The Mathematical Association of Victoria
Trial Exam 2009

SPECIALIST MATHEMATICS

STUDENT NAME ____

Written Examination 2

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of Book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	**22**	**22**	**22**
2	**5**	**5**	**58**
			Total 80

Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved graphics calculator or approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.

Students are NOT permitted to bring into the examination room: notes of any kind, a calculator of any type, blank sheets of paper and/or white out liquid/tape.

Materials supplied
Question and answer book of 24 pages with a detachable sheet of miscellaneous formulas at the back
Answer sheet for multiple-choice questions.

Instructions
Detach the formula sheet from the back of this book during reading time.
Write your name in the space provided above on this page.
All written responses must be in English.
At the end of the examination place the answer sheet for multiple-choice questions inside the front cover of this book.
Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions

Answer **all** questions on the answer sheet provided for multiple choice questions.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$

Question 1

In the Cartesian plane, a vector parallel to the line $2x+3y+3=0$ is

A. $2\underset{\sim}{i}+3\underset{\sim}{j}$

B. $-\frac{2}{3}\underset{\sim}{i}-\underset{\sim}{j}$

C. $\frac{2}{3}\underset{\sim}{i}+\underset{\sim}{j}$

D. $3\underset{\sim}{i}-2\underset{\sim}{j}$

E. $2\underset{\sim}{i}-3\underset{\sim}{j}$

Question 2

Let $\underset{\sim}{a}=\underset{\sim}{i}+2\underset{\sim}{j}-2\underset{\sim}{k}$ and $\underset{\sim}{b}=2\underset{\sim}{i}+m\underset{\sim}{j}+3\underset{\sim}{k}$

If the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ equals $\hat{\underset{\sim}{b}}$, then the magnitude of $\underset{\sim}{b}$ is equal to

A. $5+m$

B. $13+m^2$

C. $2m-4$

D. $\frac{1}{2m-4}$

E. $2m+8$

Question 3

A hyperbola has equation $\frac{4x^2}{b^2} - \frac{(y-3)^2}{a^2} = 1$. The product of the gradients of the asymptotes are

A. $\frac{b^2}{a^2}$

B. $-\frac{b^2}{a^2}$

C. $-\frac{4b^2}{a^2}$

D. $\frac{4a^2}{b^2}$

E. $-\frac{4a^2}{b^2}$

Question 4

The curve specified by the parametric equations $x = \tan(2t)$, $y = \sec(2t)$, $t \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ is situated in which quadrant/s?

A. first

B. second

C. third

D. fourth

E. first and second

Question 5

$|z| + |z+i| = 2$ represents

A. the x-axis

B. the y-axis

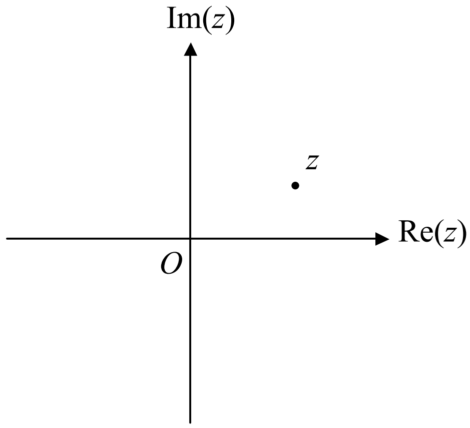
C. a parabola

D. a circle

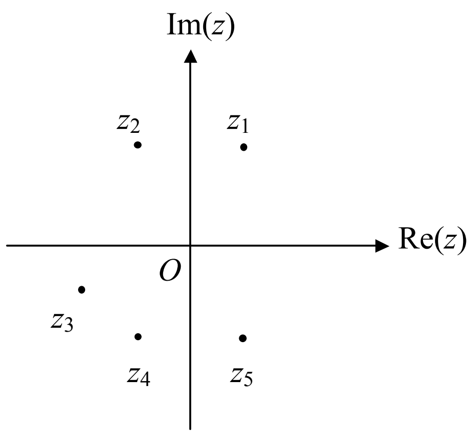
E. an ellipse

Question 6

The complex number z is shown in the diagram below.



Which one of the following could represent the complex number $\dfrac{\bar{z}}{i^3}$?



A. z_1

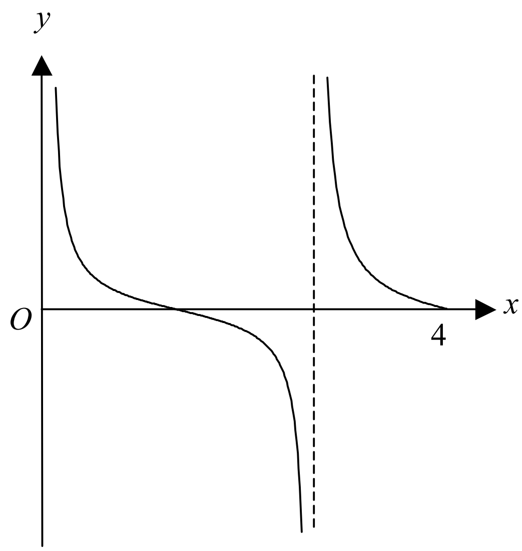
B. z_2

C. z_3

D. z_4

E. z_5

Question 7



The graph $f:[0, 4] \rightarrow R,\ f(x) = \cot(ax)$ is shown above.

The value of a is

A. $\dfrac{3\pi}{8}$

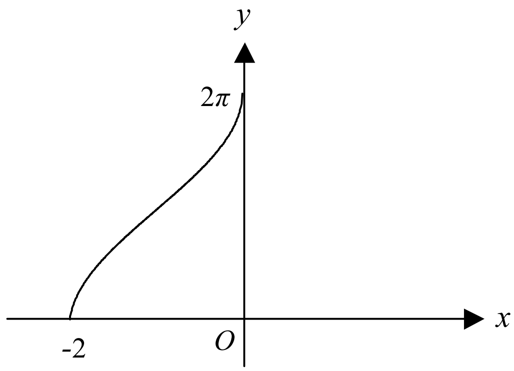
B. $\dfrac{\pi}{2}$

C. $\dfrac{3\pi}{4}$

D. $\dfrac{2\pi}{3}$

E. $\dfrac{8\pi}{3}$

Question 8



A possible equation of the curve shown above could be

A. $y = \arcsin(x+2) + \frac{3\pi}{2}$

B. $y = 2\arcsin(x+1) + \frac{\pi}{2}$

C. $y = 2\arcsin(x-1) + \pi$

D. $y = 2\arccos(1-x)$

E. $y = 2\arccos(-x-1)$

Question 9

Written as partial fractions, $\frac{2x+1}{(x-9)^2}$ would take the form

A. $\frac{A}{x-9} + \frac{B}{(x-9)^2}$

B. $\frac{A}{x-3} + \frac{B}{(x-3)^2}$

C. $\frac{A}{x-3} + \frac{B}{x+3}$

D. $\frac{A}{x-9} + \frac{Bx+C}{(x-9)^2}$

E. $\frac{A}{x-9} + \frac{B}{x+9}$

Question 10

The solution to $\frac{dy}{dx} = f(x)$ at $x = m$, given $y = b$ when $x = a$ is

A. $\int_a^b f(x)\,dx$

B. $\int_a^b f(x)\,dx + m$

C. $\int_a^m f(x)\,dx$

D. $\int_a^m f(x)\,dx + b$

E. $\int_a^m f(x)\,dx - b$

Question 11

Using a suitable substitution may $\int (x - 4\sqrt{2-x}\,)\,dx$ be expressed as

A. $\int (2 - u + 4\sqrt{u}\,)\,du$

B. $\int (2 - u - 4\sqrt{u}\,)\,du$

C. $\int (u - 2 + 4\sqrt{u}\,)\,du$

D. $\int (u - 4\sqrt{u}\,)\,du$

E. $-\int ((u+2)\sqrt{u}\,)\,du$

Question 12

Euler's method with a step size of 0.1 is used to find an approximate solution to the differential equation $\frac{dy}{dx} = \sin^{-1}(x - 2y)$ with the initial condition $y = 1$ when $x = 2$. When $x = 2.2$ the approximate value of y is

A. 1

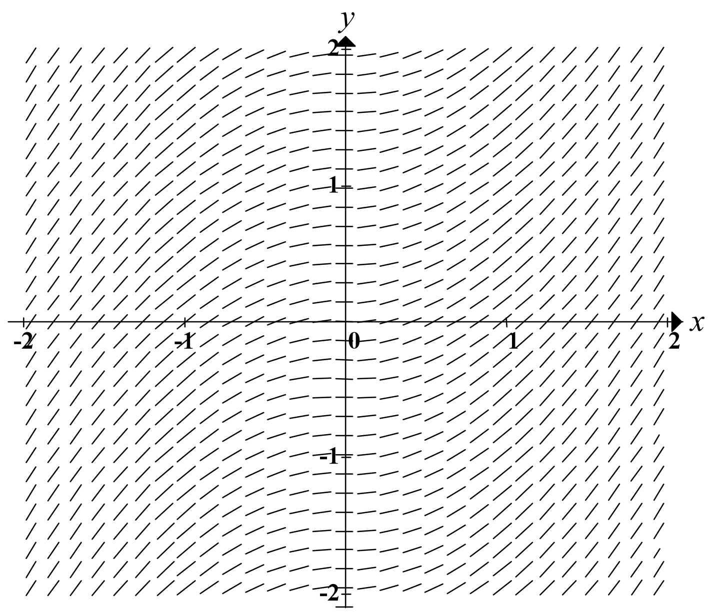
B. 1.1

C. 1.2

D. $1 + 0.1\sin^{-1}(0.1)$

E. $1.1 + \sin^{-1}(0.1)$

Question 13



The direction (slope) field for a particular first order differential equation is shown above.
The differential equation could be

A. $\dfrac{dy}{dx} = \tan(x)$

B. $\dfrac{dy}{dx} = \sec^2(x)$

C. $\dfrac{dy}{dx} = |\,x\,|$

D. $\dfrac{dy}{dx} = \dfrac{1}{2}x^3$

E. $\dfrac{dy}{dx} = 2x^4$

Question 14

The region enclosed by the curve $y = 2\cos(2x)$, the line $y = 1$ and the y-axis is rotated about the x-axis to form a solid of revolution. The volume may be found by evaluating

A. $\pi\int_0^{\frac{\pi}{6}} \left(2\cos(2x) - 1\right) dx$

B. $\pi\int_0^{\frac{\pi}{6}} \left(2\cos(4x) + 1\right) dx$

C. $\pi\int_1^{2} 4\cos^2(2x)\, dx$

D. $\pi\int_1^{2} \left(4\cos^2(2x) - 1\right) dx$

E. $\pi\int_0^{\frac{\pi}{3}} \left(4\cos^2(2x) - 1\right) dx$

Question 15

If the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel, which of the following is **NOT** necessarily true?

A. $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly dependent

B. $\underset{\sim}{a} = \mathrm{k}\underset{\sim}{b}$, where $\mathrm{k} \in \mathrm{R}$

C. $\underset{\sim}{a} = \frac{\mathrm{n}}{\mathrm{m}}\underset{\sim}{b}$, where $\mathrm{m}, \mathrm{n} \in R$

D. $\underset{\sim}{a} \cdot \underset{\sim}{b} = 1$

E. $\hat{\underset{\sim}{a}} \cdot \hat{\underset{\sim}{b}} = 1$

Question 16

A tennis player hits a tennis ball. The position vector of the tennis ball at time t, is given by

$\underset{\sim}{r}(t) = 10t\,\underset{\sim}{i} + (2 + 7t - 4t^2)\,\underset{\sim}{j}$ for $t \geq 0$, where $\underset{\sim}{i}$ is a unit vector in the forward direction and $\underset{\sim}{j}$ is a unit vector in the vertical direction.

When the tennis ball strikes the ground its velocity vector is

A. $10\,\underset{\sim}{i} - 34.75\,\underset{\sim}{j}$

B. $0\,\underset{\sim}{i} + 0\,\underset{\sim}{j}$

C. $20\,\underset{\sim}{i} - 9\,\underset{\sim}{j}$

D. $10\,\underset{\sim}{i} + 5\,\underset{\sim}{j}$

E. $10\,\underset{\sim}{i} - 9\,\underset{\sim}{j}$

Question 17

A particle is moving along the curve $y = xe^{\sin(x)}$ at a constant velocity of 2 cm/s parallel to the x-axis. The particle's velocity, in cm/s, parallel to the y-axis at $x = 2\pi$ cm is equal to

A. $\pi + 0.5$

B. 2π

C. 4π

D. $2\pi + 1$

E. $4\pi + 2$

Question 18

A bike rider travelling on a straight road with a velocity of 1 m/s accelerates at a constant rate of 2 m/s^2 over 56 m. If the mass of the bike rider is 60 kg, his change in momentum, in kg m/s, is

A. 780

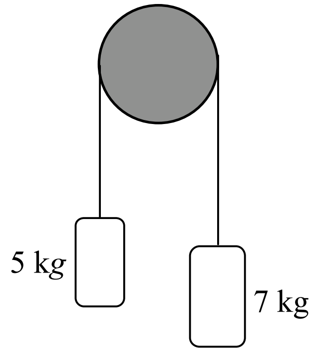
B. 840

C. 900

D. 960

E. 1020

Question 19



A 5 kg mass and a 7 kg mass are connected by a light string passing over a smooth pulley. If the connected masses are moving freely under the force of gravity, the acceleration, in m/s^2, will be

A. $\frac{g}{6}$

B. g

C. $2g$

D. $\frac{10g}{3}$

E. $6g$

Question 20

The acceleration of a body at time t seconds is given by $\frac{dv}{dt} = -\frac{4}{v}$ cm/s^2, where v is the velocity of the particle at time t. If the initial velocity of the object is −4 cm/s, the velocity at time t is given by

A. $-2\sqrt{4-2t}$

B. $2\sqrt{4-2t}$

C. $4-2\sqrt{2t}$

D. $2\sqrt{2t+4}$

E. $-\sqrt{16-2t}$

Question 21

An object of mass 2 kg, initially stationary, is acted on by a horizontal variable force of $3 + 2t$ newtons. After 4 seconds, the velocity of the object in m/s is

A. 0

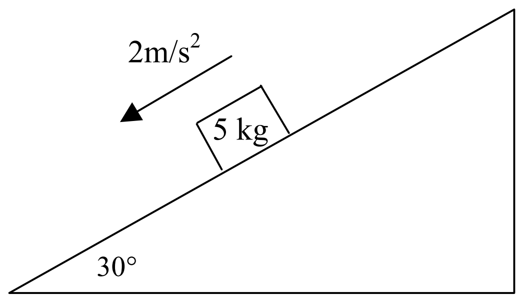
B. 5.5

C. 11

D. 14

E. 28

Question 22



A block of mass 5 kg slides down a rough slope inclined at 30° to the horizontal level. The block is sliding with an acceleration of 2 m/s^2. Correct to two decimal places, the magnitude of the coefficient of friction between the block and the rough surface is

A. 0.20

B. 0.25

C. 0.34

D. 0.58

E. 0.81

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this booklet are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude g m/s^2, where g = 9.8

Question 1

a. Given $\sin(\theta) = \frac{x}{2}$, where $0 \le \theta \le \frac{\pi}{2}$, find $\sin(2\theta)$

2 marks

b. Using the substitution $x = 2\sin(\theta)$, show that $\int \sqrt{4-x^2}\,dx = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{4-x^2} + c$

4 marks

c. **Hence** find the area bounded by the curve $y = \sqrt{4-x^2}$ and the x-axis.

3 marks

d. Find the **exact** value for the volume formed when $y = \sqrt{4 - x^2}$ is rotated about the y-axis.

3 marks

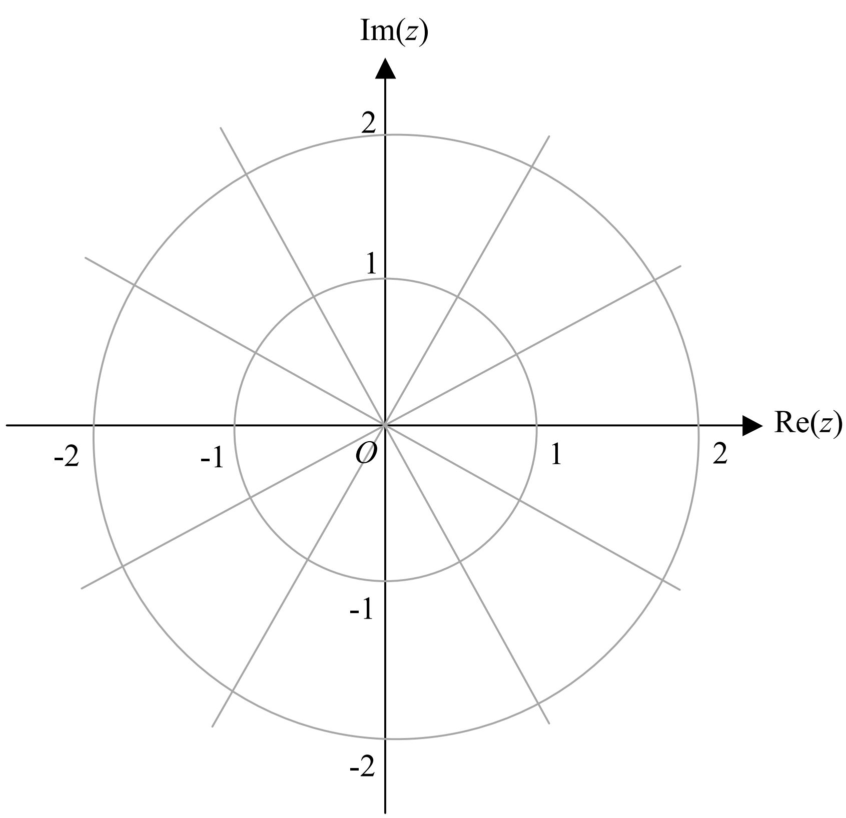
Total 12 marks

Question 2

a. **i.** Use De Moivre's theorem to find the solutions of $\{ z : z^6 - 64 = 0 \}$, $z \in C$ in polar form.

3 marks

ii. Plot and label the solutions on the Argand plane below.



1 mark

b. **i.** Find the Cartesian equation of the relation $\{ z : |z + 1 + \sqrt{3}\, i| \le |z| \}$, expressing your answer in the form $y = ax + b$, where $a, b \in R$

2 marks

ii. Illustrate $\{ z : |z + 1 + \sqrt{3}\, i| \le |z| \}$ on the Argand diagram in part a. ii.

2 marks

iii. Find $\{ z : |z + 1 + \sqrt{3}\, i| \le |z| \} \cap \{ z : z^6 - 64 = 0 \}$ in Cartesian form.

2 marks

Total 10 marks

Question 3

A paint manufacturer wants to make varying shades of yellow paint. This is achieved by mixing different ratios of green and red paint. A 180 litre vat initially contains green and red paint in the ratio of 2:1. The paint is kept well mixed by an agitator. The paint mixture is drawn off at a rate of 5 litres per minute. At the same time green and red paints are added to the mixture at the rates of 3 litres per minute and 2 litres per minute respectively.

a. If the paint mixture in the vat contains Q litres of green paint at any time t minutes, show that the rate of change of the green paint can be expressed as $\frac{dQ}{dt} = -\frac{Q-108}{36}$

2 marks

b. **i.** Solve $\frac{dQ}{dt} = -\frac{Q-108}{36}$ to find the amount of green paint in the vat at time t minutes.

3 marks

ii. Find, correct to one decimal place, the time it takes for the amount of green paint in the vat to decrease to 115 litres.

1 mark

c. i. Sketch a graph of Q against t on the axes below.

2 marks

c. ii. When the amount of green paint in the vat stabilizes, how many litres of red paint will the vat contain?

2 marks
Total 10 marks

Question 4

Angela has a model plane attached to a string and is swinging it above her head. The position of the plane, in metres, from a fixed point O at time t seconds is given by $\underset{\sim}{r}(t) = \left(1 + 3\cos(\pi t)\right)\underset{\sim}{i} + 3\sin(\pi t)\,\underset{\sim}{j} + 5\underset{\sim}{k}$

a. Find the initial position and speed of the plane.

2 marks

b. **i.** Find the Cartesian equation of the path of the plane, stating the domain of the equation.

3 marks

ii. Describe the motion of the model plane.

2 marks

c. Show that the acceleration of the model plane is perpendicular to its velocity at any time *t* seconds.

2 marks

d. After 12 seconds the string attached to Angela's plane breaks.

i. Find the velocity of the plane at the instant the string breaks.

1 mark

ii. The acceleration of the plane is now given by $\ddot{\underset{\sim}{r}}(t) = -9.8\underset{\sim}{k}$

Find the velocity which describes the motion of the plane after the string has broken.

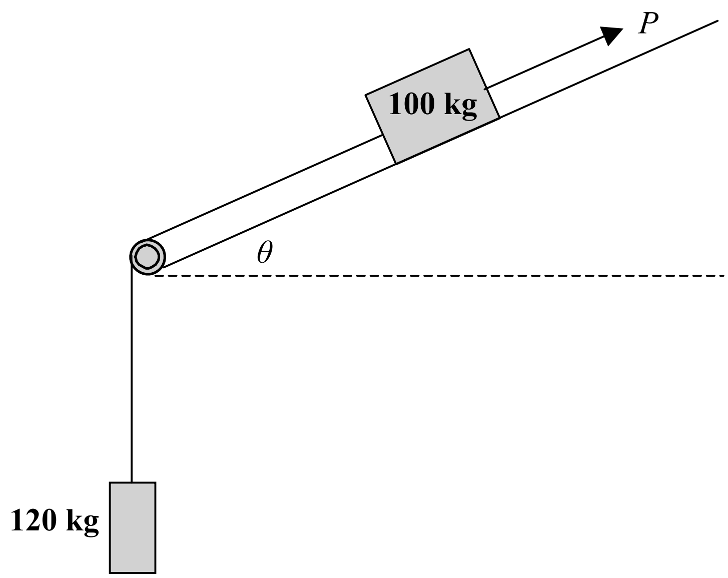
2 marks

Total 12 marks

Question 5

A 120 kg mass is connected to a 100 kg mass resting on a rough inclined plane by a rope passing over a smooth pulley. The plane is inclined at an angle of θ to the horizontal level, where $\sin(\theta) = 0.6$. The coefficient of friction between the 100 kg mass and the rough inclined plane is 0.25.

A force of P newtons parallel to the plane is applied to the connected system of masses as shown in the diagram below.



a. If the connected system of masses is on the point of moving upwards, draw all forces acting on the diagram above.

1 mark

b. Find the frictional force acting on the inclined plane.

2 marks

c. Find the force P.

3 marks

d. If P is a variable force of $55t + 200g$ newtons, the system of masses will begin to move upwards.

Show that at time t seconds, the acceleration, in m/s^2, is given by $a = \frac{t}{4}$.

2 marks

e. Assume the acceleration, in m/s^2, over the first 12 seconds of motion is given by

$$a = \begin{cases} \dfrac{t}{4} & 0 \le t \le 4 \\ 1 & 4 < t \le 6 \\ 4 - \dfrac{t}{2} & 6 < t \le 12 \end{cases}$$

i. Find the velocity of the system at 4 seconds.

1 mark

ii. Determine the total distance the 120 kg mass will rise over the 12 second interval.

5 marks

Total 14 marks

END OF SECTION 2

Specialist Mathematics Formulae Sheet

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$
cosine rule:	$c^2=a^2+b^2-2ab\cos C$

Coordinate geometry

ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ hyperbola: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Circular (trigonometric) functions

$\cos^2(x)+\sin^2(x)=1$

$1+\tan^2(x)=\sec^2(x)$ $\qquad$ $\cot^2(x)+1=\operatorname{cosec}^2(x)$

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$ $\qquad$ $\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$

$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$ $\qquad$ $\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$

$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$ $\qquad$ $\tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x)$

$\sin(2x)=2\sin(x)\cos(x)$ $\qquad$ $\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$

function	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1, 1]$	$[-1, 1]$	R
range	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0,\pi]$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Algebra (Complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\text{cis}\theta$

$|z| = \sqrt{x^2 + y^2} = r$ $\qquad -\pi < \text{Arg}\, z \le \pi$

$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$ $\qquad \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$

$z^n = r^n \text{cis}(n\theta)$ (de Moivre's theorem)

Calculus

$\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$ $\qquad \displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \ne -1$

$\dfrac{d}{dx}\left(e^{ax}\right) = ae^{ax}$ $\qquad \displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$

$\dfrac{d}{dx}\left(\log_e(x)\right) = \dfrac{1}{x}$ $\qquad \displaystyle\int \frac{1}{x}\,dx = \log_e|x| + c$

$\dfrac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$ $\qquad \displaystyle\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$

$\dfrac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$ $\qquad \displaystyle\int \cos\,(ax)\,dx = \frac{1}{a}\sin(ax) + c$

$\dfrac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$ $\qquad \displaystyle\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$

$\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad \displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$

$\dfrac{d}{dx}\left(\cos^{-1}(x)\right) = \dfrac{-1}{\sqrt{1-x^2}}$ $\qquad \displaystyle\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$

$\dfrac{d}{dx}\left(\tan^{-1}(x)\right) = \dfrac{1}{1+x^2}$ $\qquad \displaystyle\int \frac{a}{a^2 + x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c$

product rule: $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

quotient rule: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

Euler's method: If $\dfrac{dy}{dx} = f(x),\ x_0 = a$ and $y_0 = b$,
then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$

acceleration: $a = \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at \qquad s = ut + \dfrac{1}{2}at^2 \qquad v^2 = u^2 + 2as \qquad s = \dfrac{1}{2}(u+v)t$

Vectors in two and three dimensions

$$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$$

$$\left|\underset{\sim}{r}\right| = \sqrt{x^2 + y^2 + z^2} = r \qquad \underset{\sim}{r_1} \cdot \underset{\sim}{r_2} = r_1 r_2 \cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\underset{\sim}{\dot{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$$

Mechanics

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

friction: $F \le \mu N$

END OF FORMULA SHEET