Working space

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1

The range of the graph with equation $y = a\cos^{-1}(3x) + \frac{\pi}{2}$ is:

A. $\left(\frac{\pi}{2}, \frac{\pi}{2} + a\right)$

B. $\left[\frac{\pi a}{2}, \frac{3\pi a}{2}\right]$

C. $\left[-\frac{1}{3}, \frac{1}{3}\right]$

D. $\left[\frac{\pi}{2}, (2a+1)\frac{\pi}{2}\right]$

E. $[2\pi a, 6\pi a]$

Question 2

The distance between vertices, and the equations of the asymptotes for the following hyperbola with equation $9(x-1)^2 - 4y^2 = 36$ are, respectively:

A. $2;\ 3x - 2y - 3 = 0,\ -3x - 2y + 3 = 0$

B. $4;\ 3x - 2y - 3 = 0,\ -3x - 2y + 3 = 0$

C. $3;\ 3x - 2y = 0,\ 3x + 2y = 0$

D. $2;\ 3x - 2y = 0,\ 3x + 2y = 0$

E. $4;\ 3x - 2y - 2 = 0,\ 3x + 2y + 2 = 0$

Question 3

If $8\text{cis}\left(-\frac{5\pi}{6}\right)$ is one of the cube roots of the complex number z, then $|z|$ and $\text{Arg}(z)$ are:

A. 2 and $-\frac{5\pi}{18}$

B. 2 and $-\frac{\pi}{2}$

C. 512 and $-\frac{\pi}{2}$

D. 512 and $\frac{\pi}{2}$

E. $\frac{8}{3}$ and $-\frac{\pi}{2}$

Question 4

The following equation $z^4 - 4z^3 + 8z^2 - 8z + 16 = 0$, where $z \in C$, has:

A. three complex and one real solution
B. two real and two complex solutions
C. three real and one complex solution
D. no complex solutions
E. four complex solutions

Question 5

The unit vector in the direction of vector $\underset{\sim}{a} = \sqrt{2}\,\underset{\sim}{i} - \underset{\sim}{j} + 7\underset{\sim}{k}$ is:

A. $\frac{1}{2\sqrt{13}}\left(\sqrt{2}\,\underset{\sim}{i} - \underset{\sim}{j} + 7\underset{\sim}{k}\right)$

B. $\frac{1}{2\sqrt{13}}\left(-\sqrt{2}\,\underset{\sim}{i} + \underset{\sim}{j} - 7\underset{\sim}{k}\right)$

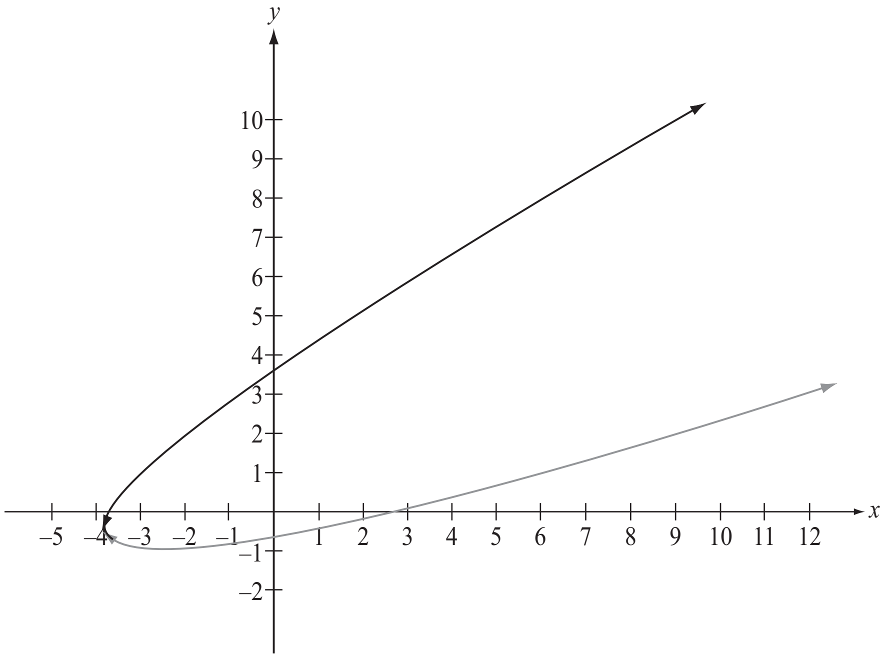
C. $\frac{1}{2\sqrt{3}}\left(-\sqrt{2}\,\underset{\sim}{i} + \underset{\sim}{j} - 7\underset{\sim}{k}\right)$

D. $\frac{1}{\sqrt{10}}\left(\sqrt{2}\,\underset{\sim}{i} - \underset{\sim}{j} + \sqrt{7}\,\underset{\sim}{k}\right)$

E. $\frac{1}{\sqrt{54}}\left(-2\underset{\sim}{i} + \underset{\sim}{j} + 7\underset{\sim}{k}\right)$

Question 6

The graph below shows the implicit relation $x^2 - 4xy + 4y^2 + x - 12y - 10 = 0$.



The gradient of the tangent at the point $(6, 1)$ is equal to:

A. $\frac{19}{28}$

B. $\frac{9}{28}$

C. $\frac{31}{4}$

D. $-\frac{9}{28}$

E. $-\frac{19}{28}$

Question 7

The set of points in the complex plane defined by $|z| = |z + 4|$ is:

A. the circle with centre $(-2, 0)$ and radius 2

B. the circle with centre $(2, 0)$ and radius 2

C. the line $\text{Re}(z) = 2$

D. the line $\text{Re}(z) = -2$

E. the line $\text{Im}(z) = -2$

Question 8

Using a suitable substitution, $\int_0^{\frac{\pi}{6}} \left(\cos(3x)e^{\sin(3x)}\right)dx$ is equal to:

A. $\int_0^{\frac{\pi}{6}} e^u\,du$

B. $\int_0^{1} e^u\,du$

C. $3\int_0^{1} e^u\,du$

D. $\frac{1}{3}\int_0^{1} e^u\,du$

E. $-\frac{1}{3}\int_0^{1} e^u\,du$

Question 9

Points A, B and C have position vectors $2\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}$, $\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}$ and $3\underset{\sim}{i}+\underset{\sim}{j}$ respectively, with respect to an origin O. The cosine of $\angle ABC$ is:

A. $\frac{8}{9}$

B. $-\frac{8}{9}$

C. $\frac{4}{\sqrt{17}}$

D. $\frac{5}{\sqrt{30}}$

E. $\frac{8}{81}$

Question 10

Euler's method, with a size step of 0.1, is used to solve the differential equation $\frac{dy}{dx} = \log_e(x)$ with $y = 3$ at $x = 1$. The value of y at $x = 1.3$, correct to four decimal places, is:

A. 3.0095

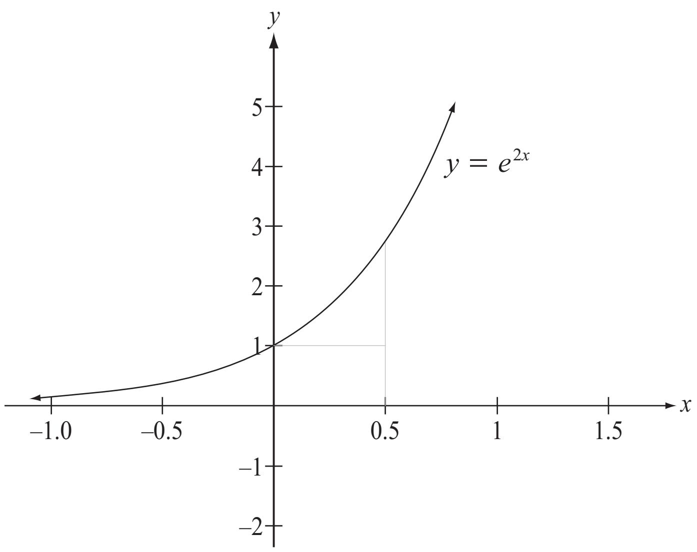
B. 3.0277

C. 3.0278

D. 3.2602

E. 3.2603

Question 11



The region enclosed by the curve $y = e^{2x}$ and the lines $x = \frac{1}{2}$ and $y = 1$ is rotated about the y-axis. The volume of revolution is given by:

A. $\pi\int_1^e \left(\frac{1}{2} - \frac{1}{2}\log_e y\right)^2 dy$

B. $\frac{\pi}{4}\int_1^e \left(1 - \log_e^2 y\right) dy$

C. $\pi\int_0^{\frac{1}{2}} \left(e^{4x} - 1\right) dx$

D. $\frac{\pi}{4}\int_1^e \left(\log_e y\right)^2 dy$

E. $\int_1^e \frac{1 - (\log_e y)^2}{4} dy$

Question 12

The graph of the function $y = \frac{x^2 + 9}{3x}$ has:

A. no asymptotes and no turning points
B. no asymptotes and one turning point
C. a single asymptote $x = 0$ and no turning points
D. a single asymptote $x = 0$ and two turning points
E. two asymptotes and two turning points

Question 13

The necessary and sufficient conditions for a point P to be a point of inflection are that, as the curve passes through P:

A. $\frac{d^2y}{dx^2} = 0$ and $\frac{dy}{dx} = 0$

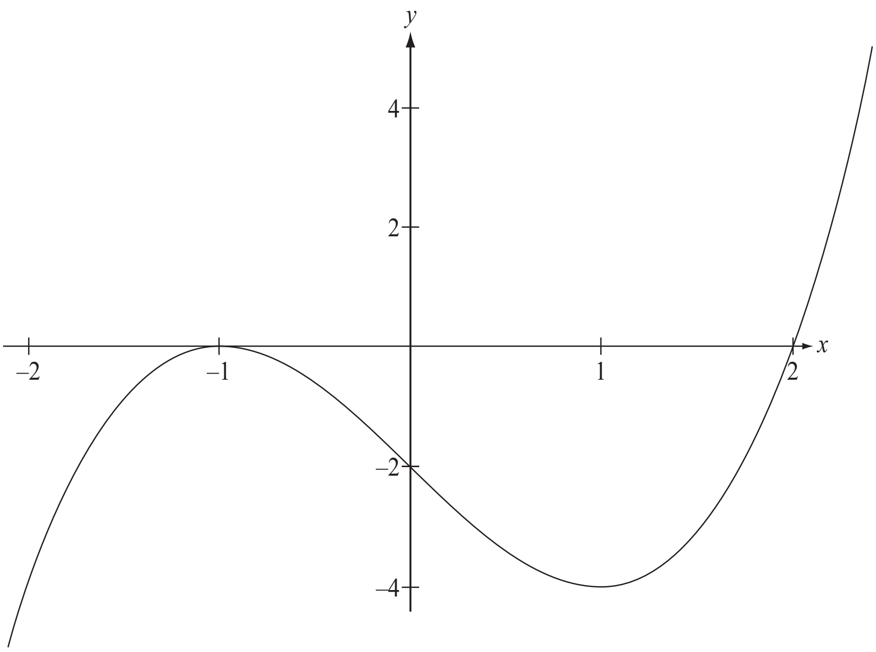
B. $\frac{d^2y}{dx^2} = 0$ and $\frac{dy}{dx}$ changes sign

C. $\frac{d^2y}{dx^2} = 0$ and $\frac{dy}{dx} > 0$

D. $\frac{d^2y}{dx^2}$ changes sign and $\frac{dy}{dx}$ does not change sign

E. $\frac{d^2y}{dx^2} = 0$ and $\frac{dy}{dx} < 0$

Question 14



The graph of $y = f'(x)$ is shown above. Which one of the following statements is true for the graph of $y = f(x)$?

A. The graph has a local maximum at $x = -1$ and a local minimum at $x = 1$.

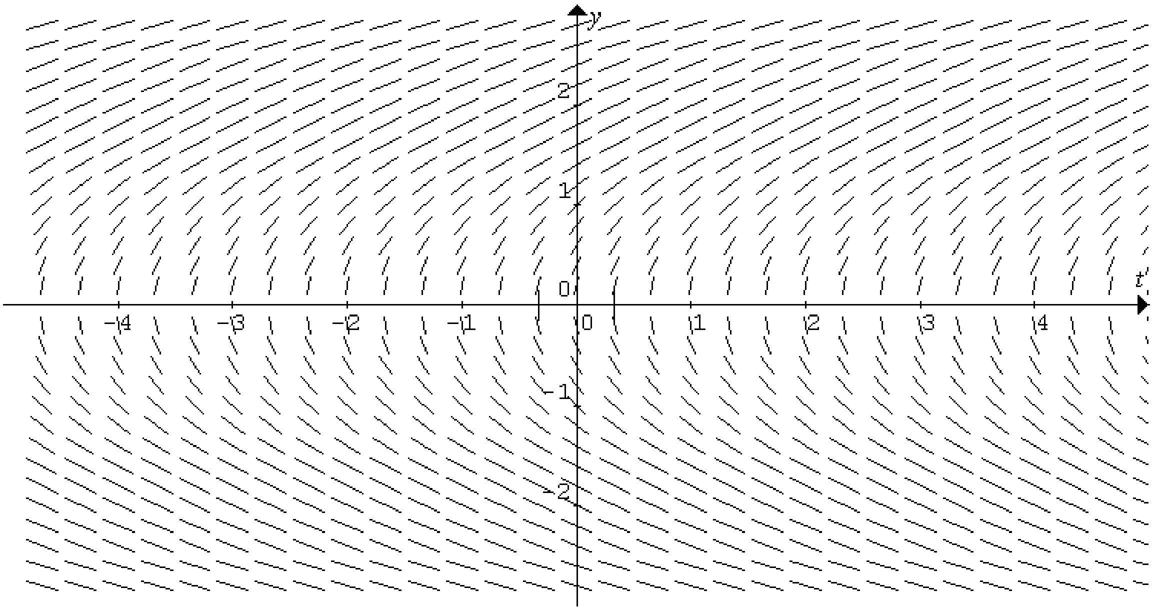
B. The graph has a stationary point of inflection at $x = -1$, a local minimum at $x = 1$ and a local minimum at $x = 2$.

C. The graph has a stationary point of inflection at $x = -1$, a point of inflection at $x = 1$ and a local minimum at $x = 2$.

D. The graph has a stationary point of inflection at $x = -1$ as well as a local maximum at $x = 2$.

E. The graph has a stationary point of inflection at $x = 2$ as well as a local maximum at $x = -1$.

Question 15



Given the slope field diagram shown above, the possible differential equation could be:

A. $\dfrac{dy}{dt} = \dfrac{1}{y}$

B. $\dfrac{dy}{dt} = t^2 - 2$

C. $\dfrac{dy}{dt} = \sin(t)$

D. $\dfrac{dy}{dt} = t$

E. $\dfrac{dy}{dt} = e^t$

Question 16

The position vector of a particle at time t seconds, where $t \geq 0$, is given by
$\underset{\sim}{r}(t) = (2 + 3t - 2t^2)\underset{\sim}{i} + 6t\underset{\sim}{j} + \sqrt{t}\,\underset{\sim}{k}$.

The direction of motion at $t = 2$ is:

A. $-4\underset{\sim}{i} + 6\underset{\sim}{j} + \dfrac{1}{4}\underset{\sim}{k}$

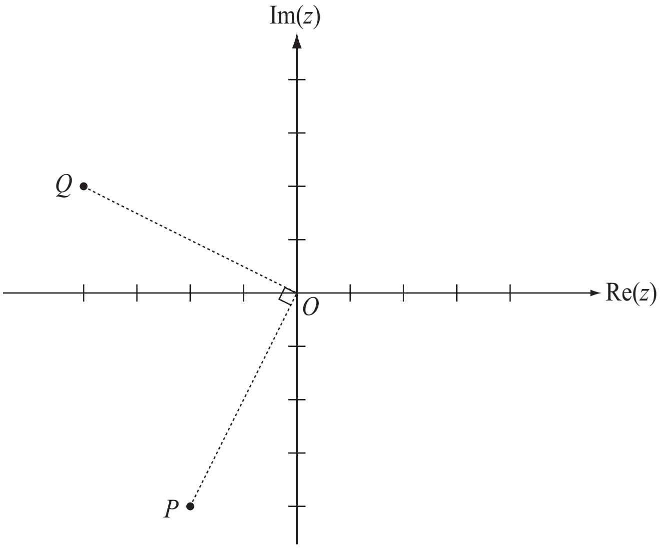
B. $-5\underset{\sim}{i} + 6\underset{\sim}{j} + \dfrac{1}{4}\underset{\sim}{k}$

C. $-5\underset{\sim}{i} + 6\underset{\sim}{j} + \dfrac{1}{2}\underset{\sim}{k}$

D. $-5\underset{\sim}{i} + 6\underset{\sim}{j} + \dfrac{\sqrt{2}}{4}\underset{\sim}{k}$

E. $12\underset{\sim}{j} + \sqrt{2}\,\underset{\sim}{k}$

Question 17



The point P on the Argand plane represents a complex number z. If $OP = OQ$ and the angle POQ is 90°, then point Q represents:

A. iz

B. $-iz$

C. $-\overline{z}$

D. $\overline{z}$

E. $-z$

Question 18

A particle starts from rest at $t = 0$ and moves in a straight line with acceleration a given by the equation $a = 5\sin(2t) - 1$. The velocity of the particle at $t = 1$, correct to 2 decimal places, is closest to:

A. –14.16

B. –4.54

C. –3.54

D. 1.54

E. 2.54

Question 19

Jimmy, whose mass is 75 kg, is skiing down a hill which is inclined to the horizontal at an angle of $\arctan\left(\frac{1}{5}\right)$. He starts from rest on the top of the hill and skis down the total distance of 250 metres. Assuming that friction can be neglected, the magnitude of his momentum at the bottom of the hill is closest to:

A. 2326 $kgms^{-1}$

B. 2325 $kgms^{-1}$

C. 2324 $kgms^{-1}$

D. 333 $kgms^{-1}$

E. 34 $kgms^{-1}$

Question 20

A particle is acted upon by two forces, one of magnitude 8 newton acting due north and the other of magnitude 6 newton acting S60°W. The magnitude of the resultant force (in newton) acting on the particle is:

A. $2\sqrt{13}$

B. $2\sqrt{37}$

C. $\sqrt{34}$

D. 14

E. 52

Question 21

A bullet of mass m kg is shot vertically up with an initial speed of 400 m/s. It is subjected to an air resistance of $\frac{mv^2}{1000}$ newton where v m/s is the speed of the bullet. The maximum height (in metres) that the bullet reaches is closest to:

A. 8163

B. 1426

C. 1417

D. 1396

E. 785

Question 22

A balloon is ascending with a uniform velocity of 20 m/s. When the balloon is 60 m above the ground, a ball is released from it. Assume that air resistance can be neglected. The time (in seconds) after which the ball will reach the ground is closest to:

A. 2.01

B. 3.50

C. 6.09

D. 2.04

E. 5.71

END OF SECTION 1

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

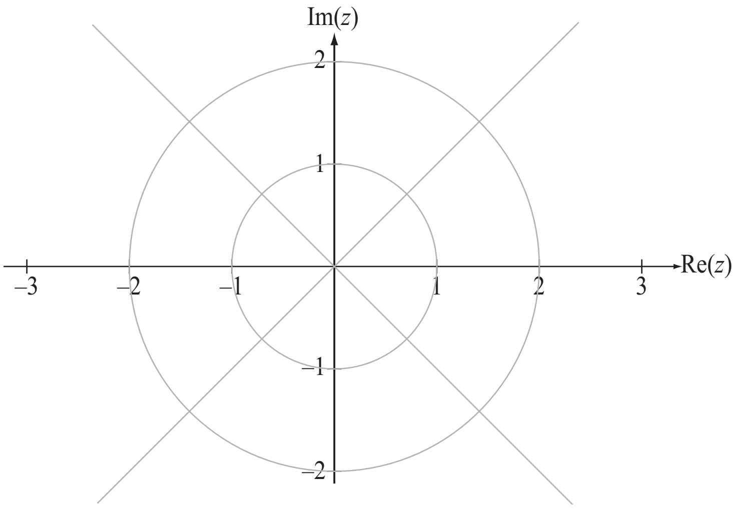
Question 1

Let u and v be the square roots of the complex number $-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$, where $\text{Arg}(u) > \text{Arg}(v)$.

a. Use de Moivre's theorem to find u and v in polar form.

3 marks

b. Mark u and v on the Argand diagram below.



2 marks

c. On the same diagram, draw and clearly label the following region T where $T = \{z : |z - u| = |z - v|\} \cap \{z : |z| \leq 2\}$.

3 marks

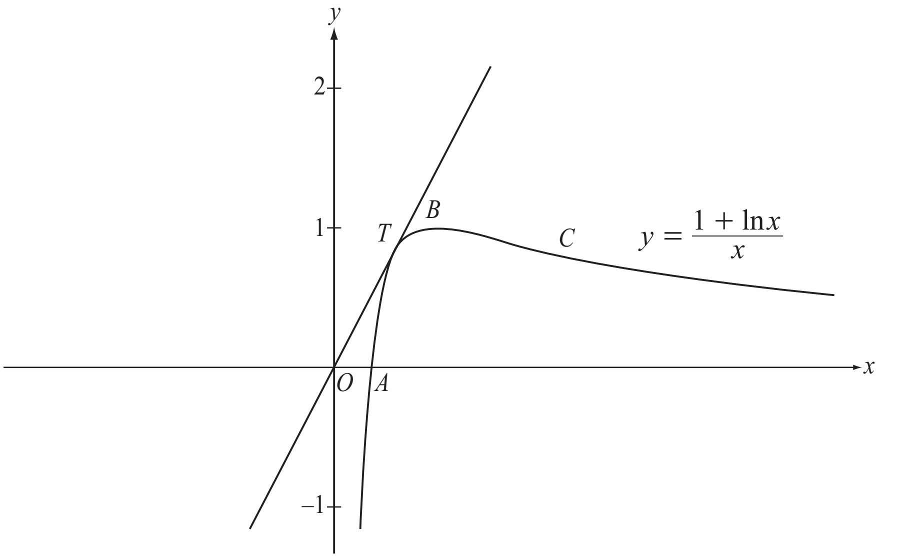
d. For what values of n is u^n purely imaginary?

3 marks

Total 11 marks

Question 2

The graph of $y = f(x) = \dfrac{1 + \log_e x}{x}$ is given below.



The line *OT*, where *O* is the origin, is the tangent to the graph at *T*. The graph of $f(x)$ meets the *x*-axis at *A*.

a. Find the **exact** coordinates of the *x*-intercept *A*.

1 mark

b. **i.** Find $f'(x)$.

1 mark

ii. Hence find the coordinates of the maxium turning point *B*.

1 mark

c. Find the **exact** coordinates of the point of inflection C.

3 marks

d. Verify that point C is a point of inflection.

1 mark

e. Show that the x-coordinate of point T is $x_T = \dfrac{1}{\sqrt{e}}$.

2 marks

f. Using calculus, find the **exact** area between the tangent *OT*, the graph of $f(x)$ and the x-axis.

4 marks

Total 13 marks

Question 3

A triangle has its vertices at $A(-1, 3, 2)$, $B(3, 6, 1)$ and $C(-4, 4, 3)$.

a. Show that $\overrightarrow{BC} \cdot \overrightarrow{BA} = 36$.

2 marks

Let D be a point on the side BC and let $\overrightarrow{BC} = \underset{\sim}{p}$, $\overrightarrow{BA} = \underset{\sim}{q}$ and $\overrightarrow{BD} = k\underset{\sim}{p}$ where $k \in R$.

b. Express $\overrightarrow{AD}$ in terms of $\underset{\sim}{p}$, $\underset{\sim}{q}$ and k.

1 mark

c. Find the value of k if $\overrightarrow{AD}$ is perpendicular to $\overrightarrow{BC}$.

3 marks

d. Hence, or otherwise, find the shortest distance from vertex A to the side BC, correct to 2 decimal places.

3 marks

Total 9 marks

Question 4

A large tank initially contains 800 litres of water in which is dissolved 10 kg of salt. A salt solution of concentration 0.1 kg per litre flows into the tank at the rate of 5 litres per minute. The mixture, which is kept uniform by stirring, flows out of the tank at the rate of 5 litres per minute. After t minutes the tank contains x kg of salt.

a. **i.** Show that the differential equation for x in terms of t is of the form

$$\frac{dx}{dt} = \frac{a-x}{b}$$

where a and b are positive constants.

3 marks

ii. Hence, write down the values of a and b.

1 mark

iii. Using calculus, solve the differential equation to find x in terms of t.

4 marks

iv. Find, correct to 3 decimal places, the number of kilograms of salt present in the solution after 12 minutes.

1 mark

b. If, instead, the mixture flows out of the tank at a rate of 1 litre per minute, set up ***but do not solve***, the differential equation for x in terms of t.

2 marks

Total 11 marks

Question 5

A toy plane of mass 1.5 kg is projected horizontally from the top of a vertical cliff 40 metres high with a speed of 5 m/s. Take O as the origin and T as the top of the cliff with $\underset{\sim}{i}$ directed horizontally to the right and $\underset{\sim}{j}$ directed vertically upwards. Ignore air resistance.

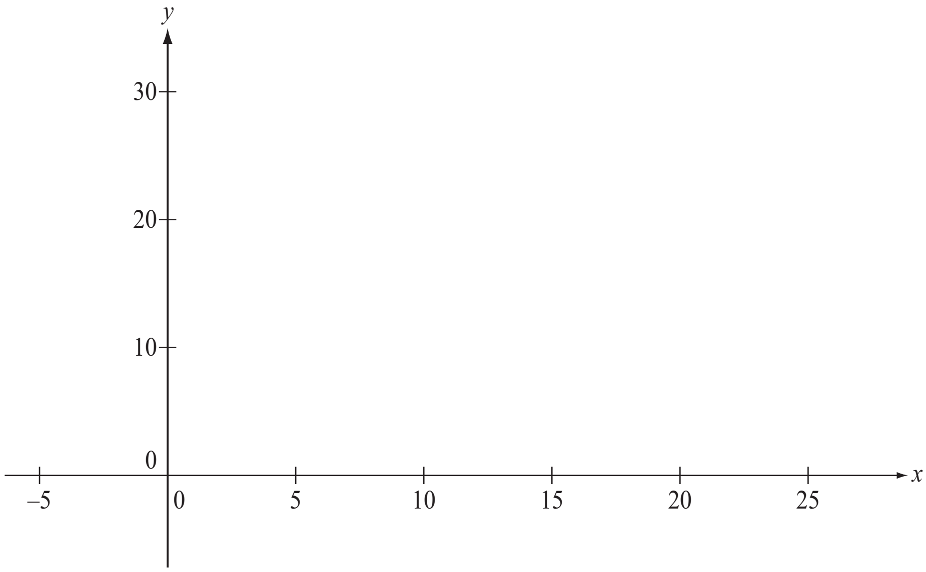
a. Show that the position vector of the toy plane is given by $\underset{\sim}{a} = 5t\underset{\sim}{i} + (40 - 4.9t^2)\underset{\sim}{j}$.

2 marks

b. Find the time taken to reach the ground and the horizontal distance travelled. Give your answers correct to 2 decimal places.

2 marks

c. Find the Cartesian equation of the path of the toy plane and sketch its path on the axes below, showing all relevant features.

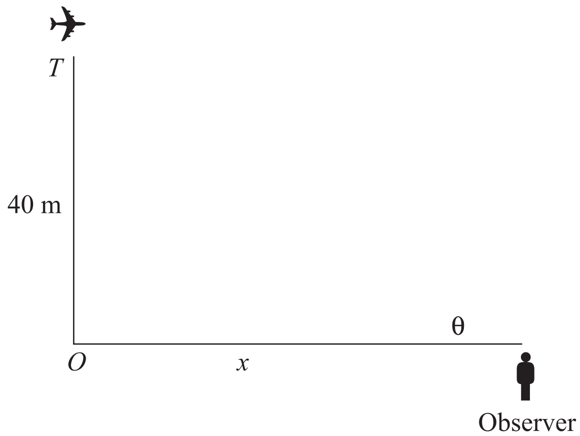


4 marks

d. Find the magnitude of the momentum of the toy plane when it hits the ground. Give your answer to 3 significant figures.

3 marks

Another toy plane is battery operated so it can maintain a constant altitude. It also starts from the top of the cliff but flies horizontally at a constant speed of 5 m/s at a constant height of 40 metres in a straight line so that it will eventually fly directly over an observer who is at ground level.



e. Differentiate implicitly the equation $\tan(\theta) = \frac{40}{x}$ to find $\frac{d\theta}{dt}$.

2 marks

f. Hence evaluate $\frac{d\theta}{dt}$, in radian per second, when $x = 10$ metres, expressing the answer in **exact** form.

2 marks

Total 15 marks

END OF QUESTION AND ANSWER BOOK

Specialist Mathematics Exam 2: SOLUTIONS

Multiple-choice Answers

1. D	**2.** B	**3.** C	**4.** E	**5.** A
6. B	**7.** D	**8.** D	**9.** A	**10.** C
11. B	**12.** E	**13.** D	**14.** C	**15.** A
16. D	**17.** B	**18.** E	**19.** B	**20.** A
21. B	**22.** C			

Section 1: Multiple-choice Solutions

Question 1 Answer D

The range of $\cos^{-1}(3x)$ is $[0, \pi]$ and so the range of $y = a\cos^{-1}(3x) + \frac{\pi}{2}$ will be:
$\left[a \times 0 + \frac{\pi}{2}, a \times \pi + \frac{\pi}{2}\right] = \left[\frac{\pi}{2}, (2a+1)\frac{\pi}{2}\right]$

Question 2 Answer B

The hyperbola is $\frac{(x-1)^2}{4} - \frac{y^2}{9} = 1$ which has centre at $(1, 0)$. The vertices are 2 units either side of the centre, given by $\sqrt{4}$ in the equation. Hence they are 4 units apart.

Equations of asymptotes: $y = \pm\frac{3}{2}(x-1)$ which can be written as: $3x - 2y - 3 = 0$ and $-3x - 2y + 3 = 0$.

Question 3 Answer C

$$
\begin{aligned}
z &= \left[8\text{cis}\left(-\frac{5\pi}{6}\right)\right]^3 \\
&= 512\text{cis}\left(-\frac{5\pi}{2}\right) \\
&= 512\text{cis}\left(-\frac{\pi}{2}\right)
\end{aligned}
$$

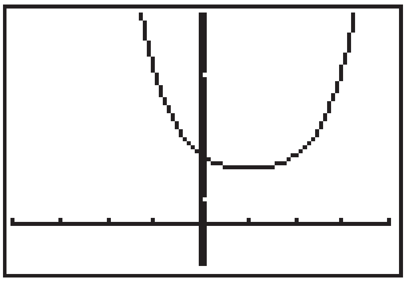
and so $|z|$ and $\text{Arg}(z)$ are 512 and $-\frac{\pi}{2}$

Question 4 Answer E

$z^4 - 4z^3 + 8z^2 - 8z + 16 = 0, z \in C$

If there are any complex solutions then they will be in pairs as all of the coefficients are real. This eliminates alternatives A and C.

Graphing this polynomial onto the graphics calculator shows no x-intercepts and so there are 4 complex solutions.



Question 5 Answer A

$\underset{\sim}{a} = \sqrt{2}\,\underset{\sim}{i} - \underset{\sim}{j} + 7\underset{\sim}{k}$ has a magnitude of $\sqrt{2 + 1 + 49} = \sqrt{52}$ or $2\sqrt{13}$.

$\therefore$ unit vector $= \frac{1}{2\sqrt{13}}\left(\sqrt{2}\,\underset{\sim}{i} - \underset{\sim}{j} + 7\underset{\sim}{k}\right)$

Question 6 Answer B

From the graph, the gradient is clearly positive and so alternatives D and E are incorrect. Differentiate the equation:

$x^2 - 4xy + 4y^2 + x - 12y - 10 = 0$

$2x - \left(4y + 4x\frac{dy}{dx}\right) + 8y\frac{dy}{dx} + 1 - 12\frac{dy}{dx} = 0$

Substituting $x = 6$ and $y = 1$ gives:

$12 - \left(4 + 24\frac{dy}{dx}\right) + 8\frac{dy}{dx} + 1 - 12\frac{dy}{dx} = 0$

$9 = 28\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{9}{28}$

Question 7 Answer D

Let $z = x + iy$ in $|z| = |z + 4|$.

$|x + iy| = |(x + 4) + iy|$

$\sqrt{x^2 + y^2} = \sqrt{(x + 4)^2 + y^2}$

$x^2 + y^2 = (x + 4)^2 + y^2$

$x^2 + y^2 = x^2 + 8x + 16 + y^2$

$8x = -16$ and so $x = -2$

Now $x = \text{Re}(z)$ and so the answer is D.

Question 8 Answer D

$$\int_0^{\frac{\pi}{6}} \cos(3x)e^{\sin(3x)}\,dx$$

Let $u = \sin(3x)$ so $\dfrac{du}{dx} = 3\cos(3x)$

If $x = 0$ then $u = 0$ and if $x = \dfrac{\pi}{6}$ then $u = 1$.

$\dfrac{1}{3}\displaystyle\int_0^1 e^u\,du$ is the result.

Question 9 Answer A

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = \left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|\cos\angle ABC$$

$$\begin{aligned}\overrightarrow{BA} &= \overrightarrow{BO} + \overrightarrow{OA}\\ &= -\left(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) + 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\\ &= \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k} \text{ and so } \left|\overrightarrow{BA}\right| = 3\end{aligned}$$

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC}\\ &= -\left(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) + 3\underset{\sim}{i} + \underset{\sim}{j}\\ &= 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} \text{ and so } \left|\overrightarrow{BC}\right| = 3\end{aligned}$$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = 2 + 4 + 2$ and therefore $\cos\angle ABC = \dfrac{8}{9}$

Question 10 Answer C

$$\begin{aligned}f(x+h) &\approx f(x) + h\log_e(x) \text{ where}\\ f(1+0.1) &\approx f(1) + 0.1 \times \ln(1)\\ &= 3\end{aligned}$$

$$\begin{aligned}f(1.1+0.1) &\approx f(1.1) + 0.1 \times \ln(1.1)\\ &= 3 + 0.009531\\ &= 3.009531\end{aligned}$$

$$\begin{aligned}f(1.2+0.1) &\approx f(1.2) + 0.1 \times \ln(1.2)\\ &= 3.009531 + 0.01823\\ &= 3.02776\end{aligned}$$

Answer: 3.0278

Question 11 Answer B

A volume of revolution around the y-axis is found by evaluating $\int \pi x^2\, dy$. The lower limit is $y = 1$ and the upper is e.

$y = e^{2x}$

$2x = \log_e(y)$

$x = \frac{1}{2}\log_e(y)$

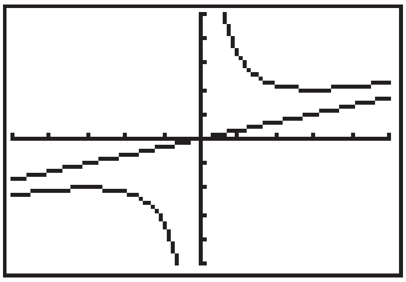
The volume formed by rotating the curve around the y-axis $= \frac{\pi}{4}\int_1^e (\log_e(y))^2\, dy$

The volume formed by rotating the line $x = \frac{1}{2}$ around the y-axis $= \pi\int_1^e \left(\frac{1}{2}\right)^2 dy$

Required volume $\frac{\pi}{4}\int_1^e (1 - \log^2(y))dy$

Question 12 Answer E

The graph of the function $y = \frac{x^2 + 9}{3x}$ has two asymptotes, one vertical $(x = 0)$ and the other oblique $\left(y = \frac{x}{3}\right)$. It also has two turning points as can be seen from the graph of this function. The oblique asymptote with equation $y = \frac{x}{3}$ is also shown.



Question 13 Answer D

A point of inflection occurs where the second derivative is zero and the first derivative does **not** change sign. So, at the point of inflection $\frac{d^2y}{dx^2}$ changes sign and $\frac{dy}{dx}$ does not change sign.

Question 14 Answer C

A stationary point of inflection occurs at $x = -1$ since $f'(x) = 0$ and the derivative of $f'(x)$ is zero as well at that point. $f'(x)$ changes from negative to positive through zero at $x = 2$ and so there is a local minimum at this point for the graph of $y = f(x)$. The rate of change of the gradient of $f'(x)$ is zero at $x = 1$ (but the gradient of $f(x)$ is about -4) and so there is a point of inflection at this point.

Question 15 Answer A

It should be noted that $\frac{dy}{dx}$ is

- positive if $y > 0$
- negative if $y < 0$
- infinite (shown by vertical steepness) as y approaches zero.

Only alternative A displays these properties.

Question 16 Answer D

The direction of motion at $t = 2$ is given by $\dot{\underset{\sim}{r}}(2)$.

$$\dot{\underset{\sim}{r}}(t) = (3 - 4t)\underset{\sim}{i} + 6\underset{\sim}{j} + \frac{1}{2\sqrt{t}}\underset{\sim}{k}$$

Substituting $t = 2$: $-5\underset{\sim}{i} + 6\underset{\sim}{j} + \frac{1}{2\sqrt{2}}\underset{\sim}{k}$ which is equivalent to $-5\underset{\sim}{i} + 6\underset{\sim}{j} + \frac{\sqrt{2}}{4}\underset{\sim}{k}$.

Question 17 Answer B

Let the coordinates of P be $(-a, -b)$ where a and b are both positive. Hence $z = -a - bi$.
Q (by inspection) is $-b + ai$.

Testing each of the alternatives given:

A. $iz = i(-a - bi) = b - ai$ which is not Q. However, it is the negative of what is required and so the answer is **B**.

Alternatively, rotation clockwise through a right angle is equivalent to multiplication by $-i$.

Question 18 Answer E

$$v = \int (5\sin(2t) - 1)dt$$

$$= -\frac{5}{2}\cos(2t) - t + c$$

At $t = 0$, $v = 0$ and so $c = 2.5$

$$v = -\frac{5}{2}\cos(2t) - t + 2.5$$

At $t = 1$, $v = -2.5\cos(2) - 1 + 2.5$

Answer: $v = 2.54$

Question 19 Answer B

Acceleration down slope $= g\sin(\theta)$ where

$\theta = \arctan\left(\frac{1}{5}\right)$.

$\therefore a = \dfrac{g}{\sqrt{26}}$ is Jimmy's uniform acceleration.

$u = 0$, $s = 250$, $a = \dfrac{g}{\sqrt{26}}$ and $v^2 = u^2 + 2as$

$\therefore v^2 = 0 + 2 \times \dfrac{g}{\sqrt{26}} \times 250$ and so $v = 30.9995$

$$\begin{aligned}\text{Momentum} &= mv \\ &= 75 \times 30.9995 \\ &= 2325 \text{ kgms}^{-1}\end{aligned}$$

Question 20 Answer A

Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors acting east and north respectively.

The sum of the two forces are:

$8\underset{\sim}{j} + \left(-6\cos 60°\underset{\sim}{j} - 6\sin 60°\underset{\sim}{i}\right)$

$= -6\sin 60°\underset{\sim}{i} + (8 - 6\cos 60°)\underset{\sim}{j}$

$= -3\sqrt{3}\,\underset{\sim}{i} + 5\underset{\sim}{j}$

$$\begin{aligned}\text{Magnitude} &= \sqrt{(27 + 25)} \\ &= \sqrt{52} \\ &= 2\sqrt{13}\end{aligned}$$

Question 21 Answer B

$$-mg = \frac{mv^2}{1000} = ma$$

$$a = -\frac{1000g + v^2}{1000}$$

$$v\frac{dv}{dx} = -\frac{1000g + v^2}{1000}$$

$$\frac{dv}{dx} = -\frac{1000g + v^2}{1000v}$$

$$\frac{dx}{dv} = -\frac{1000v}{1000g + v^2}$$

$$x = \int -\frac{1000v}{1000g + v^2}dv$$

$$x = -500\ln\left(1000g + v^2\right) + c$$

When $x = 0,\ v = 400$

$$x = 500\ln\frac{9800 + 400^2}{1000g + v^2}$$

When $v = 0, x = 1426$ m

Question 22 Answer C

$s = -60, a = -9.8, u = 20, t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$-60 = 20t - 4.9t^2$$

$$4.9t^2 - 20t - 60 = 0$$

Solving this quadratic using TI-83 QUAD PRGM: $t = -2.01, 6.09$

Ignoring the negative solution, $t = 6.09$ seconds.

Specialist Mathematics Exam 2: SOLUTIONS

Section 2

Question 1

a. $-\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}i = 1\text{cis}\left(\dfrac{3\pi}{4}\right)$

Apply de Moivre's theorem:

$z^2 = -\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}i$

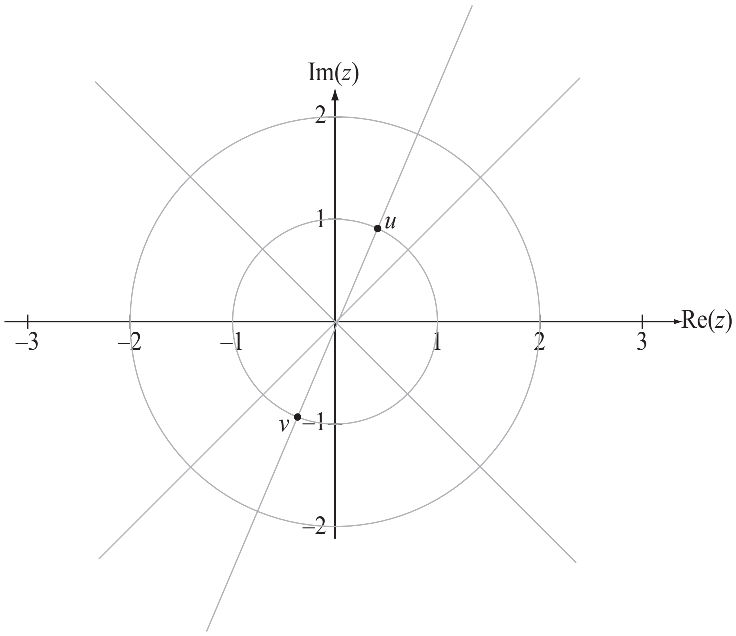
$r^2\,\text{cis}\,(2\theta) = 1\text{cis}\left(\dfrac{3\pi}{4}\right)$ **M1**

$r^2 = 1 \therefore r = 1, u = 1\text{cis}\left(\dfrac{3\pi}{4}\right)$

$2\theta = \dfrac{3\pi}{4} + 2k\pi, k = 0, 1$

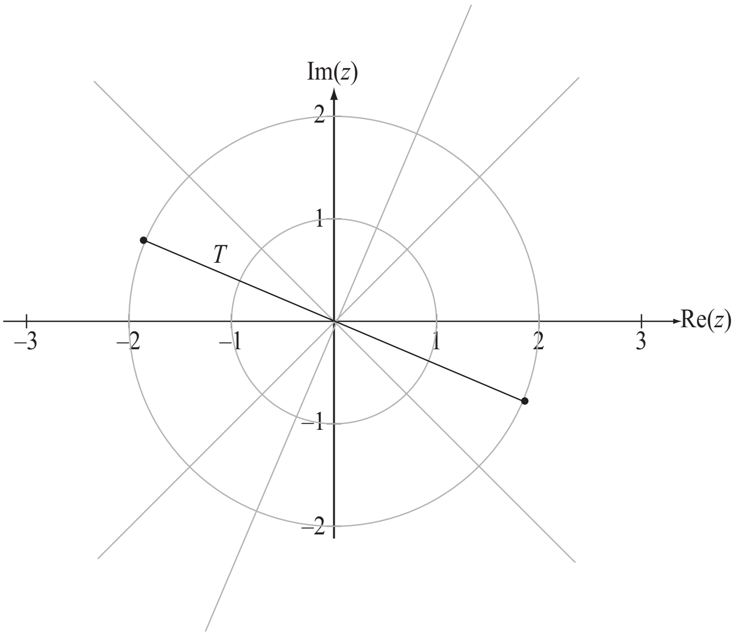
$u = 1\text{cis}\left(\dfrac{3\pi}{8}\right)$ and $v = 1\text{cis}\left(-\dfrac{5\pi}{8}\right)$ **A2**

b.



A2 for u and v in correct positions
A1 for u and v swapped

c.



Region T is the intersection of the inside of the circle centre $(0, 0)$ and radius 2 with the perpendicular bisector of the line segment joining u and v.

A1 perpendicular bisector

A1 inside of the circle

A1 intersection of the two clearly labelled.

d. For what values of n is u^n purely imaginary?

$$u^n = \left[1\text{cis}\left(\frac{3\pi}{8}\right)\right]^n$$
$$= 1\text{cis}\left(\frac{3n\pi}{8}\right)$$

If this is purely imaginary then $\cos\left(\frac{3n\pi}{8}\right) = 0$ **M1**

$\frac{3n\pi}{8} = \frac{(2k-1)\pi}{2}$ where $k \in Z$ or alternatively

$\frac{3n\pi}{8} = \frac{(2k+1)\pi}{2}$ **M1** for equating angle to $\frac{\text{odd } \pi}{2}$

Hence $n = \frac{4}{3}(2k-1)$ where $k \in Z$ **or** $n = \frac{4}{3}(2k+1)$ for any multiple of $\frac{\pi}{2}$ **A1**

Question 2

a. $1 + \log_e x = 0$

$\log_e x = -1$ so $x = e^{-1}$

Coordinates of $A = \left(e^{-1}, 0\right)$ **A1** for $\left(e^{-1}, 0\right)$

b. **i.**
$$f'(x) = \frac{x\left(\frac{1}{x}\right) - 1\left(1 + \log_e(x)\right)}{x^2}$$
$$= \frac{1 - 1 - \log_e(x)}{x^2}$$
A1
$$= \frac{-\log_e(x)}{x^2}$$

ii. $f'(x) = 0$

$\therefore \frac{-\log_e(x)}{x^2} = 0$

$\log_e(x) = 0$

When $x = 1$, $y = 1$

Coordinates of $B = (1, 1)$ **A1** for $(1, 1)$

c.
$$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) + 2x\log_e(x)}{x^4}$$
$$= \frac{-x + 2x\log_e(x)}{x^4}$$
$$= \frac{-1 + 2\log_e(x)}{x^3}$$
$$= 0 \text{ if } x = e^{0.5}$$

A1
M1 for equating second derivative to zero
A1 for both coordinates correct

Coordinates of $C = \left(e^{0.5}, 1.5e^{-0.5}\right)$

d. $\sqrt{e} \approx 1.65$

$f''(1.6) = -0.0146$

$f''(1.7) = 0.0125$

$f''(x)$ changes sign so the concavity changes, hence the point of inflection. **A1**

It should be noted that the first derivative does **not** change sign at $x = 1.65$.

e. Equate the first derivative to the gradient between the points $(0, 0)$ and $\left(\frac{1+\log_e(x)}{x}\right)x$.

$$-\frac{\log_e(x)}{x^2} = \frac{\frac{1+\log_e(x)}{x} - 0}{x - 0}$$

$$-\frac{\log_e(x)}{x^2} = \frac{1+\log_e(x)}{x^2}$$

$$-\log_e(x) = 1 + \log_e(x)$$

$$-2\log_e(x) = 1 \text{ and so } \log_e(x) = -\frac{1}{2}$$

$$\therefore x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

f. $$\int_0^{\frac{1}{\sqrt{e}}} \left(\frac{e}{2}x\right)dx - \int_{\frac{1}{e}}^{\frac{1}{\sqrt{e}}} \left(\frac{1+\log_e(x)}{x}\right)dx$$

(This first integral represents the area under the tangent which passes through the origin.)

M1 the difference of two integrals

A1 correct limits

$$= \left[\frac{ex^2}{4}\right]_0^{\frac{1}{\sqrt{e}}} - \frac{1}{2}\left[(1+\log_e(x))^2\right]_{\frac{1}{e}}^{\frac{1}{\sqrt{e}}}$$ **A1** (correct anti-derivative of the $\frac{1+\log_e(x)}{x}$ function)

$$= 0.25 - \frac{1}{2}\left[(1-0.5)^2 - (1-1)^2\right]$$

$$= \frac{1}{8}$$

Question 3

a. $\overrightarrow{OA} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$, $\overrightarrow{OB} = 3\underset{\sim}{i} + 6\underset{\sim}{j} + \underset{\sim}{k}$ and $\overrightarrow{OC} = -4\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$

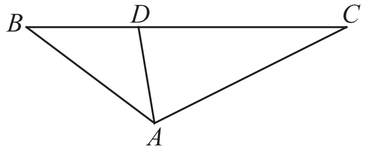
$$\overrightarrow{BC}\cdot\overrightarrow{BA} = \left(\overrightarrow{BO} + \overrightarrow{OC}\right)\cdot\left(\overrightarrow{BO} + \overrightarrow{OA}\right)$$ **M1** dot product

$$= \left(-3\underset{\sim}{i} - 6\underset{\sim}{j} - \underset{\sim}{k} - 4\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}\right)\cdot\left(-3\underset{\sim}{i} - 6\underset{\sim}{j} - \underset{\sim}{k} - \underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

$$= \left(-7\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)\cdot\left(-4\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}\right)$$ **A1** for these vectors

$$= 28 + 6 + 2$$

$$= 36$$



b. $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$ **A1**

$$= -\underset{\sim}{q} + k\underset{\sim}{p}$$

Let D be a point on the side BC and let $\overrightarrow{BC} = \underset{\sim}{p}$, $\overrightarrow{BA} = \underset{\sim}{q}$ *and* $\overrightarrow{BD} = k\underset{\sim}{p}, k \in R$.

c. If $\overrightarrow{AD} \perp \overrightarrow{BC}$ then $\overrightarrow{AD} \bullet \overrightarrow{BC} = 0$.

Find k such that $\left(-\underset{\sim}{q} + k\underset{\sim}{p}\right).\underset{\sim}{p} = 0$

So $-\underset{\sim}{q}.\underset{\sim}{p} + k\underset{\sim}{p}.\underset{\sim}{p} = 0$

$$k = \frac{\underset{\sim}{q}.\underset{\sim}{p}}{\underset{\sim}{p}.\underset{\sim}{p}}$$

$\underset{\sim}{q}.\underset{\sim}{p} = 36$ (from part (a))

$\underset{\sim}{p}.\underset{\sim}{p} = \left(-7\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)\bullet\left(-7\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$ **M1**

$= 57$

So $k = \frac{36}{57} = \frac{12}{19}$ **A1**

d. Find the magnitude of $\overrightarrow{AD}$ with $k = \frac{12}{19}$ for the shortest distance because we need $\overrightarrow{AD} \perp \overrightarrow{BC}$.

$\overrightarrow{AD} = -\underset{\sim}{q} + k\underset{\sim}{p}$ with $k = \frac{12}{19}$ **M1**

$\overrightarrow{AD} = 4\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k} + \frac{12}{19}\left(-7\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}\right)$ **M1**

$= -\frac{8}{19}\underset{\sim}{i} + \frac{33}{19}\underset{\sim}{j} + \frac{5}{19}\underset{\sim}{k}$ **A1**

$$\left|\overrightarrow{AD}\right| = \sqrt{\left(\frac{-8}{19}\right)^2 + \left(\frac{33}{19}\right)^2 + \left(\frac{5}{19}\right)^2}$$

$$= \sqrt{\frac{62}{19}}$$

$\left|\overrightarrow{AD}\right| = 1.81$ to 2 decimal places **A1**

Question 4

a. **i.** Rate of mass going in $= 0.1$ kg/litre $\times 5$ litre/min **M1**

$= 0.5$ kg/min **A1**

Rate of mass going out $= \frac{x}{800}$ kg/litre $\times 5$ litre/min

$= \frac{5x}{800}$ kg/min **A1**

$$\frac{dx}{dt} = 0.5 - \frac{x}{160} = \frac{80 - x}{160}$$

ii. $a = 80$ and $b = 160$ **A1**

iii. $\frac{dt}{dx} = \frac{160}{80 - x}$ **M1** for inversion

$t = \int \frac{160}{80 - x} dx = -160 \log_e |80 - x| + c$ **M1** log recognition

When $t = 0$, $x = 10$ so $c = 160 \log_e 70$

$t = 160 \log_e \left(\frac{70}{80 - x} \right)$ **A1**

$$\frac{80 - x}{70} = e^{-\frac{t}{160}}$$

$80 - 70e^{-\frac{t}{160}} = x$ **A1**

iv. When $t = 12$, $x = 15.058$ kg **A1**

b. Rate of mass going in $= 0.1$ kg/litre $\times 5$ litre/min

$= 0.5$ kg/min

Number of litres in the tank after t minutes $800 + 4t$ **M1** for volume $800 + At$ where A is a constant

Rate of mass going out $= \frac{x}{800 + 4t}$ kg/litre $\times 1$ litre/min

$= \frac{x}{800 + 4t}$ kg/min

$\frac{dx}{dt} = 0.5 - \frac{x}{800 + 4t}$ **A1**

Question 5

a. $\underset{\sim}{a} = 0\underset{\sim}{i} - 9.8\underset{\sim}{j}$

$\underset{\sim}{v} = \int \left(0\underset{\sim}{i} - 9.8\underset{\sim}{j}\right)dt$ **M1**

$= c\underset{\sim}{i} - (9.8t + d)\underset{\sim}{j}$ where c and d are constants.

When $t = 0$, $\underset{\sim}{v} = 5\underset{\sim}{i}$ and so $c = 5$ and $d = 0$.

$\underset{\sim}{r} = \int \left(5\underset{\sim}{i} - 9.8t\underset{\sim}{j}\right)dt$

$= (5t + e)\underset{\sim}{i} - \left(4.9t^2 + f\right)\underset{\sim}{j}$ where e and f are constants.

When $t = 0$, $\underset{\sim}{a} = 0\underset{\sim}{i} + 40\underset{\sim}{j}$ and so $e = 0$ and $f = 40$.

$\underset{\sim}{r} = 5t\underset{\sim}{i} + (40 - 4.9t^2)\underset{\sim}{j}$ as required. **A1**

b. The $\underset{\sim}{j}$ component of $\underset{\sim}{r} = 5t\underset{\sim}{i} + (40 - 4.9t^2)\underset{\sim}{j}$ is zero when the toy touches the ground.

$40 - 4.9t^2 = 0$ so $t = \sqrt{\dfrac{40}{4.9}}$

It reaches the ground after 2.86 seconds. **A1**

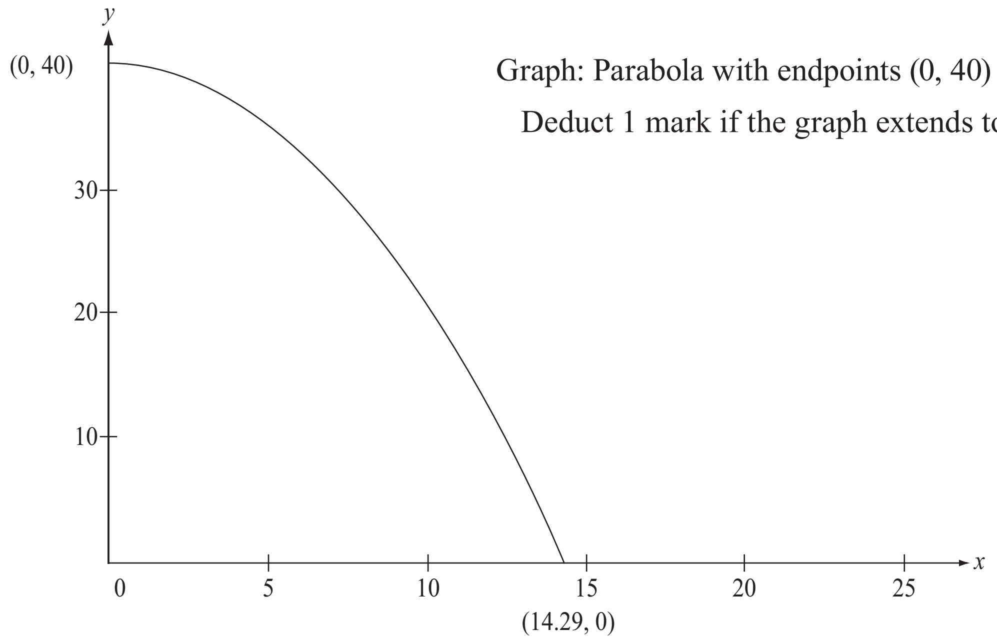
The horizontal distance reached is the $\underset{\sim}{i}$ component: $5t$

It travels 14.29 metres before reaching the ground. **A1**

c. $x = 5t$ so $t = \dfrac{x}{5}$ and $y = 40 - 4.9t^2$ **A1**

Substituting for t into y gives: $y = 40 - 4.9\left(\dfrac{x}{5}\right)^2$

$\therefore y = 40 - \dfrac{49x^2}{250}$ **A1**



Graph: Parabola with endpoints (0, 40) and (14.29, 0) **A2**

Deduct 1 mark if the graph extends to $x < 0$ or $y < 0$

d. $\underset{\sim}{v} = 5\underset{\sim}{i} - 9.8t\underset{\sim}{j}$ and $t = \sqrt{\dfrac{40}{4.9}}$ when the plane hits the ground. **M1**

The magnitude of the velocity at the point of impact with the ground is

$\sqrt{\left(25 + 9.8^2 \times \dfrac{40}{49}\right)}$ which is $\sqrt{809}$. **A1**

Magnitude of momentum $= |mv|$

$= 1.5 \times \sqrt{809}$

$= 42.7$ kg m/s to 3 significant figures. **A1**

e. $\tan(\theta) = \dfrac{40}{x}$

$$\frac{d}{dt}(\tan(\theta)) = \frac{d}{dt}\left(\frac{40}{x}\right)$$

So $\dfrac{d}{d\theta}(\tan(\theta))\dfrac{d\theta}{dt} = \dfrac{d}{dx}\left(\dfrac{40}{x}\right)\dfrac{dx}{dt}$ **M1**

$\therefore \dfrac{d\theta}{dt} = -\dfrac{40}{x^2\sec^2(\theta)}\dfrac{dx}{dt}$ **A1**

f. $\dfrac{dx}{dt} = 5$ as it is moving with a constant speed of 5 m/s.

$$\frac{d\theta}{dt} = -\frac{200\cos^2(\theta)}{x^2}$$

Now $\cos(\theta) = \dfrac{x}{\sqrt{x^2 + 1600}}$ **M1**

When $x = 10$, $\cos(\theta) = \dfrac{1}{\sqrt{17}}$, $\sec^2(\theta) = 17$ and so $\dfrac{d\theta}{dt} = -\dfrac{2}{17}$ **A1**