# *INSIGHT*

***Trial Exam Paper***

# 2009

# SPECIALIST MATHEMATICS

# Written examination 1

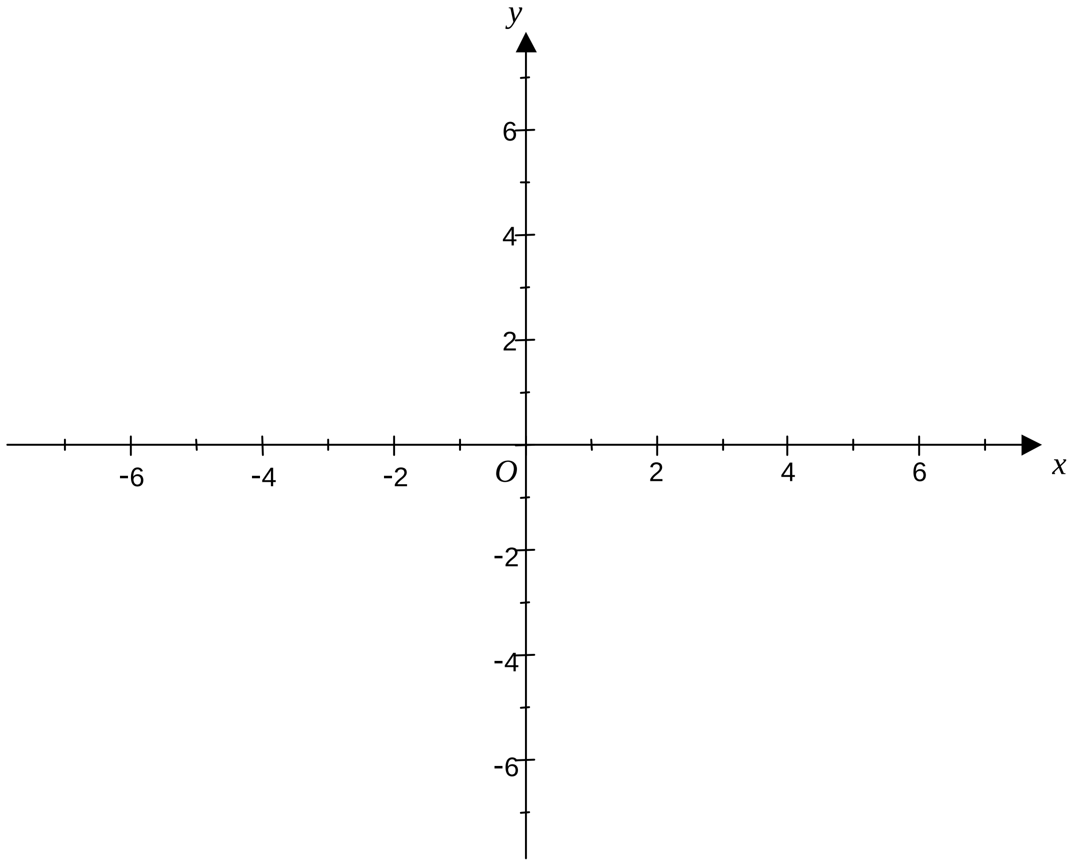
# *Worked solutions*

**This book presents:**

- worked solutions, giving you a series of points to show you how to work through the questions.
- mark allocations
- tips on how to approach the questions.

**Question 1**

Sketch the graph of $y = \frac{1}{2-x} - x$ on the axes below. Give the exact coordinates of any stationary points and intercepts and the equations of any straight line asymptotes.




**Worked solution**

$y = \frac{1}{2-x} - x$ has a vertical asymptote at $x = 2$ and an oblique asymptote at $y = -x$.

Stationary points occur where $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{1}{(2-x)^2} - 1$$

$$0 = \frac{1}{(2-x)^2} - 1$$

$$(2-x)^2 = 1$$

$$2 - x = \pm 1$$

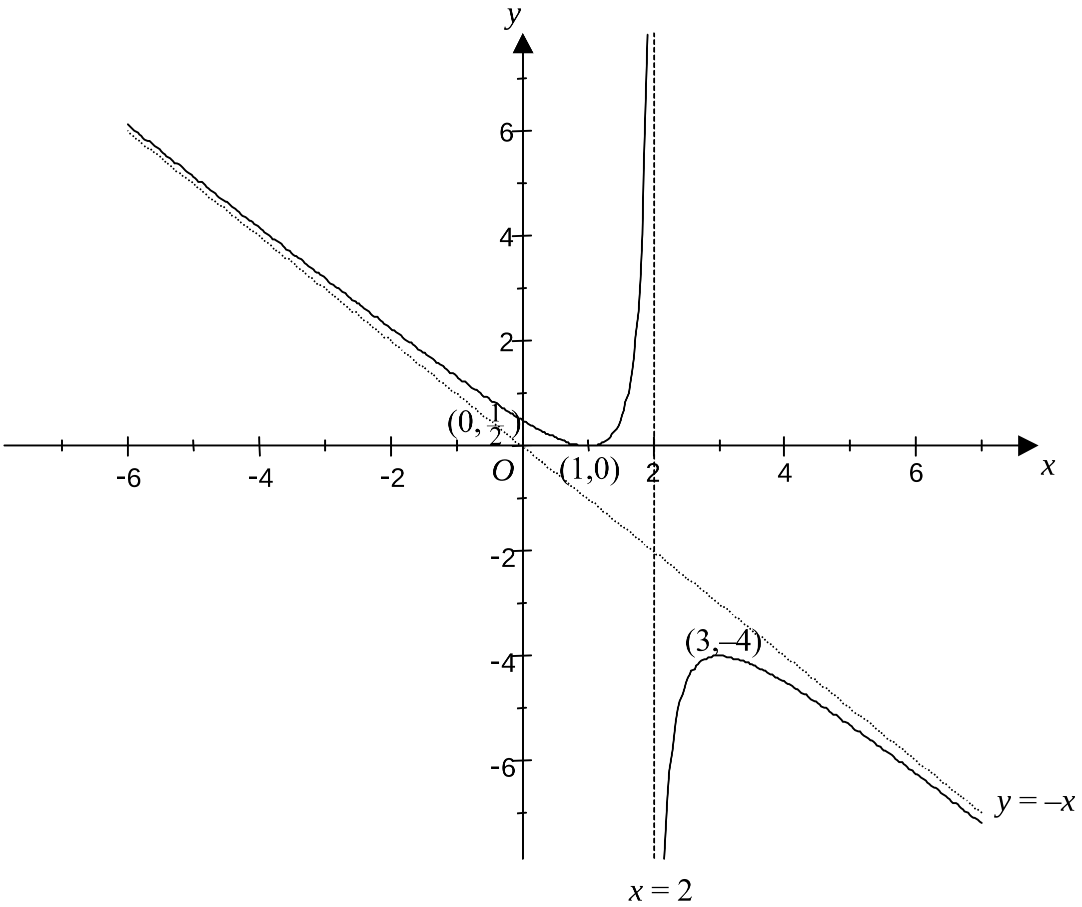
$$x = 1, 3$$

Stationary points are at (1, 0) and (3, –4).

$x$-intercept, $x = 1$, $y = 0$

$y$-intercept, $x = 0$, $y = \frac{1}{2}$

**Question 1 –** continued
**TURN OVER**




4 marks

**Mark allocation**

- 1 mark for the equations of both straight line asymptotes.
- 1 mark for the coordinates of both stationary points.
- 1 mark for correct intercepts.
- 1 mark for correct shape.

**Question 2**

Find all solutions of $z^3 + z + 10 = 0$, $z \in C$.

**Worked solution**

Let $P(z) = z^3 + z + 10$

Find a factor of $P(z)$

$P(1) \neq 0$, $P(-1) \neq 0$, $P(2) \neq 0$

$P(-2) = (-2)^3 + (-2) + 10 = 0$

$\therefore z + 2$ is a factor

Find the quadratic factor

$$\begin{array}{r|l} & z^2 - 2z + 5 \\ \hline z+2 & z^3 + 0z^2 + z + 10 \\ & \underline{z^3 + 2z^2} \\ & -2z^2 + z + 10 \\ & \underline{-2z^2 - 4z} \\ & 5z + 10 \\ & \underline{5z + 10} \\ & 0 \end{array}$$

$$\begin{aligned} P(z) &= (z+2)(z^2 - 2z + 5) \\ &= (z+2)((z^2 - 2z + 1) + 4) \\ &= (z+2)((z-1)^2 + 4) \\ &= (z+2)((z-1)^2 - (2i)^2) \\ &= (z+2)(z - 1 + 2i)(z - 1 - 2i) \end{aligned}$$

Solving $(z+2)(z-1+2i)(z-1-2i) = 0$

$z = -2$, $z = 1 - 2i$, $z = 1 + 2i$

3 marks

**Mark allocation**

- 1 mark for $z = -2$.
- 1 mark for correct method.
- 1 mark for three correct solutions.

**Question 3**

Find the cube roots of $4i-4\sqrt{3}$ in polar form.

**Worked solution**

Let $z^3 = -4\sqrt{3}+4i = r\,\text{cis}(\theta)$

Where $r=\sqrt{\left(-4\sqrt{3}\right)^2+4^2}$ and $\tan(\theta)=\dfrac{4}{-4\sqrt{3}}=-\dfrac{1}{\sqrt{3}}$

$r=\sqrt{16\times 3+16}$ $\qquad \theta=\pi-\dfrac{\pi}{6}$

$r=8$ $\qquad \theta=\dfrac{5\pi}{6}$

$$\therefore \quad z^3 = 8\,\text{cis}\left(\frac{5\pi}{6}\right)$$

$$z=\left(8\,\text{cis}\left(\frac{5\pi}{6}\right)\right)^{\frac{1}{3}}$$

$$z=8^{\frac{1}{3}}\text{cis}\left(\frac{1}{3}\left(\frac{5\pi}{6}+2k\pi\right)\right)$$ by De Moivre's Theorem

$k=0 \qquad z=2\,\text{cis}\left(\dfrac{5\pi}{18}\right)$

$k=1 \qquad z=2\,\text{cis}\left(\dfrac{5\pi}{18}+\dfrac{2\pi}{3}\right)=2\,\text{cis}\left(\dfrac{17\pi}{18}\right)$

$k=-1 \qquad z=2\,\text{cis}\left(\dfrac{5\pi}{18}-\dfrac{2\pi}{3}\right)=2\,\text{cis}\left(-\dfrac{7\pi}{18}\right)$

In polar form the cube roots of $4i-4\sqrt{3}$ are: $2\,\text{cis}\left(-\dfrac{7\pi}{18}\right)$, $2\,\text{cis}\left(\dfrac{5\pi}{18}\right)$, $2\,\text{cis}\left(\dfrac{17\pi}{18}\right)$

3 marks

**Mark allocation**

- 1 mark for finding $8\,\text{cis}\left(\dfrac{5\pi}{6}\right)$.
- 1 mark for applying De Moivre's Theorem.
- 1 mark for three correct solutions.

**Question 4**

Find the point of intersection of the normals to the curve $x^2y + y^2 = 5$ at $y = 1$.

**Worked solution**

When $y = 1$, $x^2 \times 1 + 1^2 = 5$

$$x^2 = 4$$

$$x = \pm 2$$

Need to find the gradient of the normals at the points (2, 1) and (–2, 1)

$x^2y + y^2 = 5$

Using implicit differentiation to find the gradient of the tangent at these points

$$2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(x^2 + 2y\right) = -2xy$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 + 2y}$$

At (2, 1) $\quad \frac{dy}{dx} = \frac{-2\times 2\times 1}{2^2 + 2\times 1} = -\frac{2}{3} \quad \Rightarrow \quad$ gradient of normal is $\frac{3}{2}$

At (–2, 1) $\quad \frac{dy}{dx} = \frac{-2\times -2\times 1}{(-2)^2 + 2\times 1} = \frac{2}{3} \quad \Rightarrow \quad$ gradient of normal is $-\frac{3}{2}$

Equation of the normal at (2, 1): $\quad y = \frac{3}{2}x + c \quad \Rightarrow \quad 1 = \frac{3}{2}\times 2 + c,\ c = -2$

$$y = \frac{3}{2}x - 2$$

Equation of the normal at (–2, 1): $\quad y = -\frac{3}{2}x + c \quad \Rightarrow \quad 1 = -\frac{3}{2}\times(-2) + c,\ c = -2$

$$y = -\frac{3}{2}x - 2$$

The normal equations $y = \frac{3}{2}x - 2$ and $y = -\frac{3}{2}x - 2$ intersect on the *y*-axis at (0, –2)

5 marks

**Mark allocation**

- 1 mark for finding the points (2, 1) and (–2, 1).
- 1 mark for a correct method used to find derivative.
- 1 mark for finding the correct derivative.
- 1 mark for correct normal equations.
- 1 mark for correct answer (0, –2).

**TURN OVER**

**Question 5**

Given $\frac{d}{dx}\left(\arctan\left(\frac{2x}{x^2-1}\right)\right)=\frac{a}{x^2+b}$, $|x|\neq 1$

Find the real numbers $a$ and $b$.

**Worked solution**

Let $y=\arctan(u)$ where $u=\frac{2x}{x^2-1}$ $\Rightarrow$ $\frac{du}{dx}=\frac{2(x^2-1)-2x(2x)}{(x^2-1)^2}$

$$\frac{du}{dx}=\frac{-2x^2-2}{(x^2-1)^2}$$

$$\frac{du}{dx}=\frac{-2(x^2+1)}{(x^2-1)^2}$$

Applying the chain rule

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$\frac{dy}{dx}=\frac{1}{1+u^2}\times\frac{-2(x^2+1)}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{1}{1+\left(\frac{2x}{x^2-1}\right)^2}\times\frac{-2(x^2+1)}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{-2(x^2+1)}{(x^2-1)^2+(2x)^2}$$

$$\frac{dy}{dx}=\frac{-2(x^2+1)}{(x^4-2x^2+1)+4x^2}$$

$$\frac{dy}{dx}=\frac{-2(x^2+1)}{x^4+2x^2+1}$$

$$\frac{dy}{dx}=\frac{-2(x^2+1)}{(x^2+1)^2}$$

$$\frac{dy}{dx}=\frac{-2}{x^2+1}$$

Hence $a=-2$ and $b=1$

3 marks

**Mark allocation**

- 1 mark for applying the chain rule correctly.
- 1 mark for simplifying algebra.
- 1 mark for two correct answers.

**Question 6**

Find $\int e^{6x}\sqrt{e^{2x}+1}\,dx$.

**Worked solution**

Let $u = e^{2x}+1$

$\frac{du}{dx} = 2e^{2x}$

$du = 2e^{2x}dx$

$\int e^{6x}\sqrt{e^{2x}+1}\,dx$

$= \frac{1}{2}\int e^{4x}\sqrt{e^{2x}+1}\,\left(2e^{2x}dx\right)$

$= \frac{1}{2}\int \left(e^{2x}\right)^2\sqrt{e^{2x}+1}\,\left(2e^{2x}dx\right)$ If $u = e^{2x}+1, \Rightarrow e^{2x} = u-1$

$= \frac{1}{2}\int (u-1)^2\sqrt{u}\,du$

$= \frac{1}{2}\int \left(u^2-2u+1\right)u^{\frac{1}{2}}\,du$

$= \frac{1}{2}\int \left(u^{\frac{5}{2}}-2u^{\frac{3}{2}}+u^{\frac{1}{2}}\right)du$

$= \frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}}-2\times\frac{2}{5}u^{\frac{5}{2}}+\frac{2}{3}u^{\frac{3}{2}}\right)+c$

$= \frac{1}{7}u^{\frac{7}{2}}-\frac{2}{5}u^{\frac{5}{2}}+\frac{1}{3}u^{\frac{3}{2}}+c$

$= \frac{1}{7}\left(e^{2x}+1\right)^{\frac{7}{2}}-\frac{2}{5}\left(e^{2x}+1\right)^{\frac{5}{2}}+\frac{1}{3}\left(e^{2x}+1\right)^{\frac{3}{2}}+c$ where $c$ is constant

4 marks

**Mark allocation**

- 1 mark for selecting correct substitution.
- 1 mark for simplifying integral in terms of *u*.
- 1 mark for integrating correctly.
- 1 mark for correct answer.

**Tip**

- *Use substitution.*

**Question 7**

An object in a refrigerator cools according to differential equation $\frac{dT}{dt} = -k(T-3),\ k \in R$, where $T$ °C is the temperature of the object $t$ hours after it being placed in the refrigerator.
A drink with an initial temperature of 18°C is placed in the refrigerator for 1 hour, and it cools to 8°C in that time.

**a.** Show that $T = 15e^{-kt} + 3$ is a solution to this differential equation.

**Worked solution**

Given $T = 15e^{-kt} + 3$ ..... (1)

$$\Rightarrow \quad \frac{dT}{dt} = -15k\,e^{-kt}$$

$$\frac{dT}{dt} = -k\left(15e^{-kt}\right) \text{ ..... (2)}$$

From (1) $15e^{-kt} = T - 3$

Substituting into (2)

$$\Rightarrow \quad \frac{dT}{dt} = -k(T-3)$$

Therefore $T = 15e^{-kt} + 3$ is a solution of $\frac{dT}{dt} = -k(T-3)$.

**Alternative method:**

$$\frac{dT}{dt} = -k(T-3)$$

$$\frac{dt}{dT} = \frac{1}{-k(T-3)}$$

$$t = -\frac{1}{k}\int \frac{1}{T-3}\,dt$$

$$t = -\frac{1}{k}\log_e(T-3) + c, \qquad T > 3$$

$t = 0$, $T = 18$, $\quad 0 = -\frac{1}{k}\log_e(18-3) + c$

$$c = \frac{1}{k}\log_e(15)$$

$$t = -\frac{1}{k}\log_e(T-3) + \frac{1}{k}\log_e(15)$$

$$t = -\frac{1}{k}\log_e\left(\frac{T-3}{15}\right)$$

$$e^{-kt} = \frac{T-3}{15}$$

$$T = 15e^{-kt} + 3$$

2 marks

**Mark allocation**

- 1 mark for method used.
- 1 mark for showing correct answer.

**Question 7 –** continued

**b.** Find the exact value of *k*.

**Worked solution**

When $t = 1$, $T = 8$

$$8 = 15e^{-k\times 1} + 3$$

$$e^{-k} = \frac{5}{15}$$

$$k = -\log_e\left(\frac{1}{3}\right)$$

$$k = \log_e(3)$$

1 mark

**Mark allocation**

- 1 mark for correct answer.

**c.** Find the exact temperature of the drink after 2 hours.

**Worked solution**

$$T = 15e^{-kt} + 3$$

$$T = 15e^{-\log_e(3)t} + 3$$

$$T = 15e^{-t\log_e(3)} + 3$$

$$T = 15e^{\log_e(3)^{-t}} + 3$$

$$T = 15(3)^{-t} + 3$$

When $t = 2$ $\quad T = 15(3)^{-2} + 3$

$$T = \frac{15}{9} + 3$$

$$T = 4\frac{2}{3}\ °C$$

1 mark

Total 2 + 1 + 1 = 4 marks

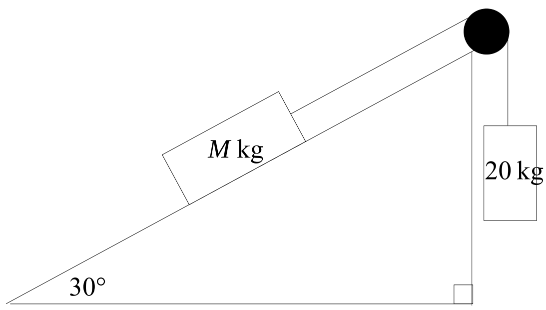
**Mark allocation**

- 1 mark for correct answer.

**Question 8**

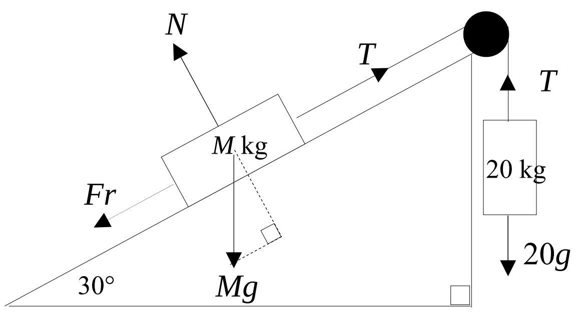
A mass of $M$ kg is connected to a 20 kg mass by a light string passing over a smooth pulley. The rough plane is inclined at 30° to the horizontal level and has coefficient of friction $\mu = \frac{1}{5}$.

The tension in the string connecting the two masses is 150 newtons.




**a.** Show that the $M$ kg mass is accelerating up the inclined plane at 2.3 m/s$^2$.

**Worked solution**




Resolving forces around the 20 kg mass. Assume this mass is moving downwards

$20g - T = 20a \qquad T = 150$ newtons

$20 \times 9.8 - 150 = 20a$

$20a = 46$

$a = 2.3\ \text{m/s}^2$

Since the acceleration is positive, the 20 kg mass is moving downwards. The masses are connected so the $M$ kg mass on the inclined plane must be moving upwards with the same acceleration.

2 marks

**Mark allocation**

- 1 mark for equation of motion for vertical mass.
- 1 mark for correct answer.

**Tip**

- *Show all forces acting on the diagram.*

**Question 8 –** continued

**b.** Determine the exact value of *M*.

**Worked solution**

Resolving forces around the *M* kg mass on the inclined plane.

This is moving upwards with an acceleration of 2.3 m/s$^2$.

$$T - Mg\sin(30°) - Fr = Ma$$

$$N = Mg\cos(30°) = \frac{\sqrt{3}Mg}{2} \text{ newtons}$$

$$Fr = \mu N = \frac{1}{5} \times \frac{\sqrt{3}Mg}{2} = \frac{\sqrt{3}Mg}{10} \text{ newtons}$$

$$150 - \frac{Mg}{2} - \frac{\sqrt{3}Mg}{10} = 2.3M$$

$$\frac{23M}{10} + \frac{Mg}{2} + \frac{\sqrt{3}Mg}{10} = 150$$

$$\frac{M}{10}\left(23 + 5g + \sqrt{3}g\right) = 150$$

$$M = \frac{1500}{23 + \left(5 + \sqrt{3}\right)g} \text{ kg}$$

3 marks
Total 2 + 3 = 5 marks

**Mark allocation**

- 1 mark for resolving forces correctly.
- 1 mark for correct substitution into the equation of motion.
- 1 mark for correct answer.

**Question 9**

The position of a particle at time *t* is given by $\underset{\sim}{r} = \left(1 - 2\sin(\pi t)\right)\underset{\sim}{i} + \left(\cos(2\pi t) + 2\right)\underset{\sim}{j}$

**a.** Find the Cartesian equation of the path of the particle.

**Worked solution**

Find *y* in terms of *x*:

$$x = 1 - 2\sin(\pi t)$$

$$\sin(\pi t) = \frac{1-x}{2} \quad \text{.....(1)}$$

$$y = \cos(2\pi t) + 2$$

$$y = 1 - 2\sin^2(\pi t) + 2$$

$$y = 3 - 2\sin^2(\pi t) \quad \text{.....(2)}$$

Substitute (1) into (2)

$$y = 3 - 2\left(\frac{1-x}{2}\right)^2$$

$$y = -\frac{1}{2}\left(x-1\right)^2 + 3$$

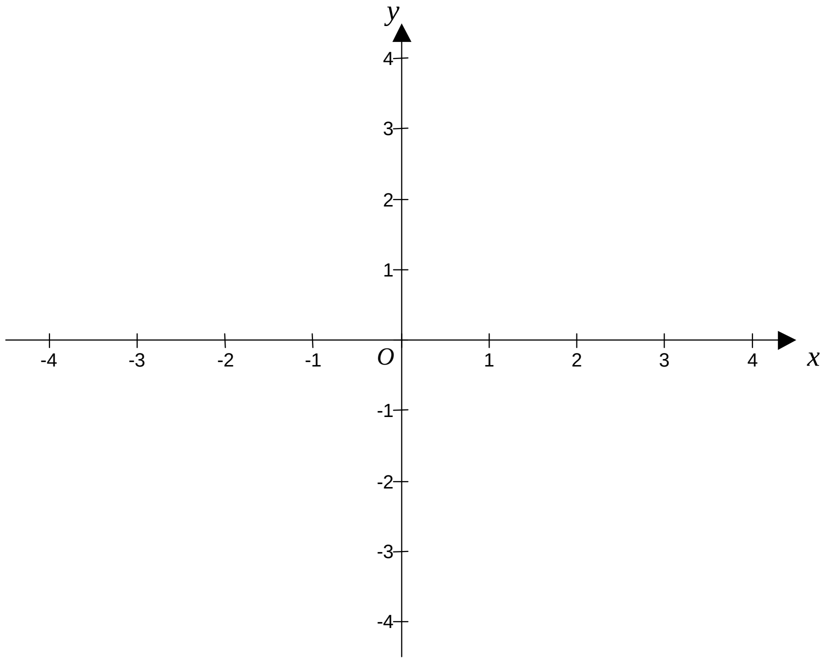
2 marks

**Mark allocation**

- 1 mark for selecting correct substitution.
- 1 mark for simplifying integral in terms of *u*.

**Question 9 –** continued
**TURN OVER**

**b.** Sketch a graph of the path of the particle for $0 \le t \le \frac{1}{2}$ indicating its direction of motion.




**Worked solution**

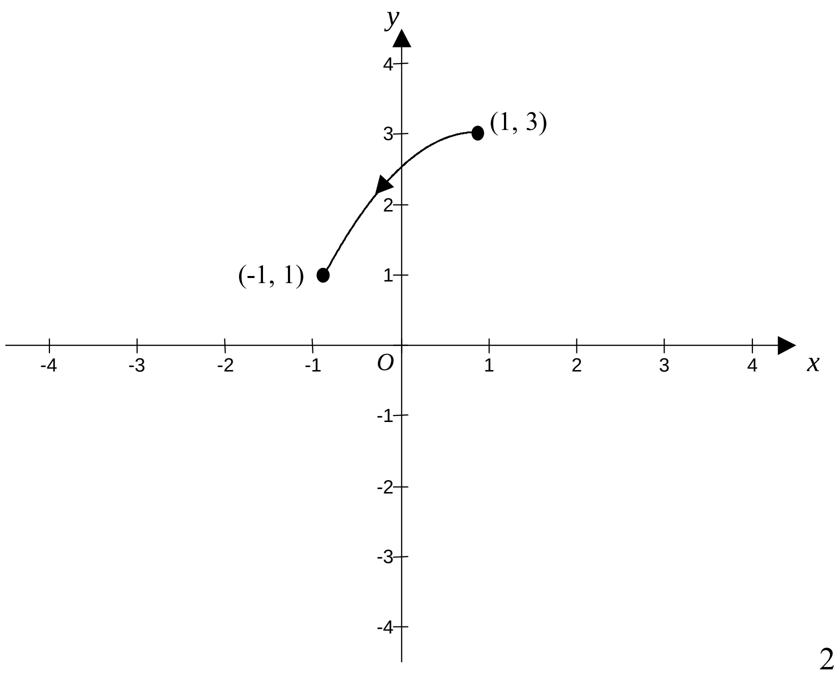
When $t = 0$ $\quad x = 1 - 2\sin(0) \qquad y = \cos(0) + 2$

$x = 1 \qquad y = 3$

When $t = \frac{1}{2}$ $\quad x = 1 - 2\sin\left(\frac{\pi}{2}\right) \qquad y = \cos(\pi) + 2$

$x = -1 \qquad y = 1$

At $t = 0$ the particle starts at the point (1, 3). It moves anticlockwise along the parabola $y = -\frac{1}{2}(x-1)^2 + 3$ to reach the point (–1, 1) at $t = \frac{1}{2}$.




2 marks

**Question 9 –** continued

**Mark allocation**

- 1 mark for drawing their curve.
- 1 mark for correct answer with direction of motion shown.

**c.** Determine the speed at which the particle is travelling when $t = \frac{1}{4}$.

**Worked solution**

Speed = $|\dot{\underset{\sim}{r}}|$

$$\underset{\sim}{r} = \left(1 - 2\sin(\pi t)\right)\underset{\sim}{i} + \left(\cos(2\pi t) + 2\right)\underset{\sim}{j}$$

$$\dot{\underset{\sim}{r}} = -2\pi\cos(\pi t)\,\underset{\sim}{i} - 2\pi\sin(2\pi t)\,\underset{\sim}{j}$$

$$|\dot{\underset{\sim}{r}}| = \sqrt{\left(-2\pi\cos(\pi t)\right)^2 + \left(-2\pi\sin(2\pi t)\right)^2}$$

$$|\dot{\underset{\sim}{r}}| = 2\pi\sqrt{\cos^2(\pi t) + \sin^2(2\pi t)}$$

When $t = \frac{1}{4}$, $|\dot{\underset{\sim}{r}}| = 2\pi\sqrt{\cos^2\left(\frac{\pi}{4}\right) + \sin^2\left(\frac{\pi}{2}\right)}$

$$|\dot{\underset{\sim}{r}}| = 2\pi\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 1^2}$$

$$|\dot{\underset{\sim}{r}}| = \sqrt{6}\,\pi$$

1 mark
Total 2 + 2 + 1 = 5 marks

**Mark allocation**

- 1 mark for correct answer.

**Tip**

- *Differentiate to find velocity vector.*

TURN OVER

**Question 10**

A cyclist of mass 72 kg is travelling on a straight track with a velocity of 1 m/s when he passes *O*. At that instant he applies a variable force of $v^3 + 3v$ newtons, where *v* m/s is his velocity *t* seconds after passing *O*. Calculate the exact distance of the cyclist from *O* when his velocity reaches $\sqrt{3}$ m/s. Assume air resistance is negligible.

4 marks

**Worked solution**

$F = ma$

$v^3 + 3v = 72a$

$a = \dfrac{v^3 + 3v}{72}$

$v\dfrac{dv}{dx} = \dfrac{v^3 + 3v}{72}$

$\dfrac{dv}{dx} = \dfrac{v^3 + 3v}{72v}$

$\dfrac{dx}{dv} = \dfrac{72}{v^2 + 3}$

$x = \int \dfrac{72}{v^2 + 3}\,dv$

$x = \dfrac{72}{\sqrt{3}}\int \dfrac{\sqrt{3}}{v^2 + 3}\,dv$

$x = \dfrac{72\sqrt{3}}{3}\tan^{-1}\left(\dfrac{v}{\sqrt{3}}\right) + c$

$x = 24\sqrt{3}\tan^{-1}\left(\dfrac{v}{\sqrt{3}}\right) + c$

When $x = 0$, $v = 1$ m/s

$0 = 24\sqrt{3}\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) + c$

$c = -24\sqrt{3} \times \dfrac{\pi}{6}$

$c = -4\sqrt{3}\pi$

$x = 24\sqrt{3}\tan^{-1}\left(\dfrac{v}{\sqrt{3}}\right) - 4\sqrt{3}\pi$

Find *x* when $v = \sqrt{3}$ m/s

$x = 24\sqrt{3}\tan^{-1}\left(\dfrac{\sqrt{3}}{\sqrt{3}}\right) - 4\sqrt{3}\pi$

$x = 24\sqrt{3}\tan^{-1}(1) - 4\sqrt{3}\pi$

$x = 24\sqrt{3} \times \dfrac{\pi}{4} - 4\sqrt{3}\pi$

$x = 6\sqrt{3}\pi - 4\sqrt{3}\pi$

**QUESTION 10 –** continued

$x = 2\sqrt{3}\pi$

The cyclist is $2\sqrt{3}\pi$ m from *O* when his velocity is $\sqrt{3}$ m/s.

**Mark allocation**

1 mark for finding the acceleration in terms of *v*.
1 mark for establishing correct integral.
1 mark for finding correct expression for *x* in terms of *v*.
1 mark for answer.

**Tip**

- *Write down the equation of motion and find the acceleration in terms of v.*

**END OF SOLUTIONS BOOK**