Specialist Mathematics Exam 1 2007 Solutions

Question 1

The equation has real coefficients therefore the conjugate root theorem applies.
So $2 - i$ is another root. **A1**

The two factors can be expressed as a quadratic as follows:

$(z-2-i)(z-2+i) = z^2 - 4z + 5$ **A1**

Divide $z^2 - 4z + 5$ into $z^4 - 4z^3 + 6z^2 - 4z + 5$ to obtain $z^2 + 1$ **M1**

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\; z^2 + 1 \\ z^2 - 4z + 5 & z^4 - 4z^3 + 6z^2 - 4z + 5 \\ & \underline{z^4 - 4z^3 + 5z^2} \\ & \quad\quad\quad\quad z^2 - 4z + 5 \\ & \quad\quad\quad\quad \underline{z^2 - 4z + 5} \end{array}$$

$\therefore (z^2 - 4z + 5)(z^2 + 1) = 0$

$(z-2-i)(z-2+i)(z-i)(z+i) = 0$

$\therefore z = 2+i, 2-i, i, -i$

Solutions are: $z = 2 \pm i$ and $z = \pm i$ **A1**

Question 2

a. $\underset{\sim}{u} = \cos(\theta)\underset{\sim}{i} + \sin(\theta)\underset{\sim}{j}$ and $\underset{\sim}{v} = \sin(\theta)\underset{\sim}{i} + \cos(\theta)\underset{\sim}{j}$

$$\begin{aligned} |\underset{\sim}{u}| &= \sqrt{\cos^2(\theta) + \sin^2(\theta)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$ **A1**

$$\begin{aligned} |\underset{\sim}{v}| &= \sqrt{\sin^2(\theta) + \cos^2(\theta)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

Hence, both $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are unit vectors.

b. $$\begin{aligned} \cos(\alpha) &= \frac{\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta)}{\sqrt{1} \times \sqrt{1}} \\ &= 2\sin(\theta)\cos(\theta) \\ &= \sin(2\theta) \end{aligned}$$ **M1**

$\alpha = \cos^{-1}(\sin(2\theta))$ or $\alpha = \frac{\pi}{2} - 2\theta$ **A1**

c. $$\begin{aligned} \alpha &= \cos^{-1}\left(\sin\left(\frac{2 \times \pi}{6}\right)\right)] \\ &= \cos^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) \\ &= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\pi}{6} \end{aligned}$$ **A1**

d. $(\underset{\sim}{v} \cdot \hat{\underset{\sim}{u}})\hat{\underset{\sim}{u}} = \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\right)\left(\frac{1}{2}\underset{\sim}{i} + \frac{\sqrt{3}}{2}\underset{\sim}{j}\right)$ **M1**

$= \frac{\sqrt{3}}{4}\underset{\sim}{i} + \frac{3}{4}\underset{\sim}{j}$ or

$= \frac{1}{4}\left(\sqrt{3}\,\underset{\sim}{i} + 3\underset{\sim}{j}\right)$ **A1**

Question 3

a. $\frac{x+2}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$ where A and B are constants. **A1**

$\therefore x + 2 = A(x+1) + B(x)$

Let $x = 0$ so $A = 2$

Let $x = -1$ so $B = -1$ **A1** (both A and B correct)

$\therefore \frac{x+2}{x^2+u} = \frac{2}{x} - \frac{1}{x+1}$

b. $\int_{-4}^{-3}\left(\frac{x+2}{x^2+x}\right)dx = \int_{-4}^{-3}\left(\frac{2}{x} - \frac{1}{x+1}\right)dx$

$= \left[2\log_e|x| - \log_e|x+1|\right]_{-4}^{-3}$ **A2** for anti-derivatives

$= (2\log_e 3 - \log_e 2) - (2\log_e 4 - \log_e 3)$ Modulus sign missing = –1

$= \log_e(27/32)$

Answer: $a = 27$, $b = 12$

Note: cannot get this mark from logs of negative numbers. Equivalent multiples of *a* and *b* in non-simplified fraction is correct.

Question 4

a. Let $u = \sqrt{3x}$ and $w = 3x$

$u = \sqrt{w}$ and so $\frac{du}{dw} = \frac{1}{2\sqrt{w}}$ and $\frac{dw}{dx} = 3$

$\frac{du}{dx} = \frac{du}{dw} \times \frac{dw}{dx}$

$= \frac{3}{2\sqrt{3x}}$ **A1**

$y = \cos^{-1}(u)$ *and so* $\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{-1}{\sqrt{1-u^2}} \times \frac{3}{2\sqrt{3x}}$ **M1**

$= \frac{-1}{\sqrt{1-3x}} \times \frac{3}{2\sqrt{3x}}$

$= \frac{-3}{2\sqrt{3x(1-3x)}}$

Hence shown.

b. $\displaystyle\int_{\frac{1}{12}}^{\frac{1}{6}} \frac{1}{\sqrt{3x(1-3x)}}dx$ **A1** for $-\frac{2}{3}$ in front

$\displaystyle = -\frac{2}{3}\int_{\frac{1}{12}}^{\frac{1}{6}} \frac{-3}{2\sqrt{3x(1-3x)}}dx$ **M1** for recognition

$\displaystyle = -\frac{2}{3}\left[\cos^{-1}\left(\sqrt{3x}\right)\right]_{\frac{1}{12}}^{\frac{1}{6}}$

$\displaystyle = -\frac{2}{3}\left(\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) - \cos^{-1}\left(\frac{1}{2}\right)\right)$

$\displaystyle = -\frac{2}{3}\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$

$\displaystyle = \frac{\pi}{18}$ **A1**

Question 5

a. $2a = 2g - 0.05v^2 \;\therefore a = g - \frac{v^2}{40}$ **A1**

b. Using $a = v\frac{dv}{dx}$ in the equation of motion gives:

$\displaystyle v\frac{dv}{dx} = \frac{2g - 0.05v^2}{2}$ **M1**

$\displaystyle \frac{dv}{dx} = \frac{2g - 0.05v^2}{2v}$

$\displaystyle \frac{dx}{dv} = \frac{2v}{2g - 0.05v^2}$ **A1**

Multiplying numerator and denominator by 20 gives

$\displaystyle \frac{dx}{dv} = \frac{40v}{40g - v^2}$ as required.

c. The required distance is given by the integral: $\displaystyle\int_0^{10} \frac{40v}{40g - v^2}dv$ **A1**

Note: The integral must have correct limits and *dv*. Does not need to have a modulus of $\frac{40v}{40g - v^2}$, since we are after distance and the graph was not asked for.

$\displaystyle x = -20\int_0^{10} \frac{-2v}{-v^2 + 40g}dv$ **M1**

$\displaystyle = \left[-20\log_e\left(40g - v^2\right)\right]_0^{10}$ **M1**

$\displaystyle = -20\log_e(40g - 100) + 20\log_e(40g)$

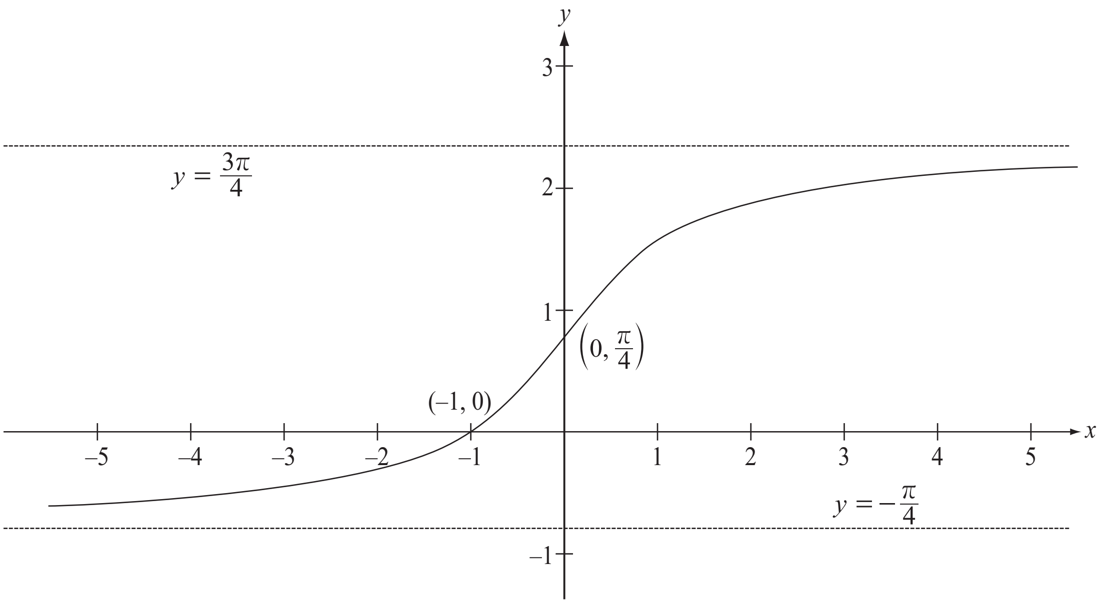
$\displaystyle = 20\log_e\left(\frac{40g}{40g - 100}\right)$

$\displaystyle = 20\log_e\left(\frac{2g}{2g - 5}\right)$

Note: $20\log_e\left(\frac{40g}{40g - 100}\right)$ can get the last A1 mark. **A1**

Question 6

a.



x-intercept $(-1, 0)$ **A1**

y-intercept $\left(0, \frac{\pi}{4}\right)$ **A1**

Asymptotes $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$ and shape. **A1**

b. $\arctan(x) + \frac{\pi}{4} = \frac{5\pi}{12}$

$\arctan(x) = \frac{\pi}{6}$

$x = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ **A1**

Question 7

a. Differentiating $\underset{\sim}{r}$ with respect to t:

$\underset{\sim}{r} = (-3\sin(t) - 2\cos(2t))\underset{\sim}{i} + (3\cos(t) - 2\sin(2t))\underset{\sim}{j}$ **A2** (1 each $\underset{\sim}{i}$, $\underset{\sim}{j}$ term)

b. Speed $= |\underset{\sim}{v}|$

$= \sqrt{(3\sin(t) + 2\cos(2t))^2 + (3\cos(t) - 2\sin(2t))^2}$ **M1**

$= \sqrt{9\sin^2(t) + 12\sin(t)\cos(2t) + 4\cos^2(2t) + 9\cos^2(t) - 12\cos(t)\sin(2t) + 4\sin^2(2t)}$

$= \sqrt{(9\sin^2(t) + 9\cos^2(t)) + 12(\sin(t)\cos(2t) - 12\cos(t)\sin(2t)) + (4\cos^2(2t) + 4\sin^2(2t))}$

$= \sqrt{9 + 4 + 12\sin(t - 2t)}$ **M1** for using the compound angle formula

$= \sqrt{13 - 12\sin(t)}$

$\therefore$ Maximum speed is $\sqrt{13 + 12}$ when $\sin(t) = -1$

$\therefore$ Maximum speed is 5. **A1**

c. $\sqrt{13-12\sin(t)}$

$-1 \le \sin(t) \le 1$

$\therefore -12 \le 12\sin(t) \le 12$

$\therefore \sqrt{13-12} = 1, \sqrt{13+12} = 5$

$\therefore$ speed will always be between 1 and 5 **A1**

$\therefore$ it never stops

Question 8

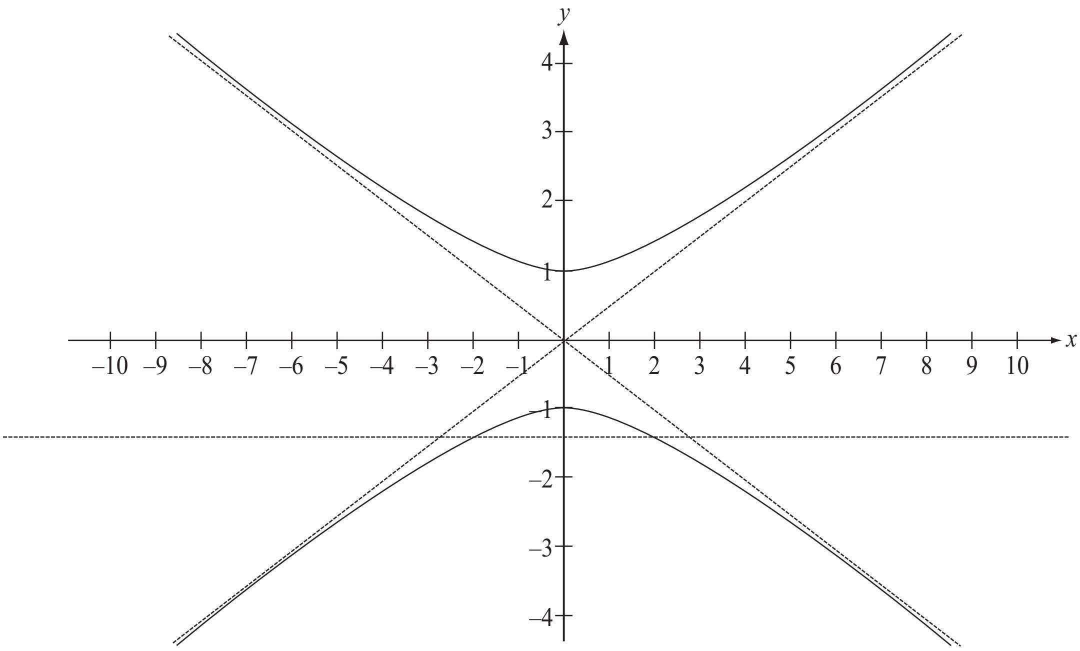
a. $\frac{x}{2} = \tan(t)$ and $y = \sec(t)$

$1 + \tan^2(t) = \sec^2(t)$ **M1**

$1 + \frac{x^2}{4} = y^2$

$1 = \frac{y^2}{1} - \frac{x^2}{4}$

b.



2 marks: A1 shape and asymptotes $y = \pm\frac{x}{2}$; **A1** y-intercepts $(0, \pm 1)$

c. $\int_1^2 \pi x^2\,dy = \int_1^2 4\pi\left(y^2-1\right)dy$

$= \left[4\pi\left(\frac{y^3}{3} - y\right)\right]_1^2$ **M1**

$= 4\pi\left[\left(\frac{8}{3}-2\right)-\left(\frac{1}{3}-1\right)\right]$

$= \frac{16\pi}{3}$ cubic units **A1**