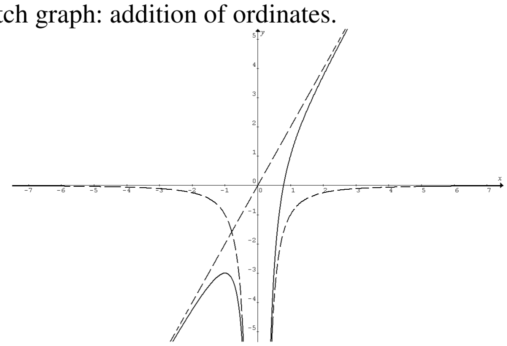
2009 VCAA Specialist Math Exam 2 Solutions

Section 1

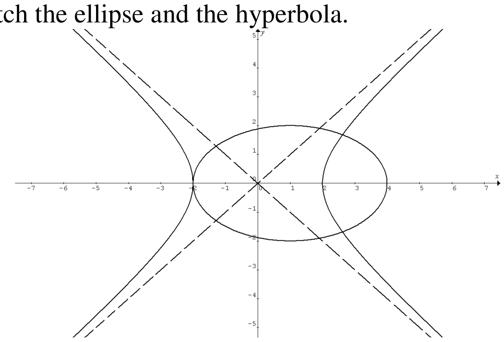
1	2	3	4	5	6	7	8	9	10	11
E	D	D	A	C	*A*	C	E	B	B	C

12	13	14	15	16	17	18	19	20	21	22
E	A	B	C	D	D	B	A	E	D	B

Q1 Sketch graph: addition of ordinates. E



Q2 Sketch the ellipse and the hyperbola. D



Q3 $-1 \le x-b \le 1$, $b-1 \le x \le b+1$, $b-1=2$, $b=3$.
$a\pi = 6\pi$, $a=6$. D

Q4 $\sec t = \frac{x+1}{2}$, $\tan t = \frac{y-1}{3}$, $\sec^2 t = 1+\tan^2 t$,

$\left(\frac{x+1}{2}\right)^2 = 1+\left(\frac{y-1}{3}\right)^2$, $\therefore \frac{(x+1)^2}{4} - \frac{(y-1)^2}{9} = 1$ A

Q5 $x^2+2ax+2y^2+4by=-16$, $x^2+2ax+2\left(y^2+2by\right)=-16$,

$(x+a)^2 + \frac{(y+b)^2}{\frac{1}{2}} = const$. $\therefore a=-3$ and $b=2$. C

Q6 Distance between z and $-\bar{z} = \left|z-(-\bar{z})\right| = \left|z+\bar{z}\right| = \left|2\operatorname{Re}(z)\right|$
Best choice A

Q7 The conjugate $z=-2-i$ is also a root of $P(z)=0$. C

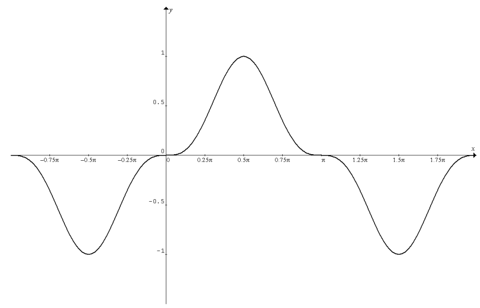
Q8 $(1+i)^n = ai$, $(1+i)^{2n+2} = \left((1+i)^n\right)^2(1+i)^2 = (ai)^2(2i) = -2a^2 i$
E

Q9 $\frac{(x-6)^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$.

Implicit differentiation: $\frac{2(x-6)}{a^2} + \frac{2(y-3)}{b^2}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{b^2(x-6)}{a^2(y-3)} = \frac{b^2(6-x)}{a^2(y-3)}$. B

Q10



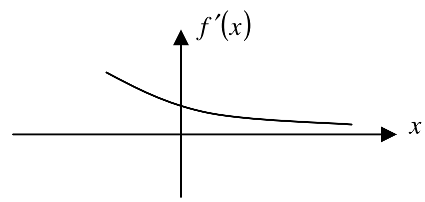
$\text{Area} = 3\times\int_0^{\pi} (\sin x)^3\,dx = 3\times\int_0^{\pi} (\sin x)^2 \sin x\,dx$

$= 3\times\int_0^{\pi} \left(1-\cos^2 x\right)\sin x\,dx$

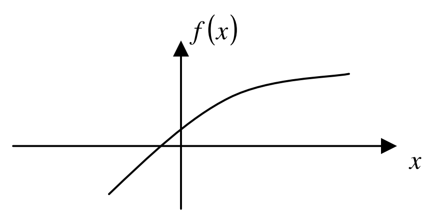
Let $u=\cos x$, $-\frac{du}{dx} = \sin x$

$= -3\times\int_1^{-1} \left(1-u^2\right)du = 3\times\int_{-1}^{1} \left(1-u^2\right)du$ B

Q11 $f'(x) > 0$ and $f''(x) < 0$, the graph of $f'(x)$ would be



and a corresponding graph of $f(x)$ would be



C

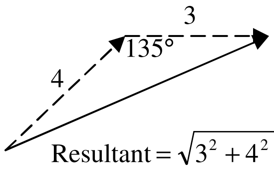
Q12 $v=f(x)$, $a = v\frac{dv}{dx} = f(x)f'(x)$ E

Q13 Let $Q=100x$ kg be the amount of salt in the tank at time t minutes (Note: t seconds in the question) . Rate of salt input = 0, and rate of salt output $= 10x$ kg per min.

$\therefore \frac{dQ}{dt} = -10x$, $100\frac{dx}{dt} = -10x$, $\therefore 10\frac{dx}{dt} + x = 0$ A

Q14 $\tilde{u} = \tilde{v} = \tilde{w}$ when $m = 1$. $\therefore 2\tilde{u} - \tilde{v} - \tilde{w} = \tilde{0}$ as an example. Hence they are linearly dependent. B

Q15



$$\text{Resultant} = \sqrt{3^2 + 4^2 - 2.3.4\cos 135°}$$
$$= \sqrt{25 + 12\sqrt{2}}$$ C

Q16 $\tilde{c} \bullet \tilde{a} = 0$, $\tilde{c} \bullet \tilde{b} = 0$, $\therefore \tilde{a}$ and $\tilde{b}$ are perpendicular to $\tilde{c}$. D

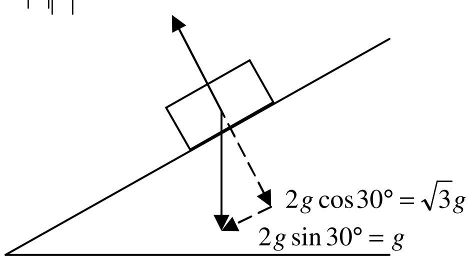
Q17 $\tilde{c} + \tilde{b} = \tilde{a}$, $\therefore \tilde{c} = \tilde{a} - \tilde{b}$

$\therefore \tilde{c} \bullet \tilde{c} = (\tilde{a} - \tilde{b}) \bullet (\tilde{a} - \tilde{b}) = \tilde{a} \bullet \tilde{a} - 2\tilde{a} \bullet \tilde{b} + \tilde{b} \bullet \tilde{b}$

$\therefore |\tilde{c}|^2 = |\tilde{a}|^2 + |\tilde{b}|^2 - 2|\tilde{a}||\tilde{b}|\cos 120°$

$\therefore |\tilde{c}|^2 = |\tilde{a}|^2 + |\tilde{b}|^2 + |\tilde{a}||\tilde{b}|$ D

Q18



Force of friction $= \mu N = 0.1 \times \sqrt{3}g = \frac{\sqrt{3}g}{10}$

$F = ma$, $g - \frac{\sqrt{3}g}{10} = 2a$, $g\left(1 - \frac{\sqrt{3}}{10}\right) = 2a$ B

Q19 Vertical: $u = 20\sin 45° = +10\sqrt{2}$, $v = -10\sqrt{2}$, $a = -g$.

Substitute into $v = u + at$, $-10\sqrt{2} = 10\sqrt{2} - gt$, $t = \frac{20\sqrt{2}}{g}$ A

Q20 $v = x$, $\frac{dx}{dt} = x$, $\frac{dt}{dx} = \frac{1}{x}$, $t = \log_e x + c$.

When $t = 3$, $x = 1$, $\therefore c = 3$ and $t = \log_e x + 3$.

Hence $x = e^{t-3}$. E

Q21 $u = +4$, $a = +2$, $s = +21$, to find *v*, substitute into $v^2 = u^2 + 2as$. $\therefore v = +10$.

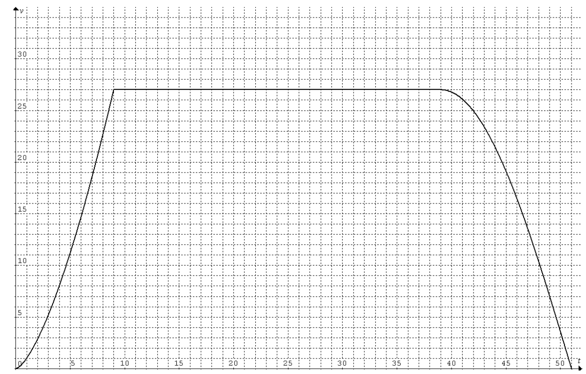
Magnitude of momentum $= 5 \times 10 = 50$ kg ms^{-1} D

Q22 To find the distance from the starting point, firstly find the displacement from the starting point = signed area bounded by the graph and the *t*-axis.

$s = \frac{1}{2} \times 10 \times 2 - \frac{1}{2}(1+4)5 + \frac{1}{2} \times 10 \times 4 = 17.5$ B

Section 2

Q1a



Q1b Distance $= \int_0^9 t^{\frac{3}{2}} dt = \left[\frac{2t^{\frac{5}{2}}}{5}\right]_0^9 = 97.2$ m

Q1c Distance $= \int_{39}^{51} 27\cos\left(\frac{\pi}{24}(t - 39)\right)dt$

$= \left[\frac{24 \times 27\sin\left(\frac{\pi}{24}(t-39)\right)}{\pi}\right]_{39}^{51} = \frac{648}{\pi} = 206.3$ m

Q1d Average speed $= \frac{total.distance}{time.taken}$

$= \frac{97.2 + 27 \times 30 + 206.3}{51} = 21.8$ ms^{-1}

Q1e Let $t^{\frac{3}{2}} = \frac{200}{9}$, $t = t_1 = \left(\frac{200}{9}\right)^{\frac{2}{3}} \approx 7.9$ s

Let $27\cos\left(\frac{\pi}{24}(t - 39)\right) = \frac{200}{9}$, use calc. to find $t = t_2 = 43.6$ s

Q1f Let *T* be the time in seconds, where $9 < T < 39$ (refer to the graph).

Distance by motorcycle = distance by car

$20T = 97.2 + 27(T - 9)$,

$T = 20.829 \approx 20.8$ s and distance $= 20T = 20 \times 20.829 \approx 417$ m

Q2a Let $z = 0$, $|-1| = 1$, $\left|-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right| = 1$.

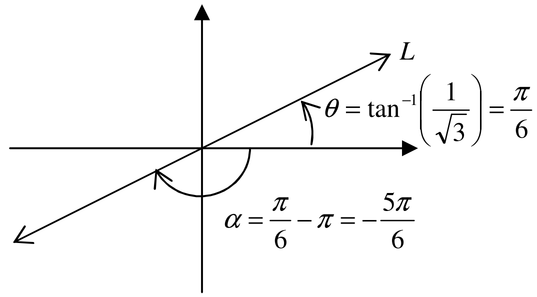
$\therefore (0,0)$ lies on *L*.

Q2b Let $z = x + yi$. $|(x-1) + yi| = \left|\left(x - \frac{1}{2}\right) + \left(y - \frac{\sqrt{3}}{2}\right)i\right|$

$(x-1)_2 + y^2 = \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{\sqrt{3}}{2}\right)^2$.

Expand and simplify to $y = \frac{1}{\sqrt{3}}x$.

Q2c



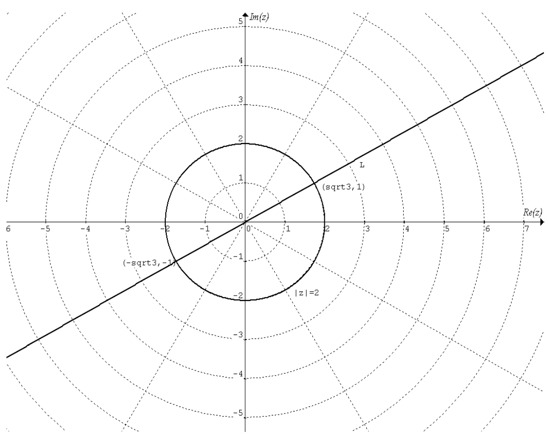
Q2d $|z| = 2$ is $x^2 + y^2 = 4$ ……(1)

L is $y = \frac{1}{\sqrt{3}}x$ ……(2)

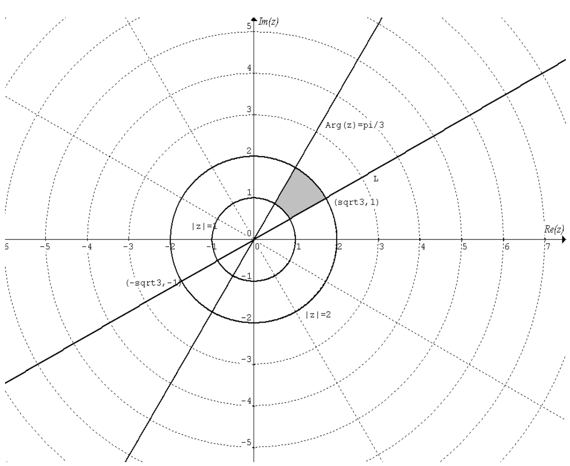
Substitute (2) into (1): $x^2 + \frac{x^2}{3} = 4$, $\therefore x^2 = 3$, $x = \pm\sqrt{3}$ and $y = \pm 1$.

The points of intersection are $\left(-\sqrt{3}, -1\right)$ and $\left(\sqrt{3}, 1\right)$.

Q2e



Q2f



Shaded area $= \frac{1}{12}\left(\pi 2^2 - \pi 1^2\right) = \frac{\pi}{4}$ square units.

Q3a $\tilde{r} = 5\sin\left(\frac{\pi}{6}t\right)\tilde{i} + 5\cos\left(\frac{\pi}{6}t\right)\tilde{j} + \left(24.5 - \frac{t^2}{8}\right)\tilde{k}$.

The height above the ground at time t is given by $24.5 - \frac{t^2}{8}$.

At $t = 0$, height $= 24.5$ metres.

Q3b Let $24.5 - \frac{t^2}{8} = 0$, $t = 14$ s.

Q3c Period of one loop $= \frac{2\pi}{\frac{\pi}{6}} = 12$ s, time taken = 12 s.

Q3d $\dot{\tilde{r}} = \frac{5\pi}{6}\cos\left(\frac{\pi}{6}t\right)\tilde{i} - \frac{5\pi}{6}\sin\left(\frac{\pi}{6}t\right)\tilde{j} - \frac{t}{4}\tilde{k}$

Q3e At $t = 14$,

$\dot{\tilde{r}} = \frac{5\pi}{6}\cos\left(\frac{7\pi}{3}\right)\tilde{i} - \frac{5\pi}{6}\sin\left(\frac{7\pi}{3}\right)\tilde{j} - \frac{7}{2}\tilde{k}$

$= 1.309\tilde{i} - 2.267\tilde{j} - 3.5\tilde{k}$.

Speed $= \left|\dot{\tilde{r}}\right| = \sqrt{1.309^2 + 2.267^2 + 3.5^2} \approx 4.4$ ms^{-1}.

Q3f $\tilde{a} = \ddot{\tilde{r}} = -\frac{5\pi^2}{36}\sin\left(\frac{\pi}{6}t\right)\tilde{i} - \frac{5\pi^2}{36}\cos\left(\frac{\pi}{6}t\right)\tilde{j} - \frac{1}{4}\tilde{k}$

$|\tilde{a}| = \sqrt{\left(\frac{5\pi^2}{36}\sin\left(\frac{\pi}{6}t\right)\right)^2 + \left(\frac{5\pi^2}{36}\cos\left(\frac{\pi}{6}t\right)\right)^2 + \frac{1}{16}}$

$= \sqrt{\left(\frac{5\pi^2}{36}\right)^2\left(\sin^2\left(\frac{\pi}{6}t\right) + \cos^2\left(\frac{\pi}{6}t\right)\right) + \frac{1}{16}}$

$= \sqrt{\left(\frac{5\pi^2}{36}\right)^2 + \frac{1}{16}}$ is a constant.

Q3gi $\dot{\tilde{r}} = \frac{5\pi}{6}\cos\left(\frac{\pi}{6}t\right)\tilde{i} - \frac{5\pi}{6}\sin\left(\frac{\pi}{6}t\right)\tilde{j} - \frac{t}{4}\tilde{k}$,

$\therefore \left|\dot{\tilde{r}}\right| = \sqrt{\frac{25\pi^2}{36} + \frac{t^2}{16}}$.

Distance from start to finish $= \int_0^{14}\sqrt{\frac{25\pi^2}{36} + \frac{1}{16}t^2}\,dt$.

Q3gii Evaluate the definite integral by graphics calc.
Distance ≈ 45.7 metres.

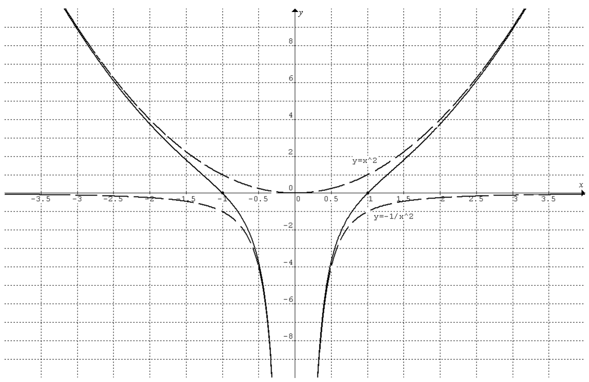
Q4a Let $\frac{x^4 - 1}{x^2} = -10$, use graphics calc. to find $x = \pm 0.315 \approx \pm 0.3$.

Let $\frac{x^4 - 1}{x^2} = 10$, use graphics calc. to find $x \approx \pm 3.2$.

$\therefore [b, a] \approx [0.3, 3.2]$, $\therefore a \approx 3.2$ and $b \approx 0.3$.

Q4b Let $\frac{x^4-1}{x^2}=0$ to find the x-intercepts. $x=\pm 1$.

$y=\frac{x^4-1}{x^2}=x^2-\frac{1}{x^2}$. Sketch by addition of ordinates.



Q4c $x^4-yx^2-1=0$, $\left(x^2\right)^2-yx^2-1=0$,

$\therefore x^2=\frac{-(-y)\pm\sqrt{(-y)^2-4(1)(-1)}}{2(1)}=\frac{y\pm\sqrt{y^2+4}}{2}$.

Since $x^2>0$, [note: $x\neq 0$ (refer to $y=\frac{x^4-1}{x^2}$)],

$x^2=\frac{y-\sqrt{y^2+4}}{2}$ is rejected.

$\therefore x^2=\frac{y+\sqrt{y^2+4}}{2}$.

Q4di $V=\int_{-10}^{10}\pi x^2dy=\int_{-10}^{10}\frac{\pi}{2}\left(y+\sqrt{y^2+4}\right)dy$

Q4dii Use graphics calc. to evaluate the definite integral, $V=174.7$ cm^3.

Q4e $\frac{dV}{dt}=+1.5$ cm^{3}s^{-1}, $\frac{dV}{dy}=\pi x^2$.

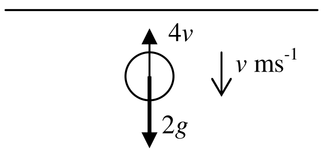
$\frac{dy}{dt}=\frac{dy}{dV}\times\frac{dV}{dt}=\frac{1}{\pi x^2}\times\frac{dV}{dt}$.

When the surface is 6 cm from the top, $y=4$ and

$x^2=\frac{4+\sqrt{16+4}}{2}=4.236$.

$\therefore\frac{dy}{dt}=\frac{1}{\pi\times 4.236}\times 1.5\approx 0.11$ cm per second.

Q5a



$a=\frac{F}{m}=\frac{2g-4v}{2}=g-2v$.

Q5b $a=g-2v$, $\frac{dv}{dt}=g-2v$, $\frac{dt}{dv}=\frac{1}{g-2v}$, $t=\int\frac{1}{g-2v}dv$,

$t=-\frac{\log_e(g-2v)}{2}+c$.

$v=0$ when $t=0$, $\therefore c=\frac{\log_e g}{2}$ and $t=0.5\log_e\left(\frac{g}{g-2v}\right)$.

Q5c $a=g-2v\to 0$ when $v\to\frac{g}{2}$, the limiting velocity.

Q5d When $v=\frac{g}{4}$,

$t=0.5\log_e\left(\frac{g}{g-\frac{g}{2}}\right)=0.5\log_e 2=\log_e\sqrt{2}$ s after its release.

Q5e $v=\frac{g}{2}\left(1-e^{-2t}\right)$, $\frac{dx}{dt}=\frac{g}{2}\left(1-e^{-2t}\right)$, where x metres is the displacement from the surface.

$x=\int_0^{180}\frac{g}{2}\left(1-e^{-2t}\right)dt=879.6$, evaluated by graphics calc.

The ocean is 880 metres at that location.

Q5f When $v=\frac{g}{3}$, $t=0.5\log_e\left(\frac{g}{g-\frac{2g}{3}}\right)=0.5\log_e 3=\log_e\sqrt{3}$,

$x=\int_0^{\log_e\sqrt{3}}\frac{g}{2}\left(1-e^{-2t}\right)dt\approx 1.1$, evaluated by graphics calc.

The device is 1.1 m below the surface.

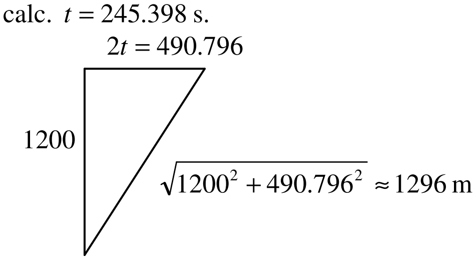
Q5g $\frac{dx}{dt}=\frac{g}{2}\left(1-e^{-2t}\right)$,

$x=\frac{g}{2}\int\left(1-e^{-2t}\right)dt=\frac{g}{2}\left(t+\frac{e^{-2t}}{2}\right)+c$.

$x=0$ when $t=0$, $\therefore x=\frac{g}{2}\left(t+\frac{e^{-2t}}{2}\right)-\frac{g}{4}$.

$x=1200$, $1200=\frac{g}{2}\left(t+\frac{e^{-2t}}{2}\right)-\frac{g}{4}$.

By graphics calc. $t=245.398$ s.



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