The Mathematical Association of Victoria
SPECIALIST MATHEMATICS
2008 Trial written examination 1 – Worked Solutions

Question 1

a. $z = 6\text{cis}\left(\frac{5\pi}{6}\right)$

$z^3 = \left(6\text{cis}\left(\frac{5\pi}{6}\right)\right)^3$

$z^3 = 6^3\text{cis}\left(\frac{5\pi}{6} \times 3\right)$

$z^3 = 216\text{cis}\left(\frac{5\pi}{2}\right)$

$z^3 = 216\left(\cos\left(\frac{5\pi}{2}\right) + i\sin\left(\frac{5\pi}{2}\right)\right)$

$z^3 = 216(0 + i)$

$z^3 = 216i$

$\therefore\ m = 216(0 + i)$ **[A1]**

b. There are three solutions of the equation $z^3 = 216i$. They are equally spaced around the circumference of a circle of radius 6. The angle between each solution is $\frac{2\pi}{3}$.

The remaining two solutions are:

$z = 6\text{cis}\left(\frac{5\pi}{6} - \frac{2\pi}{3}\right)$ and $z = 6\text{cis}\left(\frac{5\pi}{6} + \frac{2\pi}{3}\right)$

$z = 6\text{cis}\left(\frac{\pi}{6}\right)$ $\qquad z = 6\text{cis}\left(-\frac{\pi}{2}\right)$ **[A1]**

Alternative method of solution

$z^3 = 216i$

$z^3 = 216\text{cis}\left(\frac{\pi}{2}\right)$

$z = \left(216\,\text{cis}\left(\frac{\pi}{2} + 2k\pi\right)\right)^{\frac{1}{3}}$ where $k = 0, \pm 1$

$z = 216^{\frac{1}{3}}\text{cis}\frac{1}{3}\left(\frac{\pi}{2} + 2k\pi\right)$ by De Movire's Theorem

$z = 6\text{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$

$k = 0, \qquad z = 6\text{cis}\left(\frac{\pi}{6}\right)$

$k = 1, \qquad z = 6\text{cis}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) = 6\text{cis}\left(\frac{5\pi}{6}\right)$ (solution given in part **a.**)

$k = 1, \qquad z = 6\text{cis}\left(\frac{\pi}{6} - \frac{2\pi}{3}\right) = 6\text{cis}\left(-\frac{\pi}{2}\right)$

The two other solutions are: $6\text{cis}\left(\frac{\pi}{6}\right)$

and $6\text{cis}\left(-\frac{\pi}{2}\right)$

Question 2

a. $u = 1 + i$

$u = r\text{cis}(\theta)$ where $r = \sqrt{1^2 + 1^2} = \sqrt{2}$

and $\theta = \tan\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$u = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ **[A1]**

b. $uv = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right) \times 2\text{cis}\left(-\frac{\pi}{6}\right)$

$uv = 2\sqrt{2}\text{cis}\left(\frac{\pi}{4} + \left(-\frac{\pi}{6}\right)\right)$

$uv = 2\sqrt{2}\text{cis}\left(\frac{\pi}{12}\right)$ **[A1]**

c. $v = 2\text{cis}\left(-\frac{\pi}{6}\right)$

$v = 2\cos\left(-\frac{\pi}{6}\right) + 2\sin\left(-\frac{\pi}{6}\right)i$

$v = \sqrt{3} - i$ **[A1]**

d. $uv = (1 + i)\left(\sqrt{3} - i\right)$

$uv = \sqrt{3} - i + i\sqrt{3} + 1$

$uv = \left(\sqrt{3} + 1\right) + \left(\sqrt{3} - 1\right)i$ **[A1]**

e. From **b.** and **d.**

$uv = 2\sqrt{2}\text{cis}\left(\frac{\pi}{12}\right) = \left(\sqrt{3} + 1\right) + \left(\sqrt{3} - 1\right)i$

$uv = 2\sqrt{2}\cos\left(\frac{\pi}{12}\right) + 2\sqrt{2}\sin\left(\frac{\pi}{12}\right)i = \left(\sqrt{3} + 1\right) + \left(\sqrt{3} - 1\right)i$

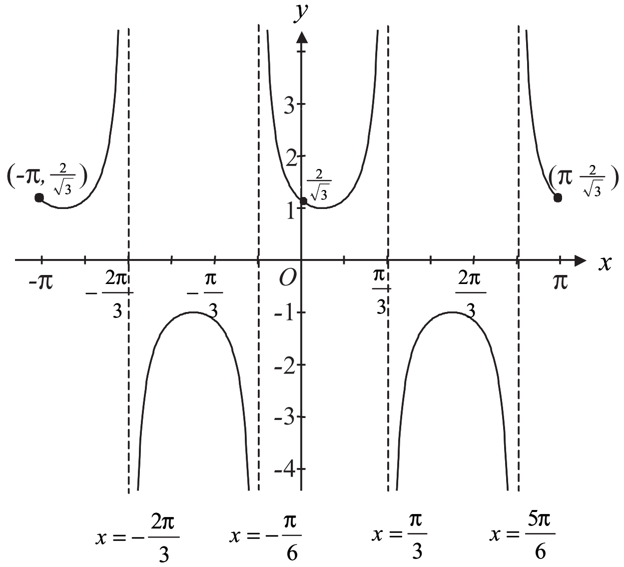
Equating imaginary components:

$2\sqrt{2}\sin\left(\frac{\pi}{12}\right) = \left(\sqrt{3} - 1\right)$

$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3} - 1}{2\sqrt{2}}$ **[A1]**

$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$ or $\frac{1}{4}\left(\sqrt{6} - \sqrt{2}\right)$ (rationalised)

Question 3



Shape **[A1]**
Asymptotes **[A1]**
Endpoints **[A1]**

Endpoint coordinates:

$x = \pi$ $\quad f(\pi) = \text{cosec}\left(2\pi + \frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}}$ $\quad \left(\pi, \frac{2}{\sqrt{3}}\right)$

$\therefore x = -\pi,$ $\quad f(-\pi) = \frac{2}{\sqrt{3}}$ $\quad \left(-\pi, \frac{2}{\sqrt{3}}\right)$

y-intercept:

$x = 0,$ $\quad f(0) = \text{cosec}\left(0 + \frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}}$ $\quad \left(0, \frac{2}{\sqrt{3}}\right)$

Asymptotes:

$$2x + \frac{\pi}{3} = -\pi, \quad 0, \quad \pi, \quad 2\pi$$

$$2x = -\frac{4\pi}{3}, \quad -\frac{\pi}{3}, \quad \frac{2\pi}{3}, \quad \frac{5\pi}{3}$$

$$x = -\frac{2\pi}{3}, \quad -\frac{\pi}{6}, \quad \frac{\pi}{3}, \quad \frac{5\pi}{6}$$

Question 4

Using implicit differentiation:

$x\sin(y) = 1$

$1.\sin(y) + x\cos(y)\dfrac{dy}{dx} = 0$ **[M1]**

$$\frac{dy}{dx} = -\frac{\sin(y)}{x\cos(y)}$$

$$\frac{dy}{dx} = -\frac{\tan(y)}{x}$$ **[A1]**

When $y = \dfrac{\pi}{6}$, $x = \dfrac{1}{\sin\left(\dfrac{\pi}{6}\right)} = 2$

$$\frac{dy}{dx} = -\frac{\tan\left(\frac{\pi}{6}\right)}{2} = -\frac{\frac{1}{\sqrt{3}}}{2}$$

$$\frac{dy}{dx} = -\frac{\sqrt{3}}{6}$$ **[A1]**

Question 5

a. Finding x-intercepts:

$$0 = x - 2\sqrt{\frac{3}{8-x^2}}$$

$$2\sqrt{\frac{3}{8-x^2}} = x$$

$$4\left(\frac{3}{8-x^2}\right) = x^2$$ **[M1]**

$$12 = x^2(8 - x^2)$$

$$x^4 - 8x^2 + 12 = 0$$

$$(x^2 - 2)(x^2 - 6) = 0$$ **[A1]**

$x = \pm\sqrt{2}$ or $x = \pm\sqrt{6}$

$\therefore\ m = \sqrt{2}$ and $n = \sqrt{6}$

b. Shaded area = $\displaystyle\int_{\sqrt{2}}^{\sqrt{6}} \left(x - 2\sqrt{\frac{3}{8-x^2}}\right) dx$

$$\int_{\sqrt{2}}^{\sqrt{6}} x\, dx - 2\sqrt{3}\int_{\sqrt{2}}^{\sqrt{6}} \frac{1}{\sqrt{8-x^2}}\, dx$$

$$= \left[\frac{1}{2}x^2 - 2\sqrt{3}\sin^{-1}\left(\frac{x}{\sqrt{8}}\right)\right]_{\sqrt{2}}^{\sqrt{6}}$$ **[A1]**

$$= \left[\frac{1}{2}\left(\sqrt{6}\right)^2 - 2\sqrt{3}\sin^{-1}\left(\frac{\sqrt{6}}{\sqrt{8}}\right)\right] - \left[\frac{1}{2}\left(\sqrt{2}\right)^2 - 2\sqrt{3}\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{8}}\right)\right]$$

$$= \left[3 - 2\sqrt{3}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] - \left[1 - 2\sqrt{3}\sin^{-1}\left(\frac{1}{2}\right)\right]$$ **[A1]**

$$= \left[3 - 2\sqrt{3}\times\frac{\pi}{3}\right] - \left[1 - 2\sqrt{3}\times\frac{\pi}{6}\right]$$

$= 2 - \dfrac{\pi\sqrt{3}}{3}$ square units **[A1]**

Question 6

$y = \int \frac{1+x}{(1-x)^2}\,dx$

Let $u = 1 - x$ $\quad x = 1 - u$

$\frac{du}{dx} = -1$ $\quad 1 + x = 2 - u$ **[A1]**

$dx = -du$

$y = \int \frac{2-u}{u^2}(-du)$ **[A1]**

$y = \int \left(\frac{1}{u} - \frac{2}{u^2}\right) du$

$y = \log_e|u| + \frac{2}{u} + c$

$y = \log_e|1-x| + \frac{2}{1-x} + c, \quad x \neq 1$ **[A1]**

When $y = 0, x = 0$

$0 = \log_e|1-0| + \frac{2}{1-0} + c$

$c = -2$

$\therefore y = \log_e|1-x| + \frac{2}{1-x} - 2, \quad x \neq 1$ **[A1]**

Since we are dealing with the part of the function where $x = 0$, we only need consider $x < 1$.

Hence solution is $y = \log_e(1-x) + \frac{2}{1-x} - 2, \quad x \neq 1$

Alternative method of solution using partial fractions:

$\frac{1+x}{(1-x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2}, \quad x \neq 1$

$1 + x = A(1-x) + B$

Let $x = 1, B = 2$

Let $x = 0, A + B = 1$ and so $A = -1$

$\int \frac{1+x}{(1-x)^2}dx = \int \left(\frac{-1}{1-x} + \frac{2}{(1-x)^2}\right)dx$

$= \log_e|1-x| + \frac{2}{1-x} - \mathrm{c}, \quad x \neq 1$

Continued as shown above to find constant, c, etc.

Question 7

Let $\underset{\sim}{u}$ be the vector resolute of $\underset{\sim}{a}$ parallel to $\underset{\sim}{b}$.

$$\underset{\sim}{u} = \left(\underset{\sim}{a}.\hat{\underset{\sim}{b}}\right).\hat{\underset{\sim}{b}} \quad \text{where } \hat{\underset{\sim}{b}} = \frac{\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)}{\sqrt{(-1)^2 + 2^2 + (-1)^2}}$$

$$= \frac{1}{\sqrt{6}}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$$

$$\underset{\sim}{u} = \left(\left(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right).\frac{1}{\sqrt{6}}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)\right)\frac{1}{\sqrt{6}}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$$

[A1]

$$\underset{\sim}{u} = \frac{1}{6}(-1 - 2 - 1)\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$$

$$\underset{\sim}{u} = -\frac{4}{6}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$$

$$\underset{\sim}{u} = \frac{2}{3}\left(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$$ **[A1]**

Let $\underset{\sim}{v}$ be the vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$.

$$\underset{\sim}{v} = \underset{\sim}{a} - \underset{\sim}{u}$$

$$\underset{\sim}{v} = \left(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) - \frac{2}{3}\left(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$$

$$\underset{\sim}{v} = \frac{1}{3}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j} + \frac{1}{3}\underset{\sim}{k}$$

$$\underset{\sim}{v} = \frac{1}{3}\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$$ **[A1]**

Question 8

Since $OABC$ is a parallelogram, $\overrightarrow{CB} = \underset{\sim}{a}$ and $\overrightarrow{CM} = \frac{1}{2}\underset{\sim}{a}$

$$\overrightarrow{OM} = \overrightarrow{OC} + \overrightarrow{CM} = \underset{\sim}{c} + \frac{1}{2}\underset{\sim}{a}$$

$$\overrightarrow{OP} = \frac{2}{3}\overrightarrow{OM} = \frac{2}{3}\left(\underset{\sim}{c} + \frac{1}{2}\underset{\sim}{a}\right) = \frac{2}{3}\underset{\sim}{c} + \frac{1}{3}\underset{\sim}{a}$$ **[M1]**

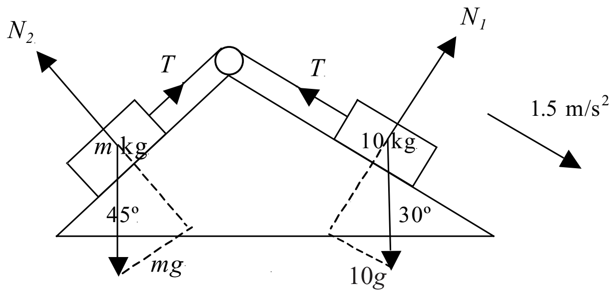
$$\overrightarrow{PC} = \overrightarrow{PO} + \overrightarrow{OC} = -\left(\frac{2}{3}\underset{\sim}{c} + \frac{1}{3}\underset{\sim}{a}\right) + \underset{\sim}{c} = -\frac{1}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{c}$$

[A1]

$$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = -\underset{\sim}{a} + \left(\frac{2}{3}\underset{\sim}{c} + \frac{1}{3}\underset{\sim}{a}\right) = -\frac{2}{3}\underset{\sim}{a} + \frac{2}{3}\underset{\sim}{c} = 2\left(-\frac{1}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{c}\right) = 2\,\overrightarrow{PC}$$ **[A1]**

$\therefore\ \overrightarrow{AP} = 2\,\overrightarrow{PC}$ as required.

Question 9



a. The 10kg mass is moving down the plane.
The component of the weight force parallel to the plane is $10g\sin(30°)$.
Resolving forces parallel to the plane:

$10g\sin(30°) - T = 10a$ **[A1]**

$T = 10 \times 9.8\sin(30°) - 10 \times 1.5$

$T = 34$ newtons **[A1]**

b. The *m* kg mass is moving up the plane.
The component of the weight force parallel to the plane is $mg\sin(45°)$.
Resolving forces parallel to the plane:

$T - mg\sin(45°) = ma$ **[A1]**

$34 - mg \times \frac{\sqrt{2}}{2} = m \times 1.5$

$68 - mg\sqrt{2} = 3m$

$m(g\sqrt{2} + 3) = 68$

$m = \frac{68}{g\sqrt{2}+3}$ **[A1]**

$\therefore a = 68, b = 2, c = 3$

Question 10

a. Differentiating the parametric equations.

$$\frac{dx}{dt} = \frac{1\left(t^2+1\right) - t\left(2t\right)}{\left(t^2+1\right)^2} \qquad \frac{dy}{dt} = \frac{-2t}{\left(t^2+1\right)^2}$$

$$\frac{dx}{dt} = \frac{1-t^2}{\left(t^2+1\right)^2}$$ **[A1]**

Finding $\frac{dy}{dx}$ using the chain rule.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{-2t}{\left(t^2+1\right)^2} \times \frac{\left(t^2+1\right)^2}{1-t^2}$$ **[M1]**

$$\frac{dy}{dx} = \frac{2t}{t^2-1}$$

When $t = 2$, $\frac{dy}{dx} = \frac{2\times 2}{2^2-1} = \frac{4}{3}$ **[A1]**

b. Finding the Cartesian equation by eliminating the parameter.

$x = \frac{t}{t^2+1} \qquad y = \frac{1}{t^2+1}$

$x = t\left(\frac{1}{t^2+1}\right)$

$x = t \times y$

$\frac{x}{y} = t$

$\frac{x^2}{y^2} + 1 = t^2 + 1$ **[M1]**

$\frac{y^2}{x^2+y^2} = \frac{1}{t^2+1}$

$\frac{y^2}{x^2+y^2} = y$

$y = x^2 + y^2$ **[A1]**

$x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4}$

$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

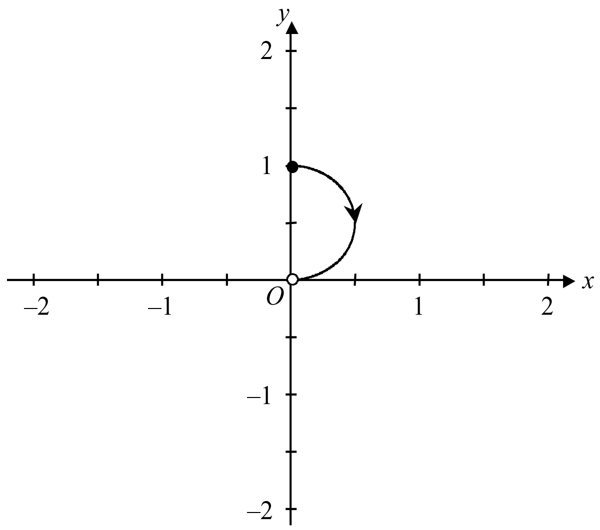
Curve is a circle with centre $\left(0, \frac{1}{2}\right)$ and radius $\frac{1}{2}$

c. Sketching circle centre $\left(0, \frac{1}{2}\right)$ and radius $\frac{1}{2}$ for $t \geq 0$

When $t = 0$ $\quad x = 0,$ $\quad y = 1$

When $t = 1$ $\quad x = \frac{1}{2},$ $\quad y = \frac{1}{2}$

As $t \to \infty,$ $\quad x \to 0,$ $\quad y \to 0$



Shape and position **[A1]**

End point (0, 1) included, end point (0, 0) excluded **[A1]**

Direction of motion not required

END OF SOLUTIONS