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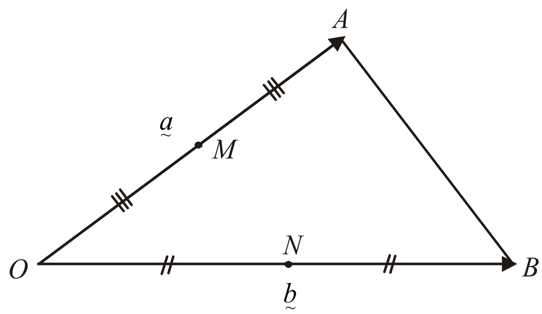
SPECIALIST MATHS TRIAL EXAMINATION 1 SOLUTIONS 2008

Question 1



To prove: $\overrightarrow{MN}$ is parallel to $\overrightarrow{AB}$

That is, $\overrightarrow{MN} = k\,\overrightarrow{AB}$ where k is a constant **(1 mark)**

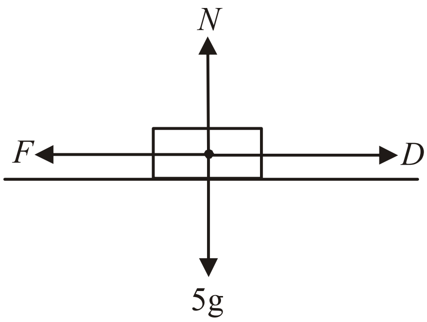
$$\begin{aligned}
LS &= \overrightarrow{MN} \\
&= \overrightarrow{MO} + \overrightarrow{ON} \\
&= -\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} \\
&= \frac{1}{2}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)
\end{aligned}$$

$$\begin{aligned}
RS &= k\,\overrightarrow{AB} \\
&= k\left(\overrightarrow{AO} + \overrightarrow{OB}\right) \\
&= k\left(-\underset{\sim}{a} + \underset{\sim}{b}\right) \\
&= k\left(\underset{\sim}{b} - \underset{\sim}{a}\right) \\
&= LS \text{ where } k = \frac{1}{2}
\end{aligned}$$

Have proved. **(1 mark)**

Question 2

a. Draw a force diagram.



The box is not about to move so,

$D = F, \quad N = 5g \quad \text{and} \quad F < \mu N$ **(1 mark)**

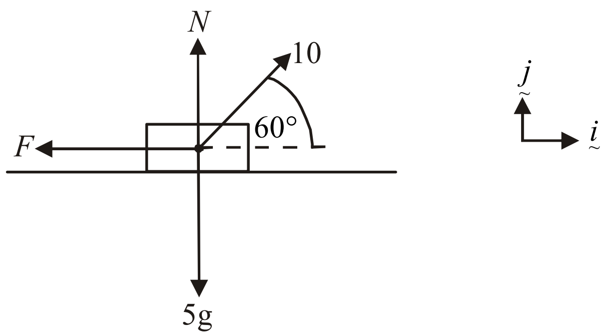
so, $D < 0 \cdot 1 \times 5g$

$< 0 \cdot 5g$

$D < 4.9$ newton

(1 mark)

b.



$$\underset{\sim}{R} = m\underset{\sim}{a}$$

$(10\cos(60^\circ) - F)\underset{\sim}{i} + (N + 10\sin(60^\circ) - 5g)\underset{\sim}{j} = 5a\underset{\sim}{i}$ **(1 mark)**

$5 - F = 5a,$ $\quad N = 5g - 5\sqrt{3}$ and $F = \mu N$

$5 - 0.5g + 0.5\sqrt{3} = 5a$ $\quad = 0 \cdot 1(5g - 5\sqrt{3})$

$a = (1 - 0 \cdot 1g + 0.1\sqrt{3})\text{ms}^{-2}$ $\quad = 0 \cdot 5g - 0.5\sqrt{3}$

or $a = (0.02 + 0.1\sqrt{3})\text{ms}^{-2}$

(1 mark)

Question 3

a. If – 1 is a solution then

$$(-1)^2 + (a-i)\times -1 + b(1-i) = 0$$

$$1 - a + i + b - bi = 0 + 0i \quad \textbf{(1 mark)}$$

$$1 - a + b = 0 \text{ and } i(1-b) = 0i$$

$$1 - b = 0$$

$$b = 1$$

so $2 - a = 0$

$$a = 2$$

So, $a = 2$ and $b = 1$. **(1 mark)**

b. Method 1

$z^2 + (2-i)z + 1 - i = 0$ is a quadratic equation

$$z = \frac{-(2-i) \pm \sqrt{(2-i)^2 - 4\times 1\times (1-i)}}{2} \quad \textbf{(1 mark)}$$

$$= \frac{-2 + i \pm \sqrt{4 - 4i - 1 - 4 + 4i}}{2}$$

$$= \frac{-2 + i \pm \sqrt{-1}}{2}$$

$$= \frac{-2+i+i}{2} \text{ or } \frac{-2+i-i}{2}$$

$$= -1 + i \text{ or } -1$$

The other solution is $-1+i$.

(1 mark)

Method 2

$z = -1$ is a solution so $z + 1$ is a factor.

We have $z^2 + (2-i)z + 1 - i = 0$.

So, by inspection, $(z+1)(z + (1-i)) = 0$ **(1 mark)**

So $z = -1 + i$ is the other solution.

(1 mark)

Question 4

The coefficients of the terms of the equation are real and hence the conjugate root theorem applies.
Since $1-i$ is a solution then its conjugate $1+i$ is also a solution. **(1 mark)**

$$(z-1+i)(z-1-i)$$
$$=z^2-z-iz-z+1+i+iz-i+1$$
$$=z^2-2z+2$$

So, z^2-2z+2 is also a factor .

$$\text{Now, } z^4-4z^3+9z^2-10z+6$$
$$=z^2(z^2-2z+2)-2z(z^2-2z+2)+3(z^2-2z+2)$$
$$=(z^2-2z+2)(z^2-2z+3)$$ **(1 mark)**
$$=(z-1+i)(z-1-i)\{(z^2-2z+1)-1+3\}$$
$$=(z-1+i)(z-1-i)\{(z-1)^2-2i^2\}$$
$$=(z-1+i)(z-1-i)(z-1-\sqrt{2}i)(z-1+\sqrt{2}i)$$

The solutions to $z^4-4z^3+9z^2-10z+6=0$ are $1\pm i$ and $1\pm\sqrt{2}i$.

(1 mark)

Question 5

Using implicit differentiation,

$$2x^2+3xy^2-4y-6=0$$
$$4x+3y^2+3x\times 2y\frac{dy}{dx}-4\frac{dy}{dx}=0$$ **(1 mark)**
$$(6xy-4)\frac{dy}{dx}=-4x-3y^2$$
$$\frac{dy}{dx}=\frac{4x+3y^2}{4-6xy}$$

At the point $(1,2)$,
$$\frac{dy}{dx}=\frac{4+12}{4-12}$$
$$=-2$$

(1 mark)

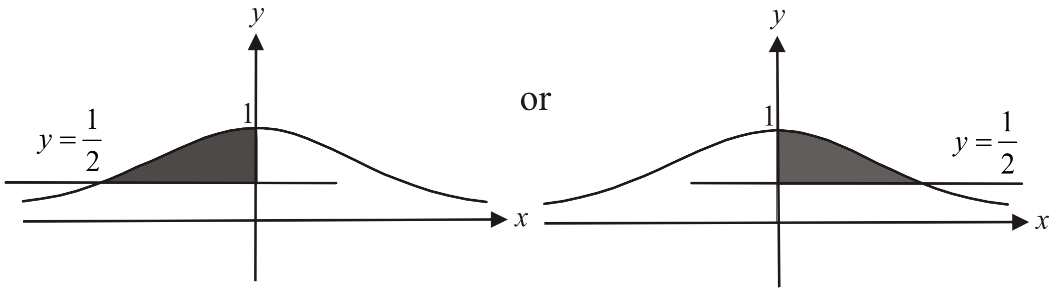
Gradient of tangent is -2 .
Gradient of normal is $\frac{1}{2}$.
Equation of normal is

$$y-2=\frac{1}{2}(x-1)$$
$$y=\frac{x}{2}+\frac{3}{2}$$

(1 mark)

Question 6

Do a quick sketch.



For either of these shaded areas the volume generated will be the same.

$$\text{volume} = \pi\int_{\frac{1}{2}}^{1} x^2\,dy$$

Since $y = \frac{1}{x^2+1}$

$$x^2 + 1 = \frac{1}{y}$$

$$x^2 = \frac{1}{y} - 1$$

$$\text{volume} = \pi\int_{\frac{1}{2}}^{1} x^2\,dy$$ **(1 mark)**

$$= \pi\int_{\frac{1}{2}}^{1}\left(\frac{1}{y} - 1\right)dy$$

$$= \pi\left[\log_e|y| - y\right]_{\frac{1}{2}}^{1}$$ **(1 mark)**

$$= \pi\left\{\left(\log_e(1) - 1\right) - \left(\log_e\left(\frac{1}{2}\right) - \frac{1}{2}\right)\right\}$$

$$= \pi\left(-1 - \log_e\left(\frac{1}{2}\right) + \frac{1}{2}\right)$$

$$= \pi\left(\log_e(2) - \frac{1}{2}\right) \text{ cubic units}$$

(1 mark)

(Note $\log_e(1) = 0$,

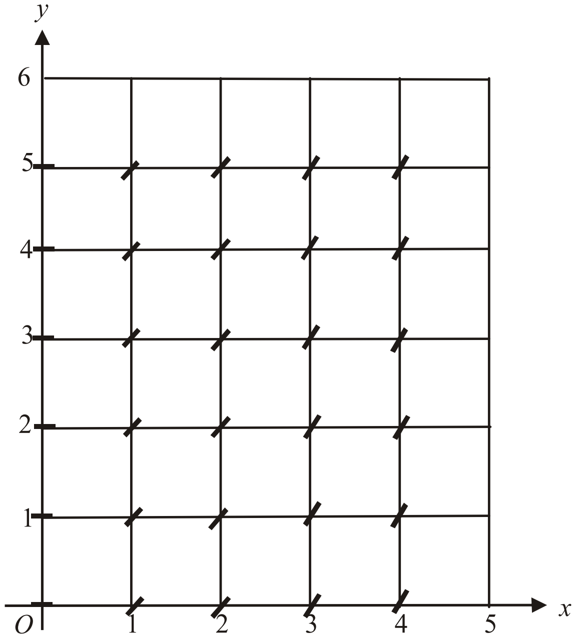
also $-\log_e\left(\frac{1}{2}\right)$

$= -\log_e\left(2^{-1}\right)$

$= \log_e(2)$)

Question 7

a.



(2 marks)

b. $\frac{dy}{dx}=\sqrt{x}, \quad x \geq 0$

$y=\int x^{\frac{1}{2}}dx$

$y=\frac{2x^{\frac{3}{2}}}{3}+c$

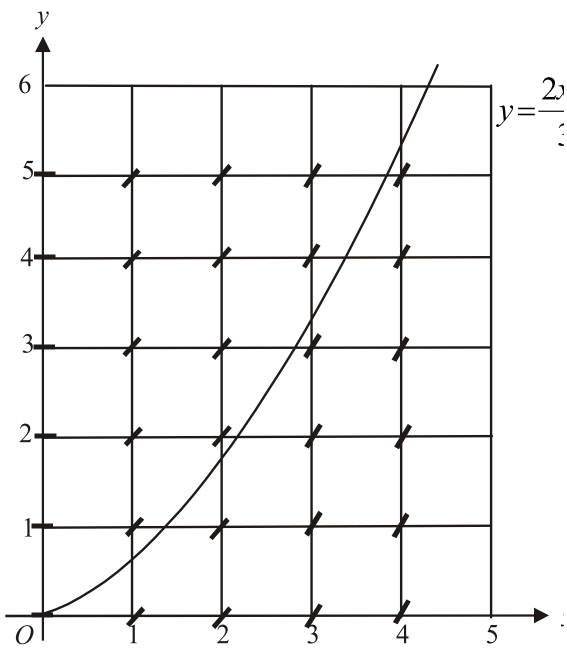
$y=0$ when $x=0$ so $c=0$

$y=\frac{2x^{\frac{3}{2}}}{3}$

(1 mark)

c. The graph of $y=\frac{2x^{\frac{3}{2}}}{3}$ passes through $(0,0)$ which was given in part **b.** as well as passing through $\left(1,\frac{2}{3}\right)$ and $\left(4,\frac{16}{3}\right)$.

It follows the gradient field shown in part **a.**. The graph is shown on the slope field.



(1 mark)

Question 8

a. Method 1

$$\int \frac{x}{3+x}\,dx$$

Let $u = 3 + x$ so $x = u - 3$

$$\frac{du}{dx} = 1$$

$$= \int (u-3)u^{-1}\frac{du}{dx}\,dx$$

$$= \int \left(u^0 - 3u^{-1}\right)du \quad \textbf{(1 mark)}$$

$$= u - 3\log_e|u| + c_1$$

$$= 3 + x - 3\log_e|3+x| + c_1$$

$$= x - 3\log_e|3+x| + c_2 \quad \text{where } c_2 = c_1 + 3$$

(1 mark) – must include | | brackets

Method 2

$$\int \frac{x}{3+x}\,dx$$

$$= \int \frac{x+3-3}{3+x}\,dx$$

$$= \int \left(1 - \frac{3}{3+x}\right)dx \quad \textbf{(1 mark)}$$

$$= x - 3\log_e|3+x| + c$$

(1 mark) – must include | | brackets

b.

$$\int \frac{x+2}{1+x^2}\,dx$$

Let $u = 1 + x^2$

$$\frac{du}{dx} = 2x$$

$$= \int \frac{x}{1+x^2}\,dx + 2\int \frac{1}{1+x^2}\,dx$$

$$= \int \frac{1}{2}\frac{du}{dx}\cdot u^{-1}\,dx + 2\int \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{2}\int u^{-1}\,du + 2\int \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{2}\log|u| + 2\tan^{-1}(x) + c$$

$$= \frac{1}{2}\log_e\left(1+x^2\right) + 2\tan^{-1}(x) \qquad (c = 0 \text{ for "an antiderivative"})$$

(1 mark) first term **(1 mark)** second term

c. Method 1

$$\int_0^4 \frac{-5}{\sqrt{16-x^2}}\,dx$$

$$= -5\int_0^4 \frac{1}{\sqrt{16-x^2}}\,dx$$

$$= -5\left[\operatorname{arsin}\left(\frac{x}{4}\right)\right]_0^4$$ **(1 mark)**

$$= -5\left(\operatorname{arsin}(1) - \operatorname{arsin}(0)\right)$$

$$= -5\left(\frac{\pi}{2} - 0\right)$$

$$= -\frac{5\pi}{2}$$

(1 mark)

Method 2

$$\int_0^4 \frac{-5}{\sqrt{16-x^2}}\,dx$$

$$= 5\int_0^4 \frac{-1}{\sqrt{16-x^2}}\,dx$$

$$= 5\left[\operatorname{arcos}\left(\frac{x}{4}\right)\right]_0^4$$ **(1 mark)**

$$= 5\left(\operatorname{arcos}(1) - \operatorname{arcos}(0)\right)$$

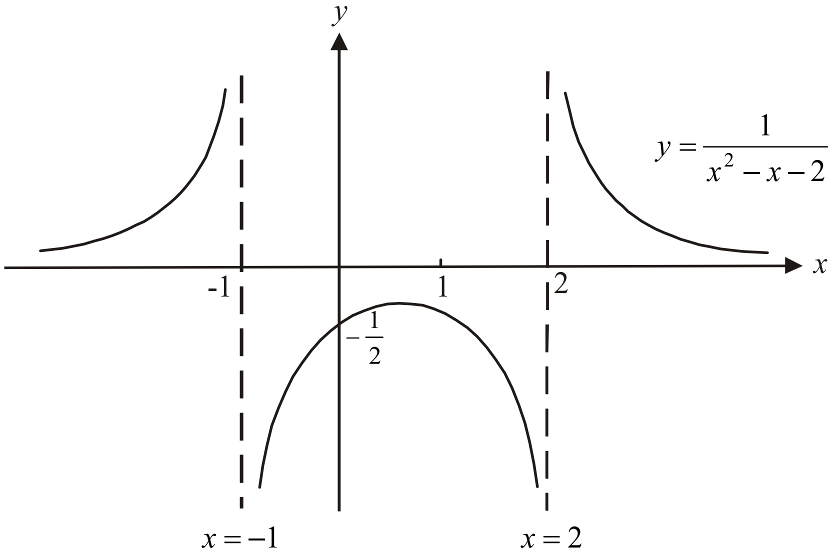
$$= 5\left(0 - \frac{\pi}{2}\right)$$

$$= -\frac{5\pi}{2}$$

(1 mark)

Question 9

a.



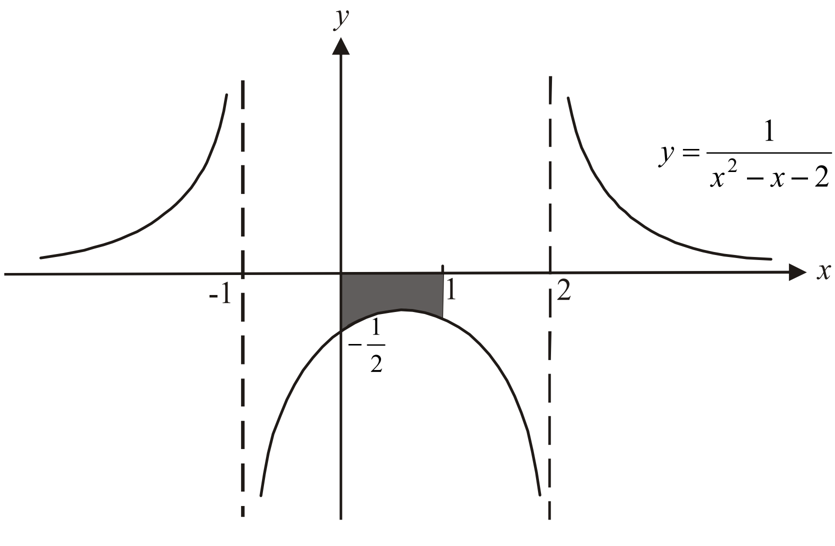
$$y=\frac{1}{x^2-x-2}$$
$$=\frac{1}{(x-2)(x+1)}$$

Asymptotes occur at $x=2,\ x=-1$ and $y=0$.

When $x=0,\ y=-\frac{1}{2}$ so $\left(0,-\frac{1}{2}\right)$ is the only axis intercept (since $y=0$ is an asymptote).

(1 mark) correct asymptotes and axis intercept
(1 mark) correct shape and position of graph

b. The area required is shaded in the graph below.



$$\text{Area} = -\int_0^1 \frac{1}{x^2 - x - 2}\, dx$$ **(1 mark)**

Note that because the region falls below the x-axis it will be negative so we multiply by -1.

Now $\frac{1}{x^2 - x - 2} = \frac{1}{(x-2)(x+1)}$

Let $\frac{1}{(x-2)(x+1)} \equiv \frac{A}{(x-2)} + \frac{B}{(x+1)}$

$$\equiv \frac{A(x+1) + B(x-2)}{(x-2)(x+1)}$$

True iff $1 \equiv A(x+1) + B(x-2)$

Put $x = -1$, $1 = -3B$, $B = -\frac{1}{3}$

Put $x = 2$, $1 = 3A$, $A = \frac{1}{3}$

So $\frac{1}{(x-2)(x+1)} \equiv \frac{1}{3(x-2)} - \frac{1}{3(x+1)}$

$$\text{Area} = -\int_0^1 \left(\frac{1}{3(x-2)} - \frac{1}{3(x+1)} \right) dx$$ **(1 mark)**

$$= -\frac{1}{3}\left[\log_e|x-2| - \log_e|x+1|\right]_0^1$$

$$= -\frac{1}{3}\left[\log_e\left|\frac{x-2}{x+1}\right|\right]_0^1$$ **(1 mark)**

$$= -\frac{1}{3}\left\{\log_e\left|\frac{-1}{2}\right| - \log_e\left|\frac{-2}{1}\right|\right\}$$

$$= -\frac{1}{3}\left(\log_e\left(\frac{1}{2}\right) - \log_e(2)\right)$$

$$= -\frac{1}{3}\left(-2\log_e(2)\right)$$

$$= \frac{2}{3}\log_e(2) \text{ square units.}$$

(1 mark)

Note that $\log_e\left(\frac{1}{2}\right) = \log_e(2)^{-1}$

$$= -\log_e(2)$$

Question 10

a. $\underset{\sim}{v}(t) = \cos(t)\underset{\sim}{i} - 2\sin(2t)\underset{\sim}{j}, \quad \underset{\sim}{r}(0) = \underset{\sim}{j}$

$\underset{\sim}{r}(t) = \sin(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j} + \underset{\sim}{c}$ **(1 mark)**

When $t = 0$

$\underset{\sim}{j} = 0\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{c}$

So $\underset{\sim}{c} = \underset{\sim}{0}$

$\underset{\sim}{r}(t) = \sin(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}$

(1 mark)

b. From **a.** $r(t) = \sin(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}$

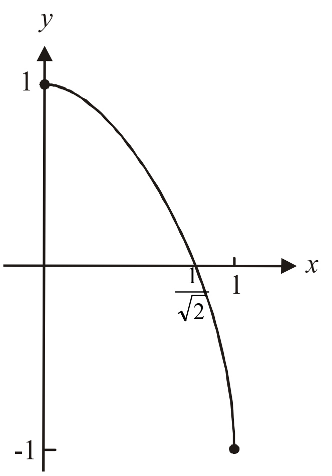
so, $x = \sin(t)$ and $y = \cos(2t)$

$= 1 - 2\sin^2(t)$

$y = 1 - 2x^2$

(1 mark)

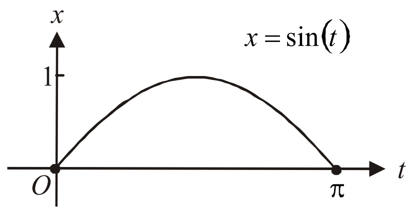
c.



(1 mark) – shape of graph
(1 mark) correct endpoints

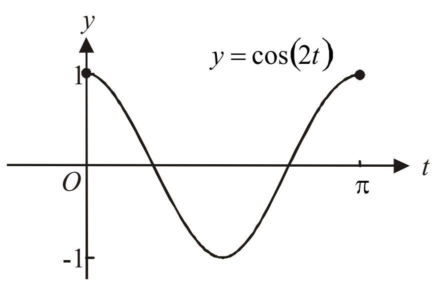
Note the original domain $0 \le t \le \pi$.

Since $x = \sin(t)$ for $0 \le t \le \pi$,
x can only be positive and has a maximum value of 1 and a minimum value of 0.



Similarly $y = \cos(2t)$ for $0 \le t \le \pi$ and so y can be positive and negative.
It has a maximum value of 1 and a minimum value of -1.

The particle travels backwards and forwards along the path shown above.



Total 40 marks