2009 Specialist Mathematics Trial Exam 1 Solutions
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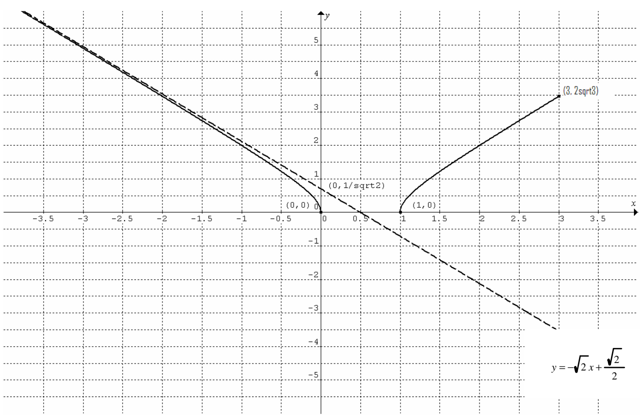
Q1a $y=\sqrt{2x(x-1)}$, $y^2=2x(x-1)$, $\frac{y^2}{2}=x^2-x$,

$\frac{y^2}{2}=\left(x-\frac{1}{2}\right)^2-\frac{1}{4}$, $\left(x-\frac{1}{2}\right)^2-\frac{y^2}{2}=\frac{1}{4}$, $\frac{\left(x-\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^2}-\frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2}=1$.

Asymptote: $y=-\frac{\frac{1}{\sqrt{2}}}{\frac{1}{2}}\left(x-\frac{1}{2}\right)=-\sqrt{2}x+\frac{\sqrt{2}}{2}$.

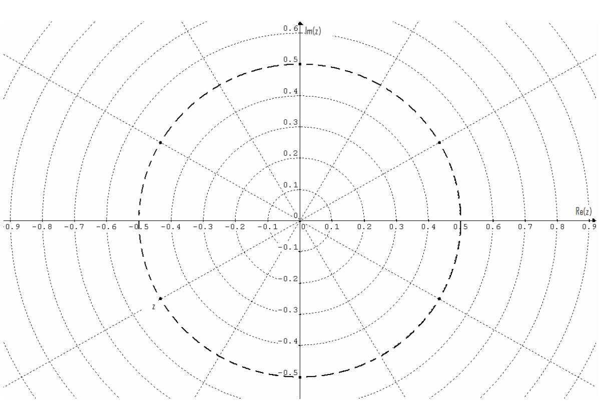


Q2a $\frac{\left(1+i\sqrt{3}\right)^5}{8(1-i)^6}=\frac{\left(2cis\frac{\pi}{3}\right)^5}{8\left(\sqrt{2}cis\left(-\frac{\pi}{4}\right)\right)^6}=\frac{1}{2}cis\left(\frac{19\pi}{6}\right)=\frac{1}{2}cis\left(-\frac{5\pi}{6}\right)$

$=\frac{1}{2}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=-\frac{\sqrt{3}}{4}-\frac{1}{4}i$.

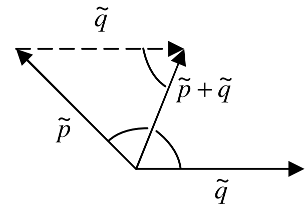
$\therefore a=-\frac{\sqrt{3}}{4}$ and $b=-\frac{1}{4}$.

Q2b Since $z=\frac{1}{2}cis\left(-\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{4}-\frac{1}{4}i$ is a solution to $64z^6+1=0$, this solution and the other 5 solutions space out equally on the circle of radius $\frac{1}{2}$ and centred at O in the complex plane.



Hence, the other 5 solutions are $-\frac{\sqrt{3}}{4}+\frac{1}{4}i$, $\frac{\sqrt{3}}{4}-\frac{1}{4}i$, $\frac{\sqrt{3}}{4}+\frac{1}{4}i$, $\frac{1}{2}i$, $-\frac{1}{2}i$.

Q3a $\tilde{p}=3\tilde{i}+4\tilde{j}$, $\tilde{q}=-3\tilde{j}+4\tilde{k}$, $\therefore |\tilde{p}|=|\tilde{q}|=5$.



$\tilde{p}+\tilde{q}$ bisects the angle between $\tilde{p}$ and $\tilde{q}$.

Let $\tilde{r}=\tilde{p}+\tilde{q}=3\tilde{i}+\tilde{j}+4\tilde{k}$, $|\tilde{r}|=\sqrt{3^2+1^2+4^2}=\sqrt{26}$.

Hence the required vector is $2\tilde{r}=6\tilde{i}+2\tilde{j}+8\tilde{k}$.

Q3b Let S be the point dividing PQ into the ratio 3:1.

$\overrightarrow{OS}=\frac{1\times\overrightarrow{OP}+3\times\overrightarrow{OQ}}{3+1}=\frac{1\left(3\tilde{i}+4\tilde{j}\right)+3\left(-3\tilde{j}+4\tilde{k}\right)}{4}$

$=\frac{3}{4}\tilde{i}-\frac{5}{4}\tilde{j}+3\tilde{k}$.

Hence S is $\left(\frac{3}{4},-\frac{5}{4},3\right)$.

Q4a Let $y=-2\left(\cos^{-1}(x+1)-\pi\right)$.

The equation of $f^{-1}(x)$ is $x=-2\left(\cos^{-1}(y+1)-\pi\right)$.

$\therefore \cos^{-1}(y+1)=-\frac{1}{2}x+\pi$, $y=\cos\left(-\frac{x}{2}+\pi\right)-1$,

$\therefore y=-\cos\left(\frac{x}{2}\right)-1$. Hence $f^{-1}(x)=-\cos\left(\frac{x}{2}\right)-1$.

Domain is $[0,2\pi]$, range is $[-2,0]$.

Q4b $g(x)=\cos x$,

$g\left(f(x)\right)=\cos(f(x))=\cos\left(-2\left(\cos^{-1}(x+1)-\pi\right)\right)$

$=\cos\left(2\left(\cos^{-1}(x+1)-\pi\right)\right)=\cos\left(2\cos^{-1}(x+1)-2\pi\right)$

$=\cos\left(2\cos^{-1}(x+1)\right)=2\left(\cos\left(\cos^{-1}(x+1)\right)\right)^2-1$

$=2(x+1)^2-1$, for $x\in[-2,0]$.

Q5 $3y^2\sqrt{x+1} = x + y$ and $\dfrac{dy}{dx} > 0$.

At $x = 0$, $3y^2 - y = 0$, $\therefore y = 0$ or $y = \dfrac{1}{3}$.

Implicit differentiation:

$$3y^2\frac{d\left(\sqrt{x+1}\right)}{dx} + \frac{d\left(3y^2\right)}{dx}\sqrt{x+1} = 1 + \frac{dy}{dx},$$

$$3y^2\frac{1}{2\sqrt{x+1}} + 6y\frac{dy}{dx}\sqrt{x+1} = 1 + \frac{dy}{dx},$$

$$6y\frac{dy}{dx}\sqrt{x+1} - \frac{dy}{dx} = 1 - \frac{3y^2}{2\sqrt{x+1}},$$

$$\left(6y\sqrt{x+1} - 1\right)\frac{dy}{dx} = 1 - \frac{3y^2}{2\sqrt{x+1}}, \ \therefore \frac{dy}{dx} = \frac{1 - \frac{3y^2}{2\sqrt{x+1}}}{6y\sqrt{x+1} - 1}.$$

$\dfrac{dy}{dx} > 0$ at $\left(0, \dfrac{1}{3}\right)$ only.

Gradient of tangent at $\left(0, \dfrac{1}{3}\right)$ is $\dfrac{dy}{dx} = \dfrac{1 - \frac{1}{6}}{2 - 1} = \dfrac{5}{6}$.

Q6a $f(x) = \tan^{-1}\left(\dfrac{1}{x}\right) + \cos^{-1}\left(\dfrac{1}{x}\right) + \sin^{-1}\left(\dfrac{1}{x}\right) = \tan^{-1}\left(\dfrac{1}{x}\right) + \dfrac{\pi}{2}$.

$$f\left(\sqrt{3}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \frac{\pi}{2} = \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3}.$$

Q6b $f(x) = \tan^{-1}\left(\dfrac{1}{x}\right) + \dfrac{\pi}{2}$, $f'(x) = \dfrac{1}{1 + \left(\frac{1}{x}\right)^2} \times \dfrac{-1}{x^2} = -\dfrac{1}{x^2 + 1}$.

$$\therefore f'\left(\sqrt{3}\right) = -\frac{1}{3+1} = -\frac{1}{4}.$$

Q7a $\displaystyle\int_0^2 f(x)dx = \int_0^2 \cos^3(x-1)dx = \int_0^2 \cos^2(x-1)\cos(x-1)dx$

$$= \int_0^2 \left(1 - \sin^2(x-1)\right)\cos(x-1)dx$$

$$= \int_{-p}^{p} \left(1 - u^2\right)du$$

$$= \left[u - \frac{u^3}{3}\right]_{-p}^{p} = 2\left(p - \frac{p^3}{3}\right).$$

Let $u = \sin(x-1)$,
$\dfrac{du}{dx} = \cos(x-1)$.
When $x = 0$,
$u = \sin(-1) = -\sin(1) = -p$.
When $x = 2$, $u = p$.

Q7b Let $u = x - 1$.

$$\int_0^2 f(x)g(x)dx = \int_0^2 \left(\cos^3(x-1)\right)(x-1)dx = \int_{-1}^{1} \left(\cos^3 u\right)u\,du.$$

Since $\cos^3 u$ is an even function and u an odd function, the product is an odd function.

$$\therefore \int_0^2 f(x)g(x)dx = \int_{-1}^{1} \left(\cos^3 u\right)u\,du = 0.$$

Q8a $\tilde{r}(0) = \tilde{i} + 2\tilde{j}$, $\tilde{r}(2) = 3\tilde{i} - 2\tilde{j} - 6\tilde{k}$.

Displacement $= \tilde{r}(2) - \tilde{r}(0) = 2\tilde{i} - 4\tilde{j} - 6\tilde{k}$.

Q8b Velocity $= \dfrac{d\tilde{r}}{dt} = \tilde{i} - 2\tilde{j} - 3\tilde{k}$ is constant.

$\therefore$ the particle moves in a straight line.

Q8c From velocity $= \tilde{i} - 2\tilde{j} - 3\tilde{k}$, $\tan\theta = \dfrac{-3}{\sqrt{1^2 + (-2)^2}} = -\dfrac{3}{\sqrt{5}}$.

Hence $\theta = \tan^{-1}\left(-\dfrac{3}{\sqrt{5}}\right) = -\tan^{-1}\left(\dfrac{3}{\sqrt{5}}\right)$.

Q9a $\tilde{v}(0) = 2\tilde{i}$, $\tilde{v}\left(\sqrt{5}\right) = 2\tilde{i} - 5\tilde{j} + 2.5\tilde{k}$.

Change in momentum $= 4.0\left(\tilde{v}\left(\sqrt{5}\right) - \tilde{v}(0)\right) = -20\tilde{j} + 10\tilde{k}$ kg ms^{-1}.

Q9b $\tilde{a}(t) = \dfrac{d\tilde{v}}{dt} = (-2t)\tilde{j} + \dfrac{\sqrt{5}}{2}\tilde{k}$, $\therefore \tilde{a}(0) = \dfrac{\sqrt{5}}{2}\tilde{k}$.

Initial resultant force $= m\tilde{a}(0) = 2\sqrt{5}\tilde{k}$ N.

Q9c Speed $= \sqrt{2^2 + \left(-t^2\right)^2 + \left(\frac{t\sqrt{5}}{2}\right)^2} = \sqrt{4 + t^4 + \frac{5t^2}{4}} = 2.5$.

$\therefore 4t^4 + 5t^2 - 9 = 0$, $\left(4t^2 + 9\right)(t+1)(t-1) = 0$.

Since $t \ge 0$, $\therefore t = 1$ s.

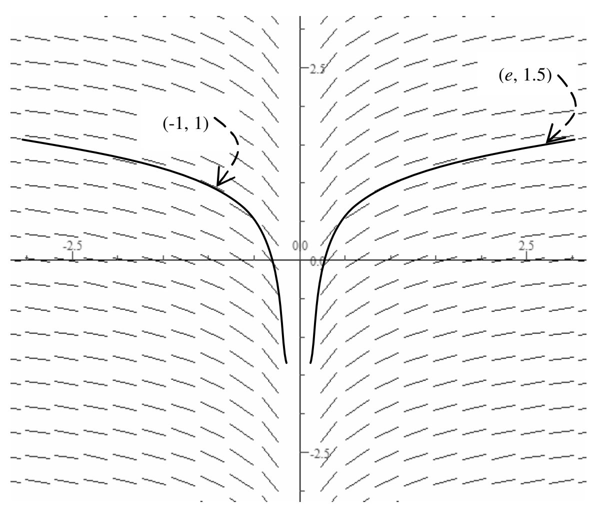
Q10a $y = a\log_e|x| + c$, where a and c are constants. $\therefore \dfrac{dy}{dx} = \dfrac{a}{x}$.

Q10b $(-1,1) \to 1 = a\log_e|-1| + c$, $\therefore c = 1$.

$(e,1.5) \to 1.5 = a\log_e e + 1$, $\therefore a = 0.5$.

The particular solution is $y = 0.5\log_e|x| + 1$.

Q10c



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