Year 2008

VCE

Specialist Mathematics
Solutions
Trial Examination 2

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SECTION 1

ANSWERS

1	A	B	C	**D**	E
2	A	**B**	C	D	E
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SECTION 1

Question 1 **Answer D**

The *x*-axis is a horizontal asymptote, option **A.** is true.
When $x = 0$ $y = \frac{1}{c}$ as the *y*-intercept, option **B.** is true.
When $-4 - 2x = 0 \Rightarrow x = -2$ is a minimum turning point. option **C.** is true.
The quadratic in the denominator $c - 4x - x^2$ has a discriminant of
$\Delta = (-4)^2 - 4 \times {}^{-}1 \times c = 16 + 4c = 4(4 + c)$ so

If $\Delta > 0$ $c > -4$ the quadratic has two real solutions, and hence $f(x)$ has two vertical asymptotes, so option **D.** is false.

If $\Delta = 0$ $c = -4$ the quadratic has one (repeated) real solution, and hence $f(x)$ has one vertical asymptote **E.** is true.

Question 2 **Answer B**

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

crosses the *x*-axis when $y = 0$ $\frac{(x-h)^2}{a^2} - \frac{(-k)^2}{b^2} = 1 \Rightarrow \frac{(x-h)^2}{a^2} = \frac{k^2}{b^2} + 1 = \frac{k^2 + b^2}{b^2}$

$(x-h)^2 = \frac{a^2(k^2 + b^2)}{b^2} \Rightarrow x = h \pm \frac{a}{b}\sqrt{k^2 + b^2}$ option **A.** is true.

crosses the *y*-axis when $x = 0$ $\frac{(-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \Rightarrow \frac{(y-k)^2}{b^2} = \frac{h^2}{a^2} - 1 = \frac{h^2 - a^2}{a^2}$

$(y-k)^2 = \frac{b^2(h^2 - a^2)}{a^2} \Rightarrow y = k \pm \frac{b}{a}\sqrt{h^2 - a^2}$ option **B.** is false.

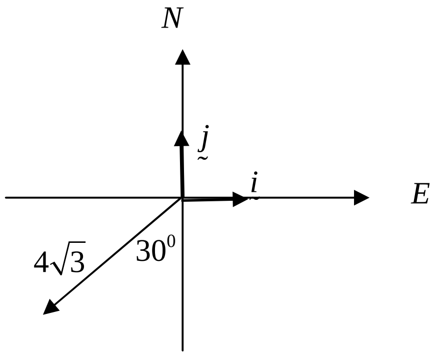
All the other options **C. D.** and **E.** are true

Question 3 **Answer C**

$\underset{\sim}{F} = -4\sqrt{3}\sin(30^0)\underset{\sim}{i} - 4\sqrt{3}\cos(30^0)\underset{\sim}{j}$

$\underset{\sim}{F} = -4\sqrt{3}\left(\frac{1}{2}\underset{\sim}{i} + \frac{\sqrt{3}}{2}\underset{\sim}{j}\right)$

$\underset{\sim}{F} = -2\sqrt{3}\,\underset{\sim}{i} - 6\,\underset{\sim}{j}$



Question 4 **Answer B**

$\underset{\sim}{r}(t) = (\cos(t) - \sin(t))\underset{\sim}{i} + (\cos(t) + \sin(t))\underset{\sim}{j}$

$x = \cos(t) - \sin(t)$ and $y = \cos(t) + \sin(t)$

$\frac{x+y}{2} = \cos(t)$ $\frac{y-x}{2} = \sin(t)$ since $\cos^2(t) + \sin^2(t) = 1$

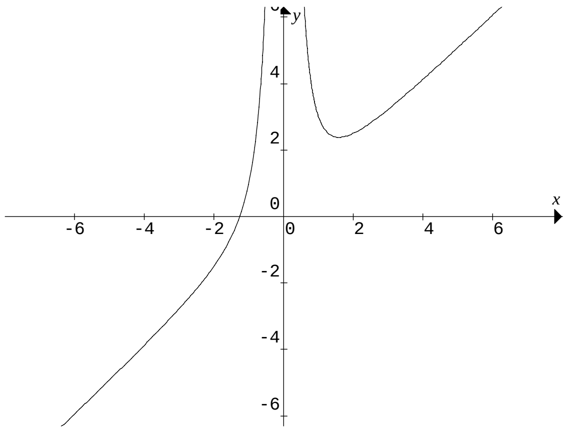
$\left(\frac{x+y}{2}\right)^2 + \left(\frac{y-x}{2}\right)^2 = 1 \quad \Rightarrow \quad x^2 + 2xy + y^2 + y^2 - 2xy + x^2 = 4$

$x^2 + y^2 = 2$ is the Cartesian equation, which is a circle.

Question 5 **Answer A**

$y = \frac{ax^3 + b}{x^2} = ax + \frac{b}{x^2}$

A sketch graph with $a = 1$ $b = 2$
shows a minimum turning point.
All the other options **C. D.** and **E.** are true
If $a < 0$ and $b > 0$ the graph has a minimum turning point. option **B.** is true
The graph has $x = 0$ as a horizontal asymptote, option **C.** is true
The graph has the line $y = ax$ as an oblique asymptote,
option **D.** is true



The graph has an x-intercept when $y = 0$ when $ax^3 + b = 0 \Rightarrow ax^3 = -b$ at $x = \sqrt[3]{-\frac{b}{a}}$,

option **E.** is true.

Question 6 **Answer D**

$z^4 + a^4 i = 0$

$z^4 = -a^4 i = a^4 \operatorname{cis}\left(-\frac{\pi}{2} + 2k\pi\right) \qquad k \in Z$

$z = a \operatorname{cis}\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right)$

so one root when $k = 0$ is $z = a \operatorname{cis}\left(-\frac{\pi}{8}\right)$, which lies on the circle of radius a at an angle of -22.5^0 from the positive real axis, all the other roots must be equally spaced and on the circle of radius a, the roots do not occur in conjugate pairs, so option **D.** is correct.

Question 7 **Answer B**

let $z = a + bi$ $\quad \text{Arg}(z) = \theta$

$a > 0 \quad b > 0 \Rightarrow 0 < \theta < \dfrac{\pi}{2}$

$i^3 z = -i(a+bi)$

$= -ai - bi^2$

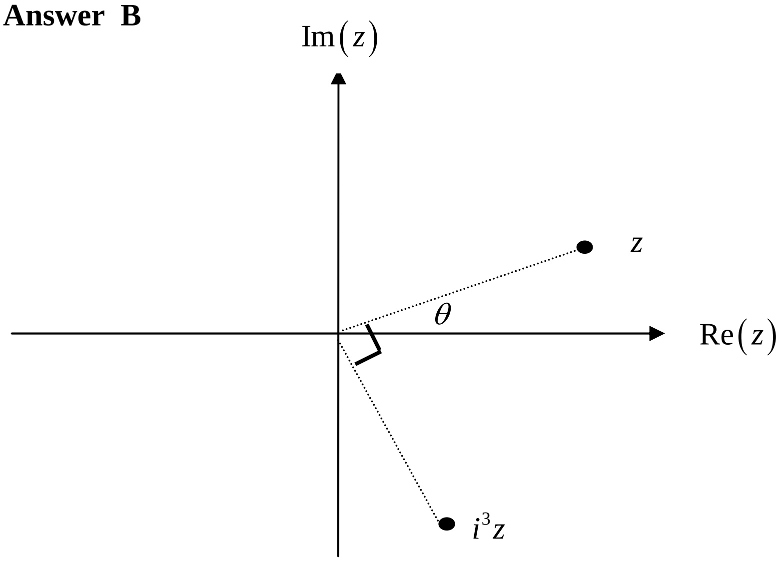
$= b - ai$

Since i is a rotation 90^0 anti-clockwise, i^3 is a rotation 270^0 anti-clockwise

$\arg(b - ai) = \theta + \dfrac{3\pi}{2}$

$\text{Arg}(b - ai) = \theta - \dfrac{\pi}{2}$

Since $\text{Arg}(z)$ must be in $(-\pi, \pi]$



Question 8 **Answer E**

$P(z) = z^3 + bz^2 + cz + d$

Since $P(ki) = 0$ and b, c, d and k are all real, by the conjugate root theorem

$P(-ki) = 0$

option **A.** is true, so $(z - ki)(z - (-ki)) = (z - ki)(z + ki) = z^2 - k^2 i^2 = z^2 + k^2$ is a factor

option **E.** is false

All the other options **B. C.** and **D.** are true, since

$$z^3 + bz^2 + cz + d = (z^2 + k^2)\left(z + \frac{d}{k^2}\right) = z^3 + \frac{d}{k^2} z^2 + k^2 z + d$$

From the coefficient of z: $k^2 = c$ option **B.** is true.

From the coefficient of z^2: $b = \dfrac{d}{k^2} = \dfrac{d}{c} \Rightarrow bc = d$ option **C.** is true.

$\left(z + \dfrac{d}{k^2}\right)$ is a factor of $P(z)$ option **D.** is true.

Question 9 **Answer E**

$$\cos\left(\pi - \cos^{-1}\left(\frac{1}{3}\right)\right) = \cos(\pi)\cos\left(\cos^{-1}\left(\frac{1}{3}\right)\right) - \sin(\pi)\sin\left(\cos^{-1}\left(\frac{1}{3}\right)\right) = -\frac{1}{3}$$

$$\cos\left(\pi - \cos^{-1}\left(\frac{1}{3}\right)\right) = -\frac{1}{3} = \frac{y}{\sqrt{x^2 + y^2 + 4}} \qquad \text{we require } y < 0$$

only $x = \pm 2$ and $y = -1$ satisfy this.

Question 10 **Answer E**

All the other options **A. B. C.** and **D.** are true, option **E.** is false

The graph of $f(x)=\dfrac{ab}{b^2x^2+1}$ has a turning point when

$f'(x)=-\dfrac{2ab^3x}{\left(b^2x^2+1\right)^2}=0 \quad \Rightarrow x=0$ and $f(0)=ab$, the range is $(0,\, ab\,]$

Question 11 **Answer D**

Option **A.** Let $s=2x+9 \quad x=8 \;\; s=25$ and $x=0 \;\; s=9 \quad x=\frac{1}{2}(s-9) \quad dx=\frac{1}{2}ds$

$\int_0^8 \frac{x}{\sqrt{2x+9}}\,dx=\frac{1}{4}\int_9^{25}\frac{s-9}{\sqrt{s}}\,ds$ is true.

Option **B.** Let $t=x \quad \int_0^8 \frac{x}{\sqrt{2x+9}}\,dx=\int_4^8 \frac{t}{\sqrt{2t+9}}\,dt$ is true.

Option **C.** Let $u=2x \quad x=8 \;\; u=16$ and $x=0 \;\; u=0 \quad x=\frac{u}{2} \quad dx=\frac{1}{2}du$

$\int_0^8 \frac{x}{\sqrt{2x+9}}\,dx=\frac{1}{4}\int_0^{16}\frac{u}{\sqrt{u+9}}\,du$ is true.

Option **D.** Let $2v=2x+9 \quad x=8 \;\; v=\frac{25}{2}$ and $x=0 \;\; v=\frac{9}{2} \quad x=\frac{(2v-9)}{2} \quad dx=du$

$\int_0^8 \frac{x}{\sqrt{2x+9}}\,dx\neq\frac{1}{2}\int_9^{25}\frac{2v-9}{\sqrt{2v}}\,dv$ **D.** is false

Option **E.** Let $z^2=2x+9$

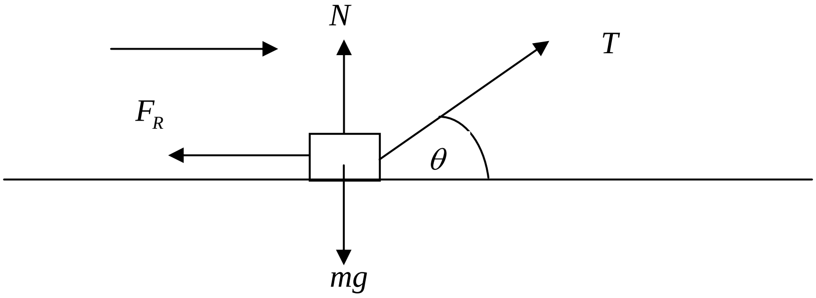
$x=8 \;\; z=5$ and $x=0 \;\; z=3 \quad x=\frac{1}{2}\left(z^2-9\right) \quad 2\,dx=2\,z\,dz$

$\int_0^8 \frac{x}{\sqrt{2x+9}}\,dx=\frac{1}{2}\int_3^5\left(z^2-9\right)dz$ is true.

Question 12 **Answer D**

The period $T=\frac{2\pi}{n}=2a \quad \Rightarrow n=\frac{\pi}{a}$ this is true for graphs of sin, cos, sec and cosec, however the period of tan and cot graphs are $T=\frac{\pi}{n}=2a \quad \Rightarrow n=\frac{\pi}{2a}$, options **A. B. C.** and **E.** are correct, option **D.** is false, its period is a

Question 13 **Answer A**



$T = 10g$ newtons $\theta = 30^0$ $m = 20$ kg $\mu = ?$

resolving parallel to the plane (1) $T\cos(\theta) - F_R = 0$

resolving perpendicular to the plane (2) $T\sin(\theta) + N - mg = 0$

from (1) $F_R = T\cos(\theta) = 10g \times \dfrac{\sqrt{3}}{2} = 5\sqrt{3}\,g$

from (2) $N = mg - T\sin(\theta) = 20g - 10g\sin(30^0) = 15g$

for equilibrium to be maintained, $F_R \le \mu N$ so that $5\sqrt{3}\,g \le 15\mu g$ $\Rightarrow \mu \ge \dfrac{\sqrt{3}}{3}$

Question 14 **Answer A**

resolving horizontally gives

$T_1\sin(46^0) = T_2\sin(44^0) \quad \Rightarrow \quad \dfrac{T_1}{T_2} = \dfrac{\sin(44^0)}{\sin(46^0)} = \dfrac{\sin(44^0)}{\cos(44^0)} = \tan(44^0)$

Question 15 **Answer A**

$\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ **A.** is false

If $x = 9$ and $y = -6$ then vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are collinear is false

If $x = 6$ and $y = -4$ the vector $\underset{\sim}{a}$ is parallel to the vector $\underset{\sim}{b}$, is true, $\underset{\sim}{a} = -2\underset{\sim}{b}$,
option **B.** is true.

If $x = y = \sqrt{5}$ then $|\underset{\sim}{a}| = |\underset{\sim}{b}|$, is true, $|\underset{\sim}{a}| = \sqrt{x^2 + y^2 + 4} = \sqrt{14}$ and $|\underset{\sim}{b}| = \sqrt{9+4+1} = \sqrt{14}$,
option **C.** is true

If $x = -4$ and $y = -5$ the vector $\underset{\sim}{a}$ is perpendicular to the vector $\underset{\sim}{b}$, is true,
since $\underset{\sim}{a}.\underset{\sim}{b} = -3x + 2y - 2 = 12 - 10 - 2 = 0$, option **D.** is true.

If $x = 0$ and $y = 2$ then the scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\underset{\sim}{a}.\hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{a}.\underset{\sim}{b}}{|\underset{\sim}{b}|} = \dfrac{2}{\sqrt{14}}$

Option **E.** is true.

Question 16 **Answer C**

When $x=2$ $y_2=\cos^{-1}(1)=\frac{\pi}{2}$, let $y_1=\cos^{-1}\left(\frac{x}{2}\right)$. The volume required is

$V_x=\pi\int_a^b\left(y_2^2-y_1^2\right)dx$ where y_1 and y_2 are the inner and outer radii respectively,

$$V=\pi\int_0^2\left(\frac{\pi^2}{4}-\left(\cos^{-1}\left(\frac{x}{2}\right)\right)^2\right)dx$$

Question 17 **Answer A**

$\frac{dQ}{dt}=\text{inflow}-\text{outflow}=bg-\frac{fQ}{V}$ and the volume $V=V(t)=V_0+(g-f)t$

$\frac{dQ}{dt}=bg-\frac{fQ}{V_0+(g-f)t}$ $\quad Q(0)=a$

Question 18 **Answer C**

Using constant acceleration formulae $a=-9.8$ $u=0$ $t=8$ $s=?$

$s=ut+\frac{1}{2}at^2$ $\qquad s=0-\frac{1}{2}\times 9.8\times 8^2=-313.6$

The height h is 313 metres. The stone hits the ground 314 metres below the initial point.

Question 19 **Answer C**

$\dot{\underset{\sim}{r}}(t)=4\sin(2t)\underset{\sim}{i}+6e^{-\frac{t}{3}}\underset{\sim}{j}$ integrating

$\underset{\sim}{r}(t)=\int 4\sin(2t)\,dt\,\underset{\sim}{i}+\int 6e^{-\frac{t}{3}}\,dt\,\underset{\sim}{j}$

$\underset{\sim}{r}(t)=-2\cos(2t)\underset{\sim}{i}-18e^{-\frac{t}{3}}\underset{\sim}{j}+\underset{\sim}{c}$ to find $\underset{\sim}{c}$

$\underset{\sim}{r}(0)=\underset{\sim}{0}=-2i-18\underset{\sim}{j}+\underset{\sim}{c}$ $\quad\Rightarrow \underset{\sim}{c}=2\underset{\sim}{i}+18\underset{\sim}{j}$

$\underset{\sim}{r}(t)=2(1-\cos(2t))\underset{\sim}{i}+18\left(1-e^{-\frac{t}{3}}\right)\underset{\sim}{j}$ since $\sin^2(A)=\frac{1}{2}(1-\cos(2A))$

$\underset{\sim}{r}(t)=4\sin^2(t)\underset{\sim}{i}+18\left(1-e^{-\frac{t}{3}}\right)\underset{\sim}{j}$

Question 20 **Answer E**

$$\frac{dx}{dt} = e^{t^2}$$

$x = \int_0^t e^{u^2}\,du + C$ now to find C, $x = 1$ when $t = 0$,

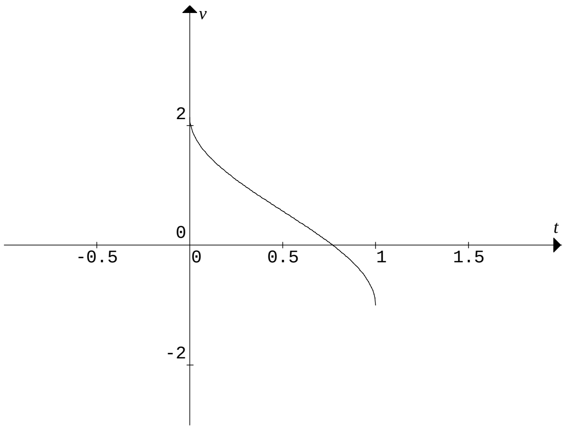
$1 = \int_0^0 e^{u^2}\,du + C \quad \Rightarrow C = 1$

$x = \int_0^t e^{u^2}\,du + 1$ now when $t = 2$ $x = \int_0^2 e^{u^2}\,du + 1$

Question 21 **Answer D**

From the velocity time graph, using a calculator, the graph crosses the *t*-axis at $t = 0.77$.
The distance travelled over $t \in [0,1]$ is the total area bounded by the graph and the *t*-axes. The area from $t = 0.77$ to $t = 1$ is below the axes and negative, the total area, or distance travelled is

$$\int_0^{0.77}\left(\cos^{-1}(2t-1)-1\right)dt - \int_{0.77}^{1}\left(\cos^{-1}(2t-1)-1\right)dt$$

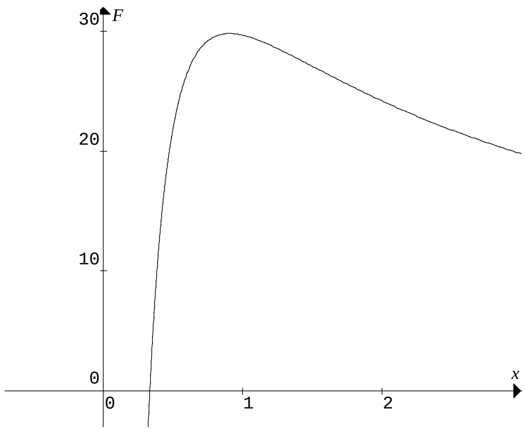


Question 22 **Answer B**

$m = 3$ kg $\quad v = 3\log_e(3x) \quad \Rightarrow \frac{dv}{dx} = \frac{3}{x}$

$F = ma = mv\frac{dv}{dx} = 3\left(3\log_e(3x)\right)\frac{3}{x} = \frac{27\log_e(3x)}{x}$

Using a calculator, this graph has a maximum at $x = 0.87 \quad F_{\max} = 29.6$,
the maximum force is closest to 30.



END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a. $u = 1 + 3i \quad \bar{u} = 1 - 3i \qquad u + \bar{u} = 2 \qquad u\bar{u} = 1 - 9i^2 = 10$

so that $z^2 - 2z + 10$ is a factor of $z^3 - 4z^2 + bz + c = 0$ M1

$z^3 - 4z^2 + bz + c = \left(z^2 - 2z + 10\right)\left(z + d\right)$

coefficient of z^2: $d - 2 = -4 \qquad \Rightarrow d = -2$

coefficient of z^1: $10 - 2d = b \qquad \Rightarrow b = 14$ A1

coefficient of z^0: $10d = c \qquad \Rightarrow c = -20$ A1

b. $u = 1 + 3i \quad iu = i + 3i^2 = -3 + i$

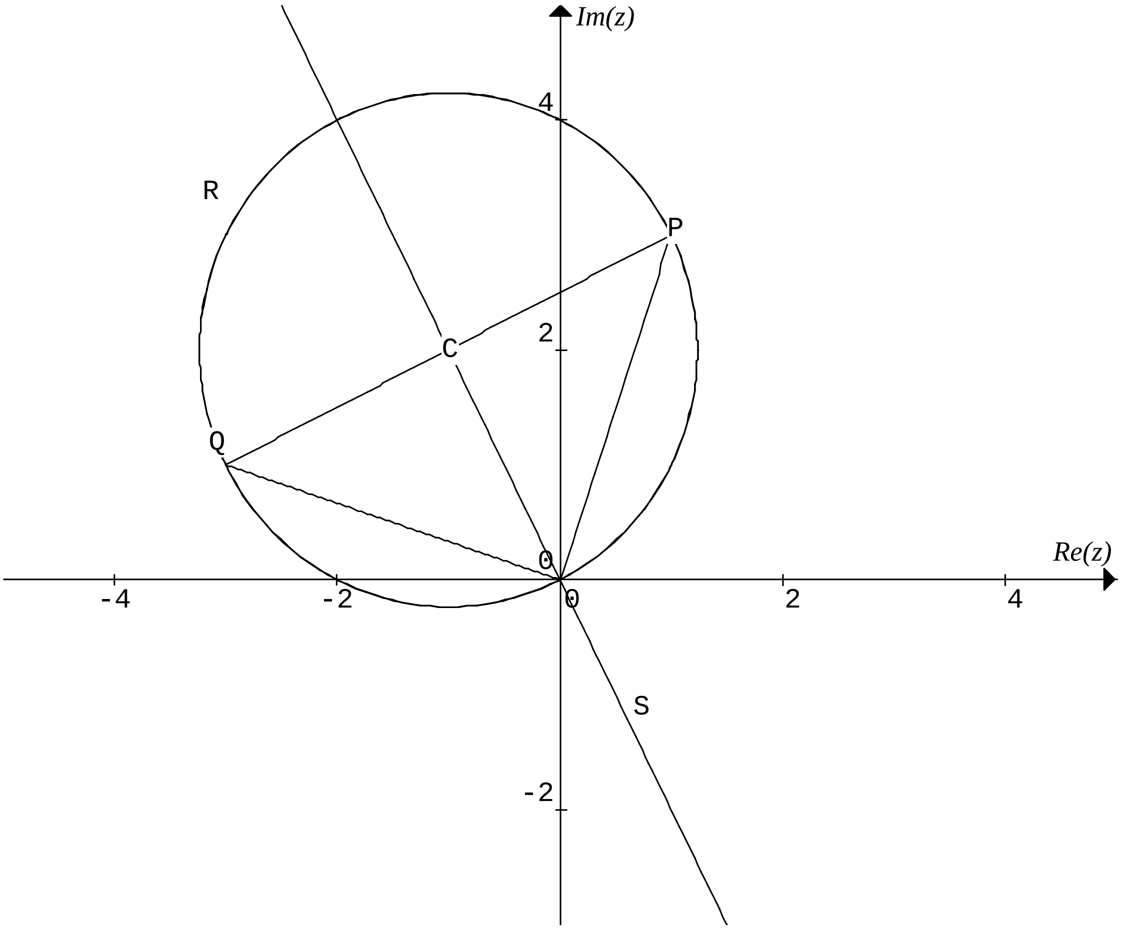


diagram A1

c. $\overrightarrow{OP} = \underset{\sim}{i} + 3\underset{\sim}{j} \qquad \overrightarrow{OQ} = -3\underset{\sim}{i} + \underset{\sim}{j}$

$\overrightarrow{OP}.\overrightarrow{OQ} = -3 + 3 = 0$ so that $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{OQ}$ A1

d. $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -4\underset{\sim}{i} - 2\underset{\sim}{j} \qquad \left|\overrightarrow{PQ}\right| = \sqrt{16 + 4} = 2\sqrt{5}$ A1

e. $p = 1 + 3i \quad q = -3 + i$

$S = \{z : |z - p| = |z - q|\}$

$|z - (1 + 3i)| = |z - (-3 + i)|$ let $z = x + yi$ M1

$\sqrt{(x-1)^2 + (y-3)^2} = \sqrt{(x+3)^2 + (y-1)^2}$ square both sides and expand

$x^2 - 2x + 1 + y^2 - 6y + 9 = x^2 + 6x + 9 + y^2 - 2y + 9$

$y = -2x$ A1

f. PQ is the diameter of the circle

the centre of the circle is at $C(-1, 2)$ the mid-point of PQ,

so that $c = -1 + 2i$ A1

and the radius of the circle is $r = \frac{1}{2}\left|\overrightarrow{PQ}\right| = |c| = \sqrt{5}$ A1

$R = \{z : |z - c| = r\}$

$|z - (-1 + 2i)| = \sqrt{5}$ let $z = x + yi$

$\sqrt{(x+1)^2 + (y-2)^2} = \sqrt{5}$ square both sides and expand

$x^2 + 2x + 1 + y^2 - 4y + 4 = 5$

$x^2 + 2x + y^2 - 4y = 0$ or $(x+1)^2 + (y-2)^2 = 5$ A1

Question 2

a. $O(0,0)$, $A(a,b)$, $B(-a,b)$, $D(0,-r)$

$\overrightarrow{OA} = a\underset{\sim}{i} + b\underset{\sim}{j} \qquad \overrightarrow{OB} = -a\underset{\sim}{i} + b\underset{\sim}{j}$

$\left|\overrightarrow{OA}\right| = \sqrt{a^2 + b^2}$ and $\left|\overrightarrow{OB}\right| = \sqrt{(-a)^2 + b^2} = \sqrt{a^2 + b^2}$

since $\left|\overrightarrow{OB}\right| = \left|\overrightarrow{OA}\right| = \sqrt{a^2 + b^2} = r$ both are radii of the circle A1

$\cos(\theta) = \frac{\overrightarrow{OA}.\overrightarrow{OB}}{\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|} = \frac{-a^2 + b^2}{r^2}$ A1

$\cos(\theta) = \frac{b^2 - a^2}{a^2 + b^2}$ A1

b. $\overrightarrow{OD} = -r\underset{\sim}{j} \qquad \left|\overrightarrow{OD}\right| = r$

$\overrightarrow{DB} = \overrightarrow{DO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OD} = -a\underset{\sim}{i} + (b + r)\underset{\sim}{j} \qquad \left|\overrightarrow{DB}\right| = \sqrt{a^2 + (b+r)^2}$

$\overrightarrow{DA} = \overrightarrow{DO} + \overrightarrow{OA} = \overrightarrow{OA} - \overrightarrow{OD} = a\underset{\sim}{i} + (b + r)\underset{\sim}{j} \qquad \left|\overrightarrow{DA}\right| = \sqrt{a^2 + (b+r)^2}$ A1

$$\cos(\alpha)=\frac{\overrightarrow{DB}.\overrightarrow{DA}}{\left|\overrightarrow{DB}\right|\left|\overrightarrow{DA}\right|}=\frac{-a^2+(b+r)^2}{a^2+(b+r)^2}$$ M1

$$\cos(\alpha)=\frac{b^2+2br+r^2-a^2}{a^2+b^2+2br+r^2} \quad \text{but} \quad r^2=a^2+b^2$$

$$\cos(\alpha)=\frac{b^2+2b\sqrt{a^2+b^2}+a^2+b^2-a^2}{a^2+b^2+2b\sqrt{a^2+b^2}+a^2+b^2}$$ M1

$$\cos(\alpha)=\frac{2b^2+2b\sqrt{a^2+b^2}}{2a^2+2b^2+2b\sqrt{a^2+b^2}}$$

$$\cos(\alpha)=\frac{2b\left(b+\sqrt{a^2+b^2}\right)}{2\sqrt{a^2+b^2}\left(b+\sqrt{a^2+b^2}\right)}$$

$$\cos(\alpha)=\frac{b}{\sqrt{a^2+b^2}}$$ A1

c. to show that $2\alpha=\theta$ or $\cos(2\alpha)=\cos(\theta)$

$$\cos(2\alpha)=2\cos^2(\alpha)-1$$

$$\cos(2\alpha)=2\left(\frac{b}{\sqrt{a^2+b^2}}\right)^2-1$$ M1

$$\cos(2\alpha)=\frac{2b^2}{a^2+b^2}-1$$

$$\cos(2\alpha)=\frac{2b^2-\left(a^2+b^2\right)}{a^2+b^2}$$ M1

$$\cos(2\alpha)=\frac{b^2-a^2}{a^2+b^2}=\cos(\theta) \quad \text{shown}$$ A1

Question 3

a. $\displaystyle\int_{2\sqrt{3}}^{5}\frac{x}{\sqrt{x^2-9}}dx$ let $u=x^2-9$ $\dfrac{du}{dx}=2x$

terminals when $x=5$ $u=16$ and $x=2\sqrt{3}$ $u=3$ M1

$$\int_{2\sqrt{3}}^{5}\frac{x}{\sqrt{x^2-9}}dx=\frac{1}{2}\int_{3}^{16}u^{-\frac{1}{2}}du=\left[\sqrt{u}\right]_3^{16}=4-\sqrt{3}$$ A1

b. $x = 3\cos(t) \qquad dx = -3\sin(t)\,dt$

$x^2 = 9\cos^2(t)$ terminals

when $x = 3$ $\cos(t) = 1$ $t = 0$ and when $x = 0$ $\cos(t) = 0$ $t = \frac{\pi}{2}$ M1

$\sqrt{9 - x^2} = \sqrt{9 - 9\cos^2(t)} = \sqrt{9(1 - \cos^2(t))} = |3\sin(t)| = 3\sin(t)$ since $0 \le t \le \frac{\pi}{2}$

$\int_0^3 \sqrt{9 - x^2}\,dx = -9\int_{\frac{\pi}{2}}^{0} \sin^2(t)\,dt = \frac{9}{2}\int_0^{\frac{\pi}{2}} (1 - \cos(2t))\,dt$ A1

$= \frac{9}{2}\left[t - \frac{1}{2}\sin(2t)\right]_0^{\frac{\pi}{2}} = \frac{9}{2}\left[\left(\frac{\pi}{2} - \frac{1}{2}\sin(\pi)\right) - \left(0 - \frac{1}{2}\sin(0)\right)\right]$ A1

$= \frac{9\pi}{4}$

c. $\frac{x^2}{9} + \frac{y^2}{4} = 1$ A1

d. $LHS = \frac{9}{x^2} + \frac{4}{y^2} = \frac{9}{9\sec^2\theta} + \frac{4}{4\text{cosec}^2(\theta)} = \frac{1}{\sec^2(\theta)} + \frac{1}{\text{cosec}^2(\theta)}$

$= \cos^2(\theta) + \sin^2(\theta) = 1 = RHS$ A1

e. when $y = 4$ $\qquad \frac{9}{x^2} + \frac{4}{16} = 1$

$\frac{9}{x^2} = 1 - \frac{1}{4} = \frac{3}{4} \qquad x^2 = 12 \Rightarrow x = \pm 2\sqrt{3}$ but in the north-west $x < 0$

so that $x = -2\sqrt{3}$ the point is $P\left(-2\sqrt{3}, 4\right)$ A1

$9x^{-2} + 4y^{-2} = 1$ using implicit differentiation

(alternate methods are to transpose and use the chain rule to find $\frac{dy}{dx}$

$-18x^{-3} - 8y^{-3}\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = -\frac{18x^{-3}}{8y^{-3}}$ M1

$\frac{dy}{dx} = -\frac{9y^3}{4x^3}$ at $P\left(-2\sqrt{3}, 4\right)$

$\left.\frac{dy}{dx}\right|_{\left(-2\sqrt{3},4\right)} = -\frac{9(4)^3}{4\left(-2\sqrt{3}\right)^3} = 2\sqrt{3}$ A1

f. $\dfrac{dy}{dx}=-\dfrac{9y^3}{4x^3}=-\dfrac{9\text{ x }8\operatorname{cosec}^3(\theta)}{4\text{ x }27\sec^3(\theta)}=-\dfrac{2\cos^3(\theta)}{3\sin^3(\theta)}=-\dfrac{2}{3}\cot^3(\theta)$ M1

$b=-\dfrac{2}{3}$ and $n=3$ A1

There are of course many other methods available to produce these results as in **g.**

g. Since $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ re-arranged becomes $\dfrac{y^2}{4}=1-\dfrac{x^2}{9}=\dfrac{9-x^2}{9}$

$y=\pm\dfrac{2}{3}\sqrt{9-x^2}$ and

since $\dfrac{9}{x^2}+\dfrac{4}{y^2}=1$ re-arranged becomes $\dfrac{4}{y^2}=1-\dfrac{9}{x^2}=\dfrac{x^2-9}{x^2}$

$y=\pm\dfrac{2x}{\sqrt{x^2-9}}$ M1

by the symmetry of the required area region and when $y=4$ $x=\pm 2\sqrt{3}$
the required area is the 4 times area of the rectangle plus the bit of area from $2\sqrt{3}$ to 5 for $y=\dfrac{2x}{\sqrt{x^2-9}}$ minus the area of the elliptical roundabout

$$A=4\left[8\sqrt{3}+\int_{2\sqrt{3}}^{5}\frac{2x}{\sqrt{x^2-9}}\,dx-\frac{2}{3}\int_0^3\sqrt{9-x^2}\,dx\right]$$ A1

h. $A=4\left[8\sqrt{3}+2\left(4-\sqrt{3}\right)-\dfrac{2}{3}\left(\dfrac{9\pi}{4}\right)\right]$ from **a.**

$A=24\sqrt{3}+32-6\pi$ A1

Question 4

a.i. now $P=200,000\,\text{W}$ $R=\dfrac{\sqrt{v}}{2}$ $m=1800\,\text{kg}$

$ma=\dfrac{P}{v}-R$

$1800a=\dfrac{200,000}{v}-\dfrac{\sqrt{v}}{2}$

$a=\dfrac{400,000-v^{\frac{3}{2}}}{3600v}$ A1

ii. Use $a = v\dfrac{dv}{dx}$ $\quad v\dfrac{dv}{dx} = \dfrac{400,000 - v^{\frac{3}{2}}}{3600v}$

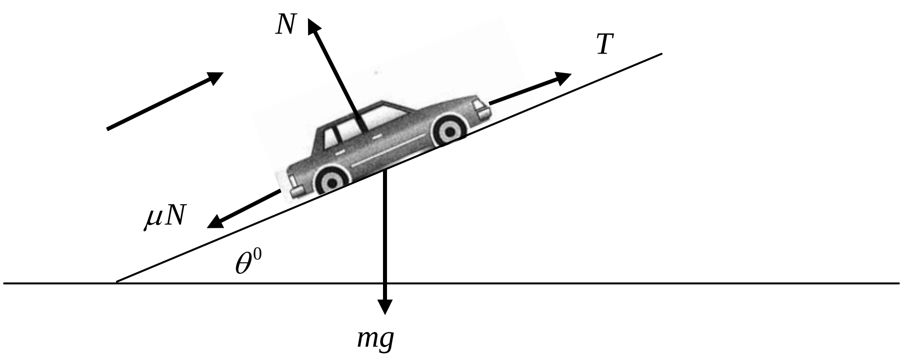
the distance travelled from rest to 10 m/s, is the definite integral

$$x = \int_0^{10} \frac{3600v^2}{400,000 - v^{\frac{3}{2}}}\, dv$$ A1

iii. using a graphics calculator the distance is 3 m A1

b. $m = 1800\,\text{kg}$ $\quad \mu = 0.2$ $\quad g = 9.8$ $\quad \theta = 50^0$

i.



A1

ii. resolving up and parallel to the roadway $(1)\ T - \mu N - mg\sin(\theta) = 0$

resolving perpendicular to the roadway $(2)\ N - mg\cos(\theta) = 0$ A1

to find T we need to eliminate N

from (2) $\quad N = mg\cos(\theta)$ substituting into (1) gives

$T - \mu mg\cos(\theta) - mg\sin(\theta) = 0$

$T = mg(\sin(\theta) + \mu\cos(\theta))$ M1

$T_{\max} = 1800 \text{x} 9.8\left(\sin\left(50^0\right) + 0.2\cos\left(50^0\right)\right) = 15,780.78$ newtons A1

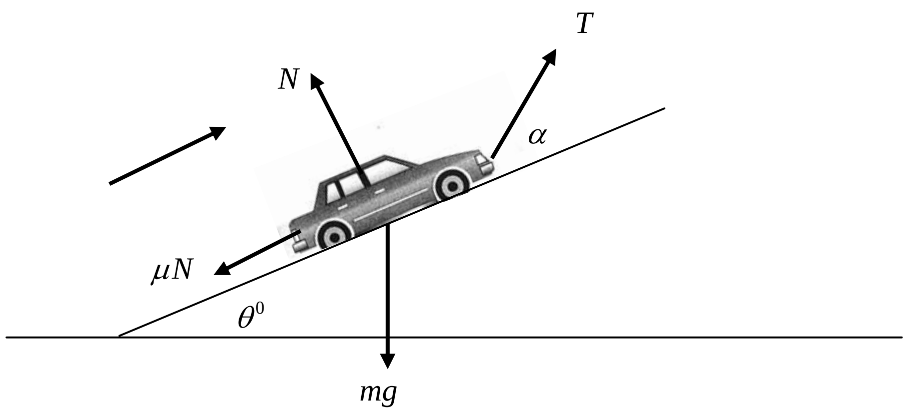
this is the maximum tension, the minimum tension is obtained when the car is on the point of moving down the roadway, reversing the sign of μ so that the frictional force μN is up and parallel to the plane.

$T_{\min} = 1800 \text{x} 9.8\left(\sin\left(50^0\right) - 0.2\cos\left(50^0\right)\right) = 11,245.27$ newtons so

$11,245.27 = T_{\min} \le T \le T_{\max} = 15,780.78$ A1

c. $m = 1800\,\text{kg} \quad \mu = 0.2 \quad g = 9.8 \quad \theta = 50^0 \quad T = 16{,}000$ newtons $\quad a = 0.1\,\text{m/s}^2 \quad \alpha = ?$

i.



A1

ii. resolving up and parallel to the roadway

$(1)\ T\cos(\alpha) - \mu N - mg\sin(\theta) = ma$

resolving perpendicular to the roadway

$(2)\ T\sin(\alpha) + N - mg\cos(\theta) = 0$ M1

to find α we need to eliminate N

from (2) $\quad N = mg\cos(\theta) - T\sin(\alpha)$ substituting into (1) gives

$T\cos(\alpha) - \mu\big(mg\cos(\theta) - T\sin(\alpha)\big) - mg\sin(\theta) = ma$ A1

$T\big(\cos(\alpha) + \mu\sin(\alpha)\big) = ma + mg\big(\sin(\theta) + \mu\cos(\theta)\big)$

$$\cos(\alpha) + \mu\sin(\alpha) = \frac{m\big(a + g\big(\sin(\theta) + \mu\cos(\theta)\big)\big)}{T}$$ A1

$$\cos(\alpha) + 0.2\sin(\alpha) = \frac{1800\big(0.1 + 9.8\big(\sin(50^0) + 0.2\cos(50^0)\big)\big)}{16{,}000} = 0.9975$$

solving on calculator gives $\alpha = 23.3^0$ A1

Question 5

a. $\dot{\underset{\sim}{r}}(t) = 8\underset{\sim}{i} + 26\underset{\sim}{j} + \frac{5\pi\sqrt{2}}{4}\cos\left(\frac{\pi t}{2}\right)\underset{\sim}{k}$ A1

$\dot{\underset{\sim}{r}}(0) = 8\underset{\sim}{i} + 26\underset{\sim}{j} + \frac{5\pi\sqrt{2}}{4}\cos(0)\underset{\sim}{k} = 8\underset{\sim}{i} + 26\underset{\sim}{j} + \frac{5\pi\sqrt{2}}{4}\underset{\sim}{k}$

initial speed is $\left|\dot{\underset{\sim}{r}}(0)\right| = \sqrt{8^2 + 26^2 + \left(\frac{5\pi\sqrt{2}}{4}\right)^2} = 27.76$ m/s A1

b. $\tan(\theta) = \dfrac{\frac{5\pi\sqrt{2}}{4}}{\sqrt{8^2 + 26^2}}$

$\theta = \tan^{-1}(0.2042) = 11.5^0$ A1

c. at maximum height $\sin\left(\frac{\pi t}{2}\right) = 1$ $\frac{\pi t}{2} = \sin^{-1}(1) = \frac{\pi}{2}$ $\Rightarrow t = 1$

the soccer ball takes one second to get to a maximum height

and its position vector at this time is $\underset{\sim}{r}(1) = 8\underset{\sim}{i} + 26\underset{\sim}{j} + \frac{5\sqrt{2}}{2}\underset{\sim}{k}$ A1

d. he heads the ball when $\frac{5\sqrt{2}}{2}\sin\left(\frac{\pi t}{2}\right) = \frac{5}{2}$ or $\sin\left(\frac{\pi t}{2}\right) = \frac{1}{\sqrt{2}}$

$\frac{\pi t}{2} = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}, \frac{3\pi}{4}$ take $\frac{3\pi}{4}$ since the ball has reached its maximum height and is on its downwards trajectory, so that $t = \frac{3}{2} = 1.5$ seconds A1

e. $\underset{\sim}{r}\left(\frac{3}{2}\right) = 12\underset{\sim}{i} + 39\underset{\sim}{j} + \frac{5}{2}\underset{\sim}{k}$ A1

the required distance is $\sqrt{12^2 + 39^2} = 41$ metres A1

f. $\dot{\underset{\sim}{r}}\left(\frac{3}{2}\right) = 8\underset{\sim}{i} + 26\underset{\sim}{j} + \frac{5\pi\sqrt{2}}{4}\cos\left(\frac{3\pi}{4}\right)\underset{\sim}{k} = 8\underset{\sim}{i} + 26\underset{\sim}{j} - \frac{5\pi}{4}\underset{\sim}{k}$ A1

$\left|\dot{\underset{\sim}{r}}\left(\frac{3}{2}\right)\right| = \sqrt{8^2 + 26^2 + \left(-\frac{5\pi}{4}\right)^2} = 27.485$ and $m = 0.43\text{kg}$

magnitude of momentum $\left|\underset{\sim}{p}\right| = 0.430 \times 27.485 = 11.82$ kg m/s A1

END OF SECTION 2 SUGGESTED ANSWERS