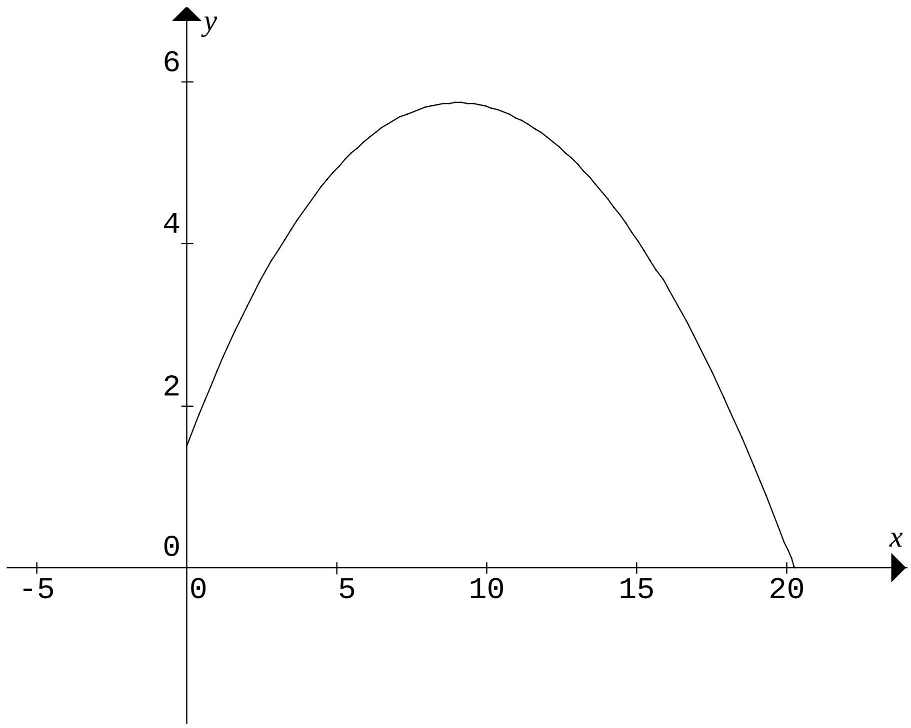


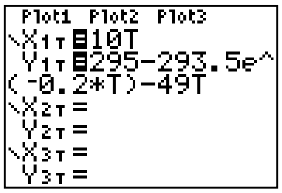
$\tan(\psi) = \frac{\dot{y}}{\dot{x}}$

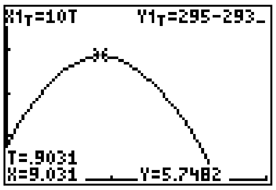
$\psi = \tan^{-1}\left(\frac{9.862}{10}\right) = 44.60^0$

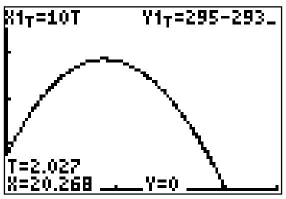
$\psi = 44^0 36'$ A1

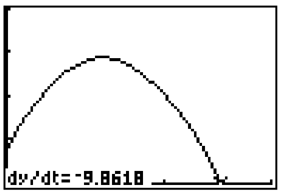
i. graph passes through y-axis at $y = 1.5$,
only for $0 \le x \le 20.268$
graph is not symmetrical, not parabolic, A1
maximum at $(9.031, 5.748)$











END OF SECTION 2 SUGGESTED ANSWERS