The Mathematical Association of Victoria
SPECIALIST MATHEMATICS
2008 Trial written examination 2 – Multiple Choice Solutions

SECTION 1
Answers:

1. B	**2.** A	**3.** E	**4.** C	**5.** A
6. D	**7.** B	**8.** B	**9.** C	**10.** A
11. C	**12.** D	**13.** B	**14.** D	**15.** E
16. E	**17.** C	**18.** A	**19.** B	**20.** E
21. D	**22.** D			

Worked Solutions:
Question 1

Answer B

Vertical asymptotes occur where $2x^2 + 3x - 5 = 0$

$(2x + 5)(x - 1) = 0$

$x = -\frac{5}{2}, x = 1$

Horizontal asymptote at $y = 0$

Question 2

Answer A

Graph is a hyperbola of the form $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$

Asymptotes intersect at $(-1, 0) \Rightarrow h = -1, k = 0$

Minimum turning point at $(-1, 2) \Rightarrow a^2 = 1, b^2 = 4$

Equation of hyperbola is $\frac{(y-0)^2}{4} - \frac{(x+1)^2}{1} = 1$

$\frac{y^2}{4} - (x + 1)^2 = 1$

Question 3 **Answer E**

$$\frac{\sin(3x)}{\sin(x)}+\frac{\cos(3x)}{\cos(x)}$$

$$=\frac{\sin(3x)\cos(x)+\cos(3x)\sin(x)}{\sin(x)\cos(x)}$$

$$=\frac{\sin(3x+x)}{\sin(x)\cos(x)}$$

$$=\frac{2\sin(4x)}{2\sin(x)\cos(x)}$$

$$=\frac{2\times 2\sin(2x)\cos(2x)}{\sin(2x)}$$

$= 4\cos(2x)$

Question 4 **Answer C**

$f(x) = \sin^{-1}(x)$ has endpoints $\left(-1, -\frac{\pi}{2}\right)$ and $\left(1, \frac{\pi}{2}\right)$

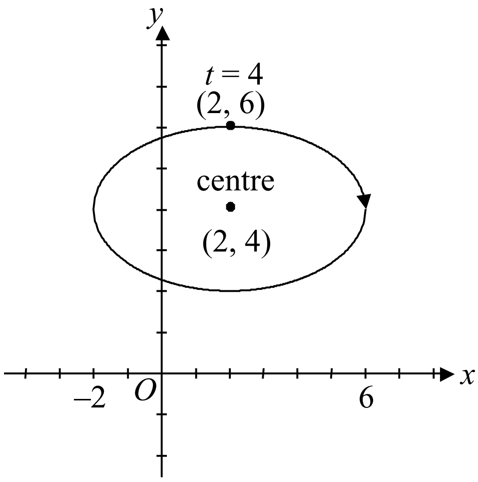
$f(x) = \sin^{-1}(a(x-b))$ has been dilated by a factor of $\frac{1}{a}$ parallel to the x-axis then translated b units right.

The endpoint coordinates will be $\left(b-\frac{1}{a},\ -\frac{\pi}{2}\right)$ and $\left(b+\frac{1}{a},\ \frac{\pi}{2}\right)$

Hence $\left(\frac{1}{a}+b,\ \frac{\pi}{2}\right)$

Question 5 **Answer A**

Graphing the curve in parametric mode on calculator will show alternatives B, C, D and E are true.



Finding the Cartesian equation

$x = 2 + 4\sin(\pi t)$ $\quad$ $y = 4 + 2\cos(\pi t)$

$\Rightarrow \dfrac{x-2}{4} = \sin(\pi t) \ \ldots(1)$ $\quad$ $\Rightarrow \dfrac{y-4}{2} = \cos(\pi t) \ \ldots(2)$

$$\left(\frac{x-2}{4}\right)^2 + \left(\frac{y-4}{2}\right)^2 = \sin^2(\pi t) + \cos^2(\pi t)$$

$$\frac{(x-2)^2}{16} + \frac{(y-4)^2}{4} = 1$$

The curve follows an elliptical path with centre (2, 4)

Question 6 **Answer D**

Let $z = x + yi$

$3z - \bar{z} = 12 - 4i$

$\Rightarrow 3(x + yi) - (x - yi) = 12 - 4i$

$3x + 3yi - x + yi = 12 - 4i$

$2x + 4yi = 12 - 4i$

$2x = 12,\ 4y = -4$

$\therefore\ x = 6,\ y = -1$

$z = 6 - i$

Question 7 **Answer B**

Since $2 - i$ and $1 - i$ are solutions of the quadratic equation

$(z - (2 - i))(z - (1 - i)) \equiv z^2 + az + b$

Expanding:

$z^2 - (1 - i)z - (2 - i)z + (2 - i)(1 - i) \equiv z^2 + az + b$

$z^2 - (3 - 2i)z + (2 - 3i - 1) \equiv z^2 + az + b$

$z^2 - (3 - 2i)z + (1 - 3i) \equiv z^2 + az + b$

Equating coefficients of powers of z:

$a = -(3 - 2i)$, $b = 1 - 3i$

$a = -3 + 2i$

Question 8 **Answer B**

$|2 - z| = 4$

$|2 + 0i - (x + yi)| = 4$

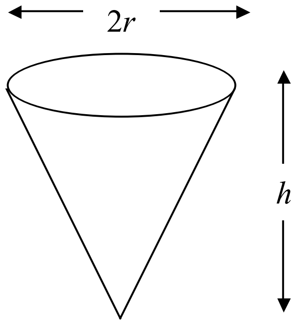
$\sqrt{(2 - x)^2 + (0 - y)^2} = 4$

$(x - 2)^2 + y^2 = 16$

Circle with centre (2, 0) and radius 4

Question 9

Answer C



$V = \frac{1}{3}\pi r^2 h$ $\qquad h = 2r$ (given)

$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$ $\qquad r = \frac{h}{2}$

$V = \frac{1}{12}\pi h^3$

$\frac{dV}{dh} = \frac{1}{4}\pi h^2$

Applying the chain rule

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $\qquad \frac{dV}{dt} = 10\,000 \text{ cm}^3\text{/min}$

$10\,000 = \frac{1}{4}\pi h^2 \times \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{40\,000}{\pi h^2}$

$\frac{dh}{dt} = \frac{40\,000}{\pi \times 50^2}$

$\frac{dh}{dt} = \frac{16}{\pi}$ cm/min

Question 10

Answer A

$\int \sin^2(x)\cos^3(x)dx$

$= \int \sin^2(x)\cos^2(x)\cos(\text{x})dx$

$= \int \sin^2(x)(1-\sin^2(x))\cos(x)dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$

$du = \cos(x)dx$

$= \int u^2(1-u^2)du$

$= \int (u^2 - u^4)du$

Question 11 **Answer C**

Finding equation of inverse $f^{-1}(x)$

$x = 2^y - 1$

$x + 1 = 2^y$

$y = \log_2 (x + 1)$

$f^{-1}(x) = \log_2 (x + 1)$

Area enclosed by curves

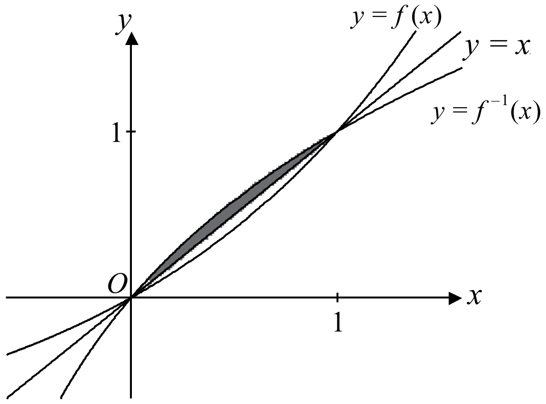
$$\int_0^1 \left(f^{-1}(x) - f(x)\right)dx$$

$$= \int_0^1 \left(\log_2(x+1) - \left(2^x - 1\right)\right)dx$$

$$= \int_0^1 \left(\log_2(x+1) - 2^x + 1\right)dx$$

A is true

The graphs are inverses of each other therefore the required area is symmetrical about the line $y = x$.



Twice $\int_0^1 \left(f^{-1}(x) - x\right)dx$ equals the required area

$$= 2\int_0^1 \left(\log_2(x+1) - x\right)dx$$

B is true

Twice $\int_0^1 \left(x - f(x)\right)dx$ equals the required area

$$= 2\int_0^1 \left(x - \left(2^x - 1\right)\right)dx$$

$$= 2\int_0^1 \left(x - 2^x + 1\right)dx$$

However C is $2\int_0^1 \left(x - 2^x - 1\right)dx$, so this is false

(Area of square side length 1) – twice $\int_0^1 f(x)\,dx$ equals the required area

$$=(1\times 1)-2\int_0^1 \left(2^x-1\right)dx$$

$$=1-2\int_0^1 \left(2^x-1\right)dx$$

D is true

Twice $\int_0^1 f^{-1}(x)\,dx$ – (area of square side length 1) equals the required area.

$$=2\int_0^1 f^{-1}(x)\,dx-(1\times 1)$$

$$=2\int_0^1 \left(\log_2(x+1)\right)dx-1$$

E is true

Question 12

Answer D

$$\frac{dy}{dx}=\sqrt{x^4+1}$$

$$y=\int \sqrt{x^4+1}\,dx$$

Let $y = f(x) + c$ ….(1) where $f'(x)=\sqrt{x^4+1}$

When $x = 1, y = 4$

$4 = f(1) + c$

$c = f(1) + 4$ ….(2)

Substitute (2) into (1)

$y = f(x) – f(1) + 4$

When $x = 3$, $y = f(3) – f(1) + 4$

$$y=\left[f(t)\right]_1^3+4$$

$$y=\int_1^3 \left(\sqrt{t^4+1}\right)dt+4$$

Question 13 **Answer B**

Let $f(x, y) = x - y$

$y_{n+1} = y_n + hf(x_n, y_n)$

Given $x_0 = 2$, $y_0 = 0$, $h = 0.5$

$y_1 = y_0 + 0.5f(x_0, y_0)$ $x_0 = 2$, $f(x_0, y_0) = x_0 - y_0 = 2 - 0 = 2$

$y_1 = 0 + 0.5 \times 2 = 1$

$y_2 = y_1 + 0.5f(x_1, y_1)$ $x_1 = 2.5$, $f(x_1, y_1) = x_1 - y_1 = 2.5 - 1 = 1.5$

$y_2 = 1 + 0.5 \times 1.5 = 1.75$

$y_3 = y_2 + 0.5f(x_2, y_2)$ $x_2 = 3$, $f(x_2, y_2) = x_2 - y_2 = 3 - 1.75 = 1.25$

$y_3 = 1.75 + 0.5 \times 1.25 = 2.375$

$y_4 = y_3 + 0.5f(x_3, y_3)$ $x_3 = 3.5$, $f(x_3, y_3) = x_3 - y_3 = 3.5 - 2.375 = 1.125$

$y_4 = 2.375 + 0.5 \times 1.125 = 2.9375$

Question 14 **Answer D**

The gradients are equal in a horizontal direction, therefore $\frac{dy}{dx} = f(y)$.

Eliminate alternatives A and B.

The field diagram shows a small positive slope at $y = 2$.

At $y = 2$, $\frac{dy}{dx} = \cos(y) = \cos(2) = -0.42$

This slope is negative, however in the field diagram the slope is positive when $y = 2$.

Eliminate E.

At $y = 2$, $\frac{dy}{dx} = y = 2$

A slope of 2 is much steeper than this field diagram shows when $y = 2$.

Eliminate C.

At $y = 2$, $\frac{dy}{dx} = \frac{1}{y^2} = \frac{1}{4}$ This is the only possible option.

The slope in the field diagram looks like the gradient is $\frac{1}{4}$ when $y = 2$.

By solving the differential equation it can be seen that the solution fits the slope field.

$\frac{dy}{dx} = \frac{1}{y^2}$

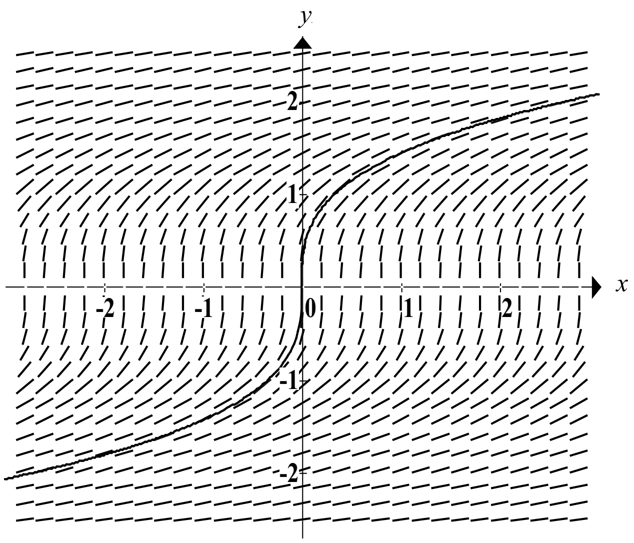
$\frac{dy}{dx} = y^2$

$x = \int y^2 dy + c$

$x = \frac{1}{3}y^3 + c$ Let $c = 0$

$y = \sqrt[3]{(3x)}$

This curve is sketched on the field diagram below.



Question 15 **Answer E**

$\underset{\sim}{a}.\underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\cos(\theta)$

$\left(2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\right)\left(\underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k}\right) = \left|2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\right|\left|\underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k}\right|\cos(\theta)$

$2 + m - 2 = \sqrt{9} \times \sqrt{\left(m^2 + 2\right)}\cos(\theta)$

$m = 3\sqrt{\left(m^2 + 2\right)}\cos(\theta)$

$\theta = \cos^{-1}\left(\dfrac{m}{3\sqrt{\left(m^2 + 2\right)}}\right)$

Question 16 **Answer E**

The dot product of perpendicular vectors is zero.

$\underset{\sim}{p}.\underset{\sim}{e} = \left(\cos(t)\underset{\sim}{i} + \sin(t)\underset{\sim}{j} - \underset{\sim}{k}\right)\left(\sin(t)\underset{\sim}{i} - \cos(t)\underset{\sim}{j} + \underset{\sim}{k}\right)$

$\underset{\sim}{p}.\underset{\sim}{e} = \cos(t)\sin(t) - \sin(t)\cos(t) - 1$

$\underset{\sim}{p}.\underset{\sim}{e} = -1$

Vectors $\underset{\sim}{p}$ and $\underset{\sim}{e}$ are not perpendicular.

Question 17 **Answer C**

Let T be the time when the particle's velocity changes from negative to positive.

Total area under graph (disregarding sign) is 305.

$8 \times 15 + \frac{1}{2}(T - 8) \times 15 + \frac{1}{2}(16 - T) \times 25 + 4 \times 25 = 305$

$120 + 7.5T - 60 + 200 - 12.5T + 100 = 305$

$T = 11$

The particle changes direction after 11 seconds.

Distance travelled with negative velocity = $8 \times 15 + \frac{1}{2}(11 - 8) \times 15 = 142.5$ metres

Distance travelled with positive velocity = $\frac{1}{2}(16 - 11) \times 25 + 4 \times 25 = 162.5$ metres

The particle is 162.5 – 142.5 = 20 metres from its starting position after 20 seconds.

Question 18 **Answer A**

$a = \frac{dv}{dt} = \frac{1}{2v+1}$

$\frac{dt}{dv} = 2v+1$

$t = \int (2v + 1)dv$

$t = v^2 + v$

When $t = 0$, $v = 0 \Rightarrow c = 0$

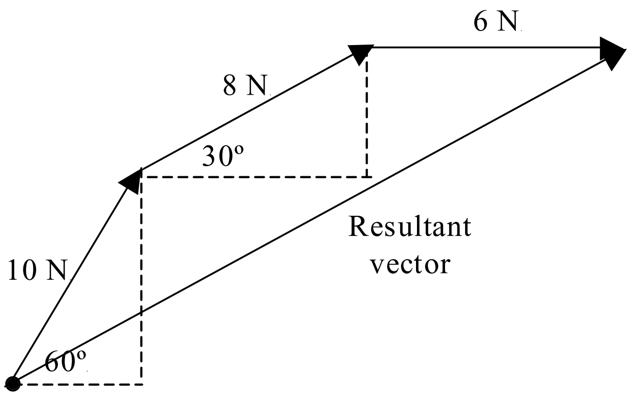
$t = v^2 + v$

When $t = 2$,
$$v^2 + v = 2$$
$$v^2 + v - 2 = 0$$
$$(v + 2)(v - 1) = 0$$
$$v = 1, v \geq 0$$

Velocity is 1 m/s after 2 seconds

Question 19 **Answer B**

Adding vectors head to tail



Horizontal component of resultant vector

$= 10\cos(60°) + 8\cos(30°) + 6$

$= 5 + 4\sqrt{3} + 6$

$= 11 + 4\sqrt{3}$

Vertical component of resultant vector

$=10\sin(60°) + 8\sin(30°)$

$= 5\sqrt{3} + 4$

Magnitude of resultant vector

$= \sqrt{\left(11+4\sqrt{3}\right)^2 + \left(5\sqrt{3}+4\right)^2}$

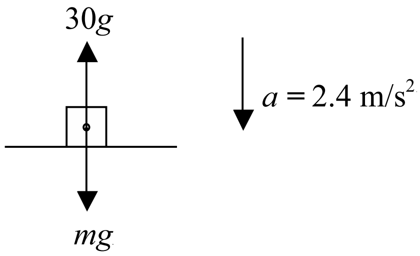
= 21.9 newtons

Question 20

Answer E

Draw diagram showing the forces acting.

Lift is accelerating downwards.



$mg - 30g = 2.4m$

$9.8m - 2.4m = 30 \times 9.8$

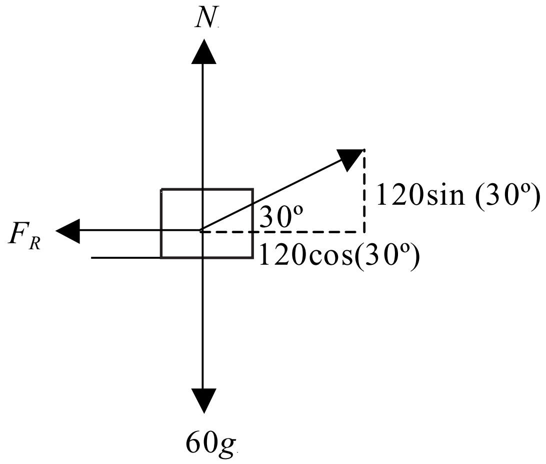
$7.4m = 294$

$m = 39.7$ kg

This is closest to 40 kg

Question 21 **Answer D**

Show all forces acting on the diagram.



Resolving vertical forces to find the normal reaction N

$N + 120\sin(30°) = 60g$

$N = 60g - 60$

Resolving horizontal forces.

$120\cos(30°) - F_R = ma$ Mass on the point of moving has a zero acceleration and so $a = 0$.

$60\sqrt{3} - \mu N = 0$ Friction: $F_R = \mu N$

$$\mu = \frac{60\sqrt{3}}{N}$$

$$\mu = \frac{60\sqrt{3}}{60g - 60}$$

$$\mu = \frac{\sqrt{3}}{g - 1}$$

Question 22 **Answer D**

The truck is towing the car and the tension in the towrope, T, represents this force. The resistance force R_1 acting on the truck is not relevant to the motion of the car.

Mathematical Association of Victoria
SPECIALIST MATHEMATICS
2008 Trial written examination 2 – Extended Analysis Questions Solutions

SECTION 2
Question 1

a. At $t = 0$

$r(t) = 30\underset{\sim}{i} - 20\underset{\sim}{j}$ **[1M]**

Speed = $\sqrt{30^2 + 20^2} = 36.06$ m/min **[1A]**

b. $\dot{\underset{\sim}{r}}(t) = 30(t+1)\underset{\sim}{i} + 20(t-1)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = 30\left(\frac{t^2}{2} + t\right)\underset{\sim}{i} + 20\left(\frac{t^2}{2} - t\right)\underset{\sim}{j} + \underset{\sim}{c}$ **[1M]**

$\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 0\underset{\sim}{j}$

$\therefore\ \underset{\sim}{c} = \underset{\sim}{0}$

$\therefore \underset{\sim}{r}(t) = 30\left(\frac{t^2}{2} + t\right)\underset{\sim}{i} + 20\left(\frac{t^2}{2} - t\right)\underset{\sim}{j}$ **[1A]**

c. $\sqrt{(100-0)^2 + (90-0)^2} = 134.5\,\text{m}$ **[1A]**

d. $\vec{AP} = \vec{AO} + \vec{OP}$

$\vec{AP} = -500\underset{\sim}{i} - 450\underset{\sim}{j} + 30\left(\frac{t^2}{2} + t\right)\underset{\sim}{i} + 20\left(\frac{t^2}{2} - t\right)\underset{\sim}{j}$

$\therefore \vec{AP} = \left(-500 + 15t^2 + 30t\right)\underset{\sim}{i} + \left(-450 + 10t^2 - 20t\right)\underset{\sim}{j}$ **[1A]**

e. $\left|\vec{AP}\right| = \sqrt{\left(-500 + 15t^2 + 30t\right)^2 + \left(-450 + 10t^2 - 20t\right)^2}$ **[1M]**

Sketch the graph of $\left|\vec{AP}\right|$ in terms of t,

From the graph of $\left|\vec{AP}\right|$ in terms of t, the minimum point is (5.5, 283.6)

To the nearest minute the airport official is closest to the plane at $t = 6$ minutes **[1A]**

f. If the plane crosses the path of the vector $\overrightarrow{OA}$ then the points *O, A* and *P* are collinear and so there exists a constant *b* such that $\overrightarrow{OP} = b\overrightarrow{OA}$

$$\underset{\sim}{r}(t) = 30\left(\frac{t^2}{2}+t\right)\underset{\sim}{i} + 20\left(\frac{t^2}{2}-t\right)\underset{\sim}{j} = 500b\underset{\sim}{i} + 450b\underset{\sim}{j}$$

Equating coefficients:

$$30\left(\frac{t^2}{2}+t\right) = 500b \qquad 20\left(\frac{t^2}{2}-t\right) = 450b$$

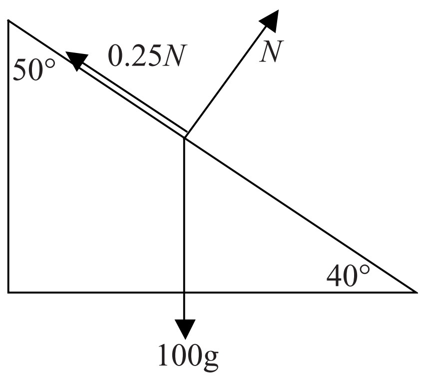
Therefore making *b* the subject of these equations

$$\frac{30}{500}\left(\frac{t^2}{2}+t\right) = \frac{20}{450}\left(\frac{t^2}{2}-t\right)$$

Solving this equation on a calculator gives no solution (except at the origin *O*)

Therefore the plane does not cross the path of the vector $\overrightarrow{OA}$ **[2A]**

g.



[1A]

h. $100g\sin40° - F = ma$ where $F = \mu N$

And

$N - 100g\cos40° = 0$

Giving

$N = 100g\cos40° = 750.724$

$\therefore\ F = \mu N = 0.25 \times 750.724 = 187.681$ **[1M]**

Using

$100g\sin40° - F = ma$

$\therefore\ 629.932 - 187.681 = 100a$

$\therefore\ a = 4.423$

To 2 decimal places

$\therefore\ a = 4.42$ m/s^2 **[1A]**

i. Using
$v^2 = u^2 + 2as$
$v^2 = 0^2 + 2 \times 4.423 \times 6$
$\therefore\ v = 7.29$
To 2 significant figures
$v = 7.3$ m/s **[1A]**

Question 2

a. $P(1 + i) = (1 + i)^4 + 2\,(1 + i)^2 - 4\,(1 + i) + 8$ **[1M]**

$(1 + i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$

$\therefore\ (1 + i)^4 = (2i)^2 = 4i^2 = -4$

$\therefore\ P(1 + i) = (2i)^2 + 2(2i) - 4 - 4i + 8$

$= -4 + 4i - 4 - 4i + 8 = 0$ **[1H]**

b. Using conjugate factors

$P(z) = (z-1-i)(z-1+i)(z+1+i\sqrt{3})(z+1-\mathrm{i}\sqrt{3})$ **[1M]**

Solutions are

$z = 1 \pm i, -1 \pm i\sqrt{3}$

In polar form

$z = \sqrt{2}cis\left(\pm\frac{\pi}{4}\right),\ 2cis\left(\pm\frac{2\pi}{3}\right)$ **[1A]**

c. $z_1z_2 = \sqrt{2}cis\frac{\pi}{4} \times 2cis\left(-\frac{2\pi}{3}\right)$ **[1M]**

$= 2\sqrt{2}cis\left(\frac{\pi}{4} + \frac{-2\pi}{3}\right)$

$\therefore\ z_1z_2 = 2\sqrt{2}cis\left(-\frac{5\pi}{12}\right)$ **[1A]**

d. $z_1 = 1 + i$ and $z_2 = -1 - i\sqrt{3}$

$\frac{z_1}{z_2} = \frac{1+i}{-1-i\sqrt{3}} \times \frac{-1+i\sqrt{3}}{-1+i\sqrt{3}}$

$\therefore\ \frac{z_1}{z_2} = \frac{\left(-1-\sqrt{3}\right) + \left(\sqrt{3}-1\right)i}{4}$

Using $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}-1}{-1-\sqrt{3}}$

$\therefore\ \tan\theta = \frac{\sqrt{3}-1}{-1-\sqrt{3}} \times \frac{-1+\sqrt{3}}{-1+\sqrt{3}} = \frac{4-2\sqrt{3}}{-2}$

Giving $\tan\theta = -2 + \sqrt{3}$ as required **[1A]**

Giving $\theta = \pi - \tan^{-1}\left(2 - \sqrt{3}\right)$

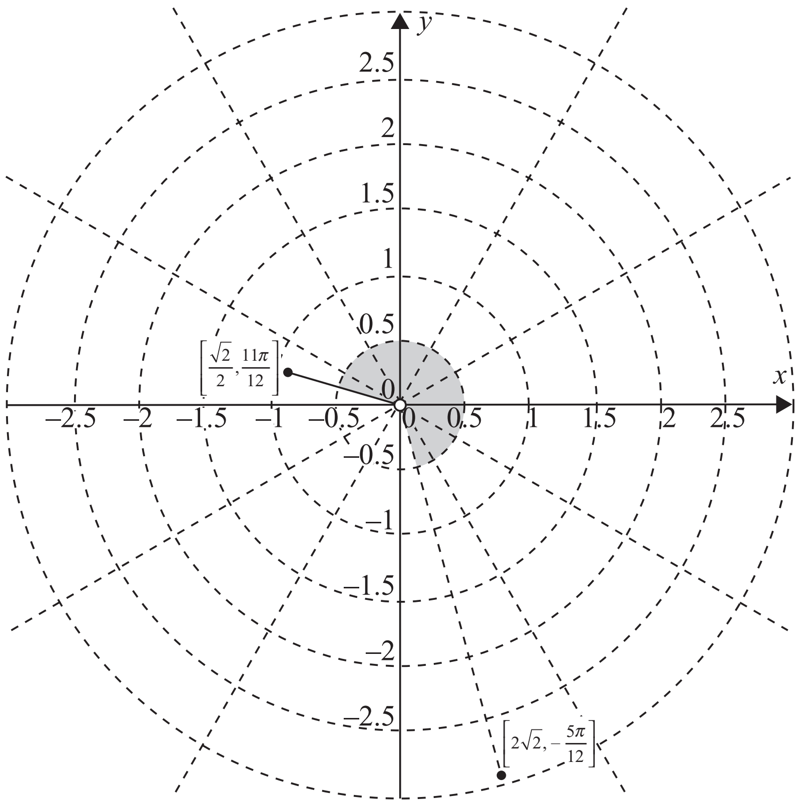
$\text{Arg}\frac{z_1}{z_2} = \theta = \pi - \frac{\pi}{12} = \frac{11\pi}{12}$

$\therefore\ \frac{z_1}{z_2} = \frac{\sqrt{2}}{2}cis\frac{11\pi}{12}$ **[1H]**

e. See solution **f.**

First point **[1A]**
Second point **[1A]**

f.



Shading **[1A]**
Boundaries **[1A]**

Question 3

a. $\frac{dy}{dt} = 80k\left(80 - y\right)$

$\therefore \frac{dt}{dy} = \frac{1}{80k(80-y)}$

$\therefore t = \frac{1}{80k}\int \frac{1}{80-y}\,dy$

$\therefore t = -\frac{1}{80k}\int \frac{-1}{80-y}\,dy$ **[1M]**

$\therefore t = -\frac{1}{80k}\ln\left|80-y\right| + c$

$\therefore -80k(t-c) = \ln\left|80-y\right|$

For $y < 80$

$\Leftrightarrow e^{-80k(t-c)} = 80 - y$ **[1A]**

$\therefore y = 80 - e^{-80k(t-c)}$

b. $y = 80 - e^{-80k(t-c)}$

$t = 0, y = 1$

$\therefore\ 1 = 80 - e^{80kc}$

$\therefore\ e^{80kc} = 80 - 1$ **[1M][1H]**

$\therefore\ 1n79 = 80kc$

$\therefore\ c = \frac{1}{80k}\,1n79$

c. $y = 80 - e^{-80k(t-c)}$ where $c = \frac{1}{80k}\,1n79$

$y = 80 - e^{-80k(t - \frac{1}{80k}\ln 79)}$

$\therefore y = 80 - e^{-80kt + \ln 79}$

$\therefore y = 80 - e^{-80kt}e^{\ln 79}$ **[1M]**

$\therefore y = 80 - e^{-80kt}(79)$

$\therefore y = 80 - 79e^{-80kt}$

$t = 10, y = 44$

$\therefore\ 44 = 80 - 79e^{-800k}$ **[1A]**

$\therefore\ 79e^{-800k} = 36$

d. $44 = 80 - 79e^{-800k}$

$\therefore\ 79e^{-800k} = 36$

$\Leftrightarrow \ln\dfrac{36}{79} = -800k$

$\therefore\ k = 0.0009824$

$\therefore\ k = 0.0010$ **[1A]**

k to 4 decimal places.

e. Using $c = \dfrac{1}{80k}\ln 79$

with $k = 0.0010$, $c = 54.618$

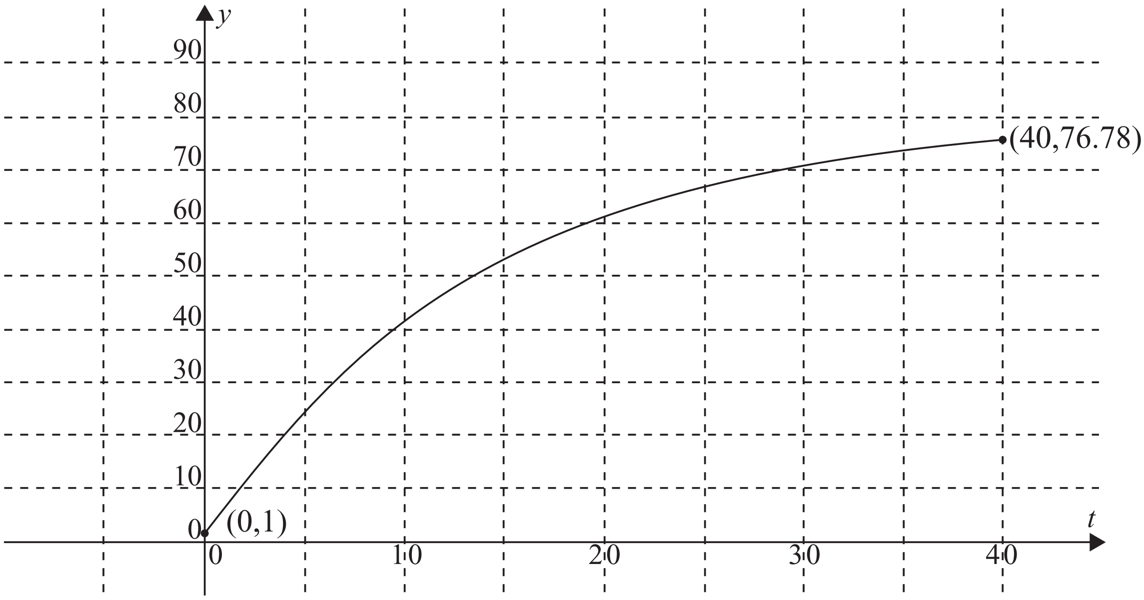
equation is $y = 80 - e^{-0.08(t - 54.618)}$

$y = 80 - e^{-0.08t}e^{4.369}$

$\therefore\ y = 80 - 78.999e^{-0.08t}$

Approximating to $y = 80 - 79e^{-0.08t}$

Check initial condition $t = 0$, $y = 1$, giving $y = 80 - 79e^{0} = 1$, which is correct



Shape **[1A]**

Correct points **[1A]**

f. $y = 64\%$ uptake after 20 months to the nearest %. **[1A]**

g. $y = 80$ represents the upper limit. 80% is the upper limit that the uptake of a certain mobile network approaches and levels out at.

Horizontal asymptote at $y = 80$ **[1A]**

Question 4

a. Convert 100 km/hr to $\frac{100}{3.6} = 27\frac{7}{9}$ m/s.

Using

$$s = \frac{u+v}{2}t$$

$$\therefore s = \frac{0 + 27\frac{7}{9}}{2} \times 6.8$$ **[1M]**

$$s = 94\frac{4}{9}$$

To 2 decimal places s = 94.44 m.

Also

$$v = u + at$$

$$\therefore 27\frac{7}{9} = 0 + 6.8a$$

$$a = 4.08497$$

To 2 decimal places a = 4.08 m/s^2 **[1A]**

b. $R = ma$ **[1A]**

$-1000 = 200a$

$\therefore\ a = -5$ m/s^2

Using constant acceleration

$v^2 = u^2 + 2as$

$\therefore\ 0 = (27\frac{7}{9})^2 - 10s$

$\therefore\ s = 77.1605$

To 2 decimal places s = 77.16 m **[1A]**

c. $a = \frac{1}{50}(v+1)^2$

$\therefore v\frac{dv}{dx} = \frac{1}{50}\left(v+1\right)^2$

$\therefore \frac{dv}{dx} = \frac{1}{50}\frac{\left(v+1\right)^2}{v}$

$\therefore \frac{dx}{dv} = \frac{50v}{\left(v+1\right)^2}$

$x = 50\int \frac{v}{\left(v+1\right)^2}dv$ **[1M]**

Let

$u = v + 1$, $v = u - 1$

$\therefore du = dv$

$\therefore x = 50\int \frac{u-1}{u^2}du$

$\therefore x = 50\int \frac{1}{u} - \frac{1}{u^2}dv$

$\therefore x = 50\left(\ln\left|u\right| + \frac{1}{u} + c\right)$ **[1H]**

$\therefore x = 50\left(\ln\left|v+1\right| + \frac{1}{v+1} + c\right)$

At

$x = 0$, $v = 0$, $\therefore c = -1$

$x = 50\left(\ln\left|v+1\right| + \frac{1}{v+1} - 1\right)$ **[1H]**

(As v is positive, students do not need to include the modulus signs)

d. $x = 50\left(\ln\left|v+1\right| + \frac{1}{v+1} - 1\right)$

Let

$x = 100$

$v = 18.1$ m/s (found on calculator) **[1A]**

e. $x = 50\left(\ln\left|v+1\right| + \frac{1}{v+1} - 1\right)$

Let

$v = 50$

$x = 147.6$ m (found on calculator) **[1A]**

Question 5

a. $-1 \le 2x + 1 \le 1$

$\therefore -2 \le 2x \le 0$

Therefore

$-1 \le x \le 0$ **[1A]**

$\therefore a = -1, b = 0$

b. $f(x) = 0.5\cos^{-1}(2x+1)$

$$\therefore f'(x) = 0.5\frac{-1}{\sqrt{1-(2x+1)^2}} \times 2$$

$$\therefore f'(x) = \frac{-1}{\sqrt{1-(2x+1)^2}}$$

For a stationary point solve

$$\frac{-1}{\sqrt{1-(2x+1)^2}} = 0$$

No solution found **[1A]**

No solution hence no stationary point

c. $\frac{d}{dx}\left(x\cos^{-1}(2x+1)\right)$

Using the Product Rule:

$$= \cos^{-1}(2x+1).1 + x.\frac{-2}{\sqrt{1-\left(2x+1\right)^2}}$$ **[1A]**

$$= \cos^{-1}\left(2x+1\right) - \frac{2x}{\sqrt{1-\left(2x+1\right)^2}}$$

d. $$\int_{-\frac{1}{2}}^{0}\left(\cos^{-1}\left(2x+1\right) - \frac{2x}{\sqrt{1-\left(2x+1\right)^2}}\right)dx = \left[x\cos^{-1}\left(2x+1\right)\right]_{-\frac{1}{2}}^{0}$$ **[2A]**

$$\therefore \int_{-\frac{1}{2}}^{0}\cos^{-1}(2x+1)dx = \int_{-\frac{1}{2}}^{0}\frac{2x}{\sqrt{1-\left(2x+1\right)^2}}dx + \left[x\cos^{-1}\left(2x+1\right)\right]_{-\frac{1}{2}}^{0}$$

e. Working below, using answer for **d.**

$$0.5\int_{-\frac{1}{2}}^{0} \cos^{-1}(2x+1)dx = 0.5\int_{-\frac{1}{2}}^{0} \frac{2x}{\sqrt{1-\left(2x+1\right)^2}}dx + 0.5\left[x\cos^{-1}\left(2x+1\right)\right]_{-\frac{1}{2}}^{0}$$

$$\therefore 0.5\int_{-\frac{1}{2}}^{0} \cos^{-1}(2x+1)dx = 0.5\int_{-\frac{1}{2}}^{0} \frac{2x}{\sqrt{1-\left(2x+1\right)^2}}dx + 0.5\left[0-\frac{-1}{2}\cos^{-1}\left(0\right)\right]$$

$$= 0.5\left(\frac{1}{2}-\frac{\pi}{4}\right)+0.5\frac{1}{2}\cos^{-1}0$$

$$= \frac{1}{4}-\frac{\pi}{8}+\frac{\pi}{8}$$

$$= \frac{1}{4}$$

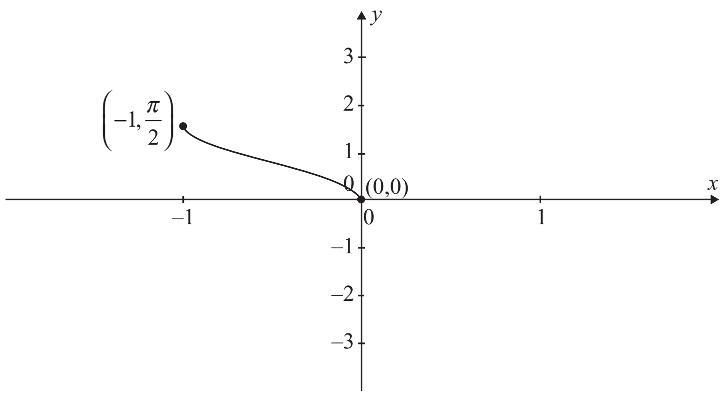
[1M][1A]

Check:

$$0.5\int_{-\frac{1}{2}}^{0} \cos^{-1}(2x+1)dx = 0.25$$

found on calculator

f.



[1A]

g. Volume $= \pi\int_{0}^{h} x^2 dy$

Using

$$y = 0.5\cos^{-1}(2x+1)$$

$$\therefore 2y = \cos^{-1}(2x+1)$$

$$\therefore \cos(2y) = 2x+1$$

$$\therefore x = \frac{\cos(2y)-1}{2}$$

$$\therefore x^2 = \left(\frac{\cos(2y)-1}{2}\right)^2$$

Volume $\frac{\pi}{4}\int_{0}^{h}\left(\cos(2y)-1\right)^2 dy$ for $0 < h < \frac{\pi}{2}$ **[1A]**

h. Volume = $\dfrac{\pi}{4}\int_0^h \left(\cos(2y)-1\right)^2 dy$

$$\frac{\pi}{4}\int_0^h \left(\cos(2y)-1\right)^2 dy$$

Using double angle formula

$\cos(4y) = 2\cos^2(2y) - 1$

$\therefore \cos^2(2y) = \dfrac{\cos(4y)+1}{2}$

$$= \frac{\pi}{4}\int_0^h \left(\cos^2(2y) - 2\cos(2y) + 1\right) dy$$

$$= \frac{\pi}{4}\int_0^h \left(\frac{\cos(4y)+1}{2} - 2\cos(2y) + 1\right) dy$$

$$= \frac{\pi}{4}\left[\frac{1}{8}\sin(4y) + \frac{y}{2} - \sin(2y) + y\right]_0^h$$ **[1M]**

$$= \frac{\pi}{4}\left(\frac{1}{8}\sin(4h) + \frac{3h}{2} - \sin(2h)\right)$$

For $h = \dfrac{\pi}{4}$ **[1M]**

Volume = $\dfrac{\pi}{4}\left(\dfrac{1}{8}\sin(\pi) + \dfrac{3\pi}{8} - \sin\left(\dfrac{\pi}{2}\right)\right)$

$= \dfrac{\pi}{4}\left(\dfrac{3\pi}{8} - 1\right)$ cm^3 **[1A]**

END OF SOLUTIONS