Student Name: ____________________

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2

2008 Trial Examination

Reading Time: 15 minutes
Writing Time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	5	5	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, an approved graphics calculator or a scientific calculator.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question book of 22 pages.

Instructions
- Print your name in the space provided on the top of this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other electronic devices into the examination room.

SECTION 1

Instructions for Section 1

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks are **not** deducted for incorrect answers.
If more than 1 answer is completed for any question, no mark will be given.
Take the **acceleration due to gravity**, to have magnitude g m/s^2, where g = 9.8

Question 1

The asymptotes of the function $y = \dfrac{2x^3 - 7x^2 + 5x}{x^2 - 3x}$ are

A. $x = 3,\ x = 0$ and $y = 2x$
B. $x = 3,\ x = 0$ and $y = 2x - 1$
C. $x = 3,\ x = 0$ and $y = 2$
D. $x = 3$ and $y = 2x - 1$
E. $x = 0$ and $y = 2x - 1$

Question 2

The equation of a hyperbola with its centre at $(-1,2)$ and an asymptote $5x - 3y + 11 = 0$ is

A. $\dfrac{(x-1)^2}{36} - \dfrac{(y+2)^2}{100} = 1$
B. $\dfrac{(x-1)^2}{9} - \dfrac{(y+2)^2}{25} = 1$
C. $\dfrac{(x+1)^2}{5} - \dfrac{(y-2)^2}{3} = 1$
D. $\dfrac{(x+1)^2}{36} - \dfrac{(y-2)^2}{100} = 1$
E. $\dfrac{(x+1)^2}{25} - \dfrac{(y-2)^2}{9} = 1$

Question 3

The graph of $y = \dfrac{-2}{x^2 - (m-1)x - m}, m > -1$, is positive when

A. $x \in (m, \infty)$
B. $x \in (-1, m)$
C. $x \in (-\infty, m)$
D. $x \in (-\infty, -1) \cup (m, \infty)$
E. $x \in (m, 1)$

SECTION 1- continued

Question 4

If z and w are complex numbers such that $|w| = |3z|$ and $Arg(w) = Arg(z) + \frac{\pi}{2}$ where $Arg(w) > 0, Arg(z) < \pi$, which one of the following must be true

A. $w = 3zi$
B. $w = 3\bar{z}$
C. $\bar{w} = 3z$
D. $Arg(w) + Arg(z) = \pi$
E. $wi = 3z$

Question 5

If $z = 2cis\left(-\frac{2\pi}{3}\right)$ then z^{-2} is

A. $-4cis\left(-\frac{2\pi}{3}\right)$
B. $-4cis\left(\frac{4\pi}{3}\right)$
C. $\frac{1}{4}cis\left(\frac{2\pi}{3}\right)$
D. $\frac{1}{4}cis\left(-\frac{2\pi}{3}\right)$
E. $\frac{1}{4}cis\left(\frac{4\pi^2}{9}\right)$

Question 6

For any $n \geq 3$, the complex roots of the equation $z^n = 1$ are vertices of a polygon with perimeter

A. $2n$
B. $n\sin\frac{2\pi}{n}$
C. $2n\sin\frac{2\pi}{n}$
D. $n\sin\frac{\pi}{n}$
E. $2n\sin\frac{\pi}{n}$

SECTION 1- continued
TURN OVER

Question 7

For the curve with equation $y = x^2 e^{-x}$, the values of x for which $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$ are

A. $\left(0,\ 2-\sqrt{2}\right)$
B. $\left(2-\sqrt{2},\ 2+\sqrt{2}\right)$
C. $\left(2-\sqrt{2},\ 2\right)$
D. $\left(2,\ 2+\sqrt{2}\right)$
E. $\left(2+\sqrt{2},\ 2\right)$

Question 8

The graph of $f(x) = a\sin^{-1}\left(x - \frac{1}{2}\right) + b$ passes through the points $\left(0, -\frac{\pi}{12}\right)$ and $\left(1, \frac{7\pi}{12}\right)$. The values of a and b are

A. $a = 2, b = \frac{\pi}{4}$
B. $a = \frac{-2}{3}, b = \frac{25\pi}{436}$
C. $a = -2, b = \frac{\pi}{4}$
D. $a = 2, b = -\frac{\pi}{4}$
E. $a = \frac{-2}{3}, b = \frac{\pi}{4}$

Question 9

If the scalar resolute of $\mathbf{a} = 2\mathbf{i} + x\mathbf{j} - \mathbf{k}$ in the direction of $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ is 2, then x is equal to

A. 0
B. $\sqrt{3}$
C. 3
D. $3\sqrt{3}$
E. 9

SECTION 1- continued

Question 10

If $\mathbf{m} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{n} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and θ is the angle between the direction of **m** and **n**, the exact value of $\cos 2\theta$ is

A. $-\frac{1}{6}$

B. $\frac{1}{6}$

C. $\frac{17}{18}$

D. $-\frac{17}{18}$

E. 3

Question 11

A solution of $\int x(2x-3)^3\,dx$ is

A. $\frac{u^5}{5} + \frac{u^4}{4}$

B. $\frac{(2x-3)^5}{20} + \frac{3(2x-3)^4}{16}$

C. $\frac{u^5}{20} + \frac{u^4}{16}$

D. $\frac{(2x-3)^5}{10} + \frac{(2x-3)^4}{8}$

E. none of these

Question 12

A tank initially contains 100 litres of solution in which 4 kg of salt is dissolved. A solution containing 5 kg of salt per litre is added at the rate of 8 litres per minute. The mixture is drained simultaneously at a rate of 10 litres per minute. There are *x* kg of salt in the tank after *t* minutes. This can be described by a differential equation

A. $\frac{dx}{dt} = 40 - \frac{10x}{100+2t}$

B. $\frac{dx}{dt} = 100 - \frac{5x}{50-t}$

C. $\frac{dx}{dt} = 40 - \frac{5x}{50-t}$

D. $\frac{dx}{dt} = 20 - \frac{5x}{50-t}$

E. $\frac{dx}{dt} = 40 + \frac{5x}{50+t}$

SECTION 1- continued
TURN OVER

Question 13

A spherical balloon is inflated at a rate $\frac{\pi}{3}cm^3$ per minute. The rate at which its surface area is increasing when the radius is $5cm$ is

A. $\frac{\pi^2}{9}$

B. $\frac{2\pi}{15}$

C. $\frac{15}{2\pi}$

D. $\frac{3}{\pi}$

E. $\frac{9}{\pi^2}$

Question 14

An approximation to the solution of the differential equation $\frac{dy}{dx} = y - x^2$ with $y(4) = 1$ is found using Euler's method with $h = 0.04$. When $x = 4.12$, the value for y is closest to

A. -0.9122
B. -0.2386
C. -0.8435
D. -1.6277
E. 0.9984

Question 15

A solution of $\int \sin 2x \sec^2 2x \ dx$ can be written as $a \sec 2x$. The value of a is

A. $\frac{1}{2}$

B. 2

C. $-\frac{1}{2}$

D. -2

E. none of these

SECTION 1- continued

Question 16

$$\int_0^2 \frac{x^2}{4+x^2}dx =$$

A. $2-\ln 2$

B. $\ln 2-\frac{1}{2}$

C. $2\tan^2 2+4\ln(\cos 2)$

D. $2\ln(\sec 2)-\sin\ \ 2$

E. $2-\frac{\pi}{2}$

Question 17

The velocity, $v\ ms^{-1}$, travelling in a straight road is given by $v=4-x^2$ where x is the position of the body at time t seconds. The acceleration of the body is equal to

A. $-2x$

B. $(4-x^2)t$

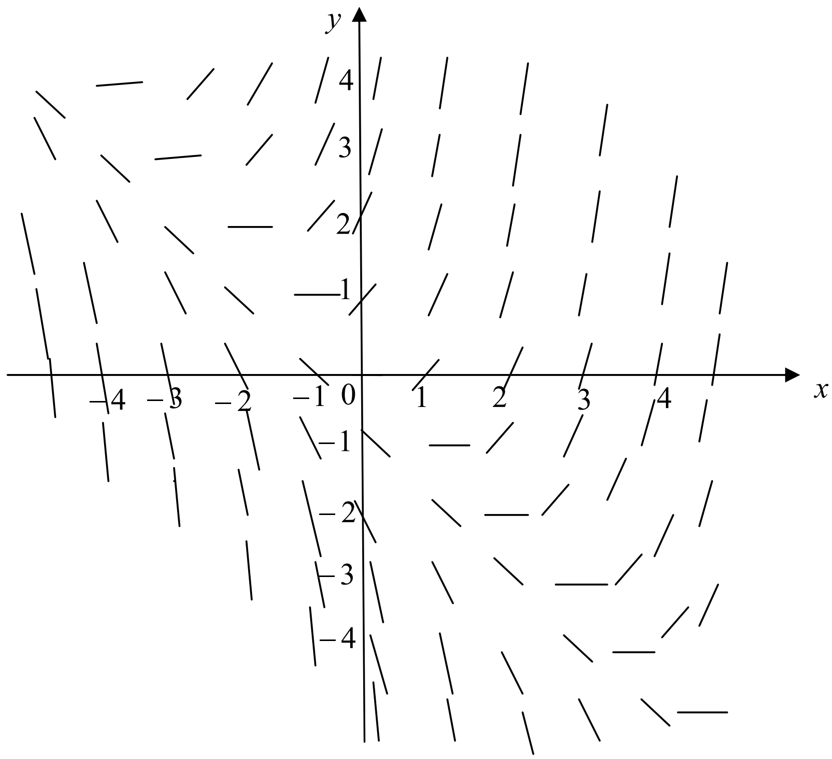
C. $-2xt$

D. $-2x^3$

E. $-8x+2x^3$

SECTION 1- continued
TURN OVER

Question 18



The family of solutions of a first order differential equation is shown above. The differential equation could be

A. $\frac{dy}{dx} = y - x$

B. $\frac{dy}{dx} = -x^2 + y$

C. $\frac{dy}{dx} = -\frac{x}{y}$

D. $\frac{dy}{dx} = x + y$

E. $\frac{dy}{dx} = -x$

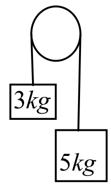
Question 19

A car of mass 1 tonne, travelling at $\frac{50}{3} m/s$ on a level road, has its speed reduced to $\frac{20}{3} m/s$ in 5 seconds when the brakes are applied. The total retarding force (assumed constant) is

A. 7200N
B. 200N
C. 2000N
D. 720N
E. 1000N

SECTION 1- continued

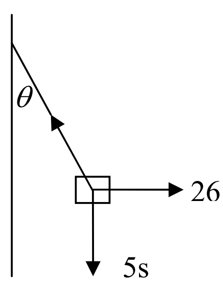
Question 20
Masses of 3kg and 5kg are hanging at the ends of a light string that passes over a smooth fixed peg as shown in the diagram.



The tension in the string is
A. $2g$
B. $\frac{15g}{4}$
C. $8g$
D. $\frac{g}{4}$
E. g

Question 21
A ball is thrown vertically upwards with an initial speed of $29.4\, ms^{-1}$ from the top of a building 20 m high. Neglecting air resistance, the ball is above the top of the building for
A. 6 seconds
B. 5 seconds
C. 4 seconds
D. 3 seconds
E. 2 seconds

Question 22
A particle of mass 5kg hangs at the end of a string attached to a fixed point. The particle is held at rest by a horizontal force of magnitude 26N so that the string makes an angle θ° with the vertical as shown in the diagram.



The angle θ is closest to
A. 89°
B. 62°
C. 28°
D. 11°
E. 45°

END OF SECTION 1
TURN OVER

SECTION 2

Instructions for Section 2
Answer **all** questions. A decimal approximation will not be accepted if the question specifically asks for an **exact** answer is required. For questions worth more than one mark, appropriate working **must** be shown. Unless otherwise indicated, the diagrams are **not** drawn to scale. Take the **acceleration due to gravity**, to have magnitude g m/s^2 , where $g = 9.8$

Question 1

Let $p(z) = z^4 - 9z^3 + az^2 - 16z - 48$, where a is a real number. The complex number w is a solution of the equation $p(z) = 0$ such that $w + \overline{w} = 4$ and $w\overline{w} = 8$.

a. Using this information, write down a quadratic factor of $p(z)$.

1 mark

b. The other quadratic factor of $p(z)$ is $z^2 - 5z - 6$. Find the value of a.

2 marks

SECTION 2- Question 1- continued

c. Show that $w = 2 + 2i$.

2 marks

d. Find, in polar form

i. w.

ii. $\overline{w}^5$.

iii. $\sqrt[3]{w}$.

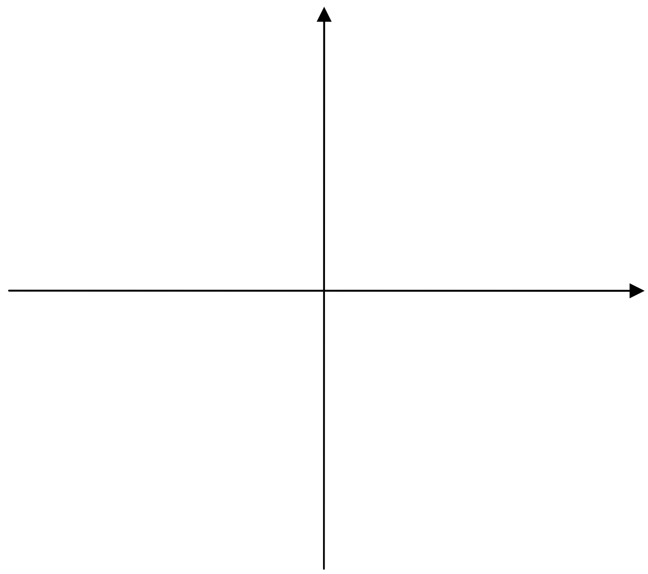
1+1+2 = 4 marks

SECTION 2- Question 1-continued

TURN OVER

e. Let S and T be subsets of the complex plane where $S = \{z : |z - w| \le |w|\}$

and $T = \{z : 0 < Arg z \le Arg w\}$. Sketch S and T and shade $S \cap T$.



3 marks
Total 12 marks

SECTION 2- continued

Question 2

a. Use integration by substitution to find the indefinite integral $\int x^{n-1}e^{-x^n}\,dx$, where n is a positive integer.

3 marks

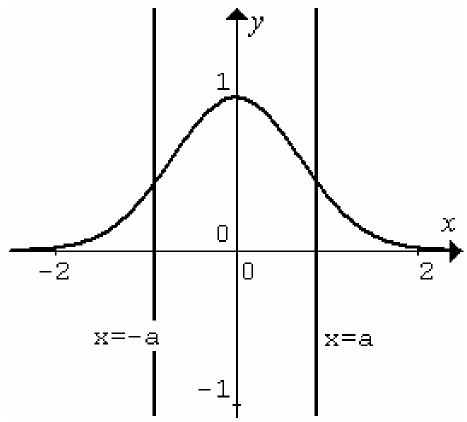
b. Evaluate the definite integral $\int_0^2 xe^{-x^2}\,dx$ giving your answer in exact form.

2 marks

SECTION 2- Question 2- continued
TURN OVER

c. The graph of $y = e^{-x^2}$ is shown below.

i. Shade the region bounded by the curve $y = e^{-x^2}$ and the lines $x = a$, $x = -a$ and $y = 0$.



ii. Write down an expression which gives the volume of the solid obtained by revolving the shaded region about the *y*-axes. Do **not** evaluate the volume at this stage.

1 + 2 = 3 marks

SECTION 2- Question 2-continued

d. Given that $\int \log_e y\,dy = y\log_e y - y + c$, find the volume of this solid of revolution.

3 marks

e. What is the limiting value of the volume when $a \to \infty$?

1 mark
Total 12 marks

SECTION 2- continued
TURN OVER

Question 3
The nautilus is a marine creature that lives around coral reefs (*Figure 1*).

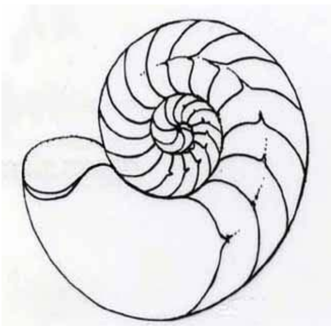


Figure 1

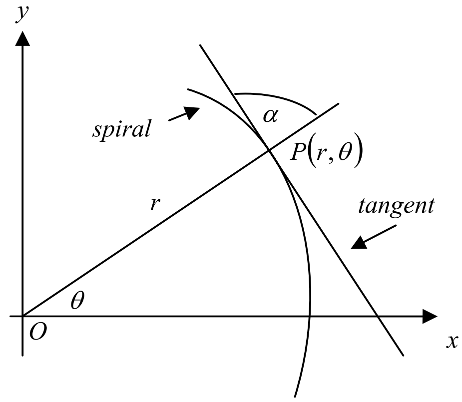


Figure 2

The mathematical model of a nautilus shell is an equiangular spiral (*Figure 2*). The equations of equiangular spirals are of the form $r = Ae^{k\theta}$, where k is a constant. At every point P, the tangent to the spiral makes the same angle, α, with the line OP. The size of the angle α depends on the constant k.
The line OP makes an angle θ with the positive part of the x-axis.

a. Show that the Cartesian coordinates of the point $P(x, y)$ are
$x = Ae^{k\theta}\cos\theta, \quad y = Ae^{k\theta}\sin\theta$.

2 marks

SECTION 2- Question 3- continued

b. Show that $\frac{dy}{dx} = \frac{k \tan\theta + 1}{k - \tan\theta}$.

4 marks

c. Explain why the gradient of the tangent to the spiral can be written as $\tan(\alpha + \theta)$.

1 mark

SECTION 2- Question 3- continued
TURN OVER

d. Hence, or otherwise, find the value of α in terms of *k.*

3 marks
Total 10 marks

SECTION 2- continued

Question 4

A ship has the position vector $\mathbf{r}_s = 0\mathbf{i} + 0\mathbf{j}$. A missile is spotted at the position $\mathbf{r}_m = 1000\mathbf{i} + 500\mathbf{j}$ moving towards a ship with velocity $\mathbf{v}_m = -30\mathbf{i} + 3\mathbf{j}$. Assume that gravity is the only force acting on the missile.

a. Show that the position vector of the missile at time *t* is given by
$\mathbf{r}_m(t) = (-30t + 1000)\mathbf{i} + (-4.9t^2 + 3t + 500)\mathbf{j}$

3 marks

An anti-missile can be fired from the ship with a velocity of $100ms^{-1}$ at an angle θ° to the horizontal. Assume that gravity is the only force acting on the anti-missile

b. Show that the position vector of the anti-missile at time *t* is given by
$\mathbf{r}_a(t) = 100t\cos\theta\,\mathbf{i} + (-4.9t^2 + 100t\sin\theta)\mathbf{j}$

3 marks

SECTION 2- Question 4 –continued
TURN OVER

c. Show that the anti-missile must be fired at an angle of 35.8° in order to intercept the missile.

5 marks

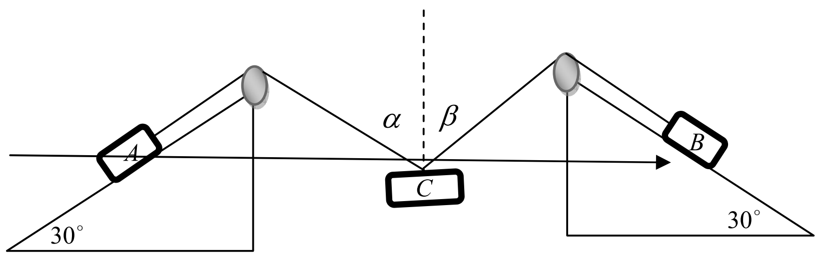
d. How many seconds after firing will the missile be intercepted?

1 mark
Total 12 marks

SECTION 2- continued

Question 5

The diagram shows particles A and B each lying on smooth planes of inclination 30° to the horizontal. A and B are attached to inextensible strings passing over smooth pulleys and are connected to a third particle C hanging freely. The strings make angles of α and β with the vertical as shown. The particles A, B and C have masses respectively 2*M*, 1.5*M* and 3*M*. The system rests in equilibrium.



a. On the diagram above, show all forces acting on these three particles.

2 marks

b. Express the tensions in the strings in terms of *M*.

2 marks

SECTION 2- Question 5 –continued
TURN OVER

c. By considering the forces acting on particle C, show that $2\sin\alpha = 3\sin\beta$ and $2\cos\alpha + 3\cos\beta = 3$

4 marks

d. Hence, find the values of α and β accurate to the nearest degree.

4 marks
Total 12 marks

END OF QUESTION AND ANSWER BOOK