**Student Name:** ______________________________

# SPECIALIST MATHEMATICS

## Units 3 & 4 – Written examination 2

## 2009 Trial Examination

Reading Time: 15 minutes
Writing Time: 2 hours

## QUESTION AND ANSWER BOOK

**Structure of book**

| *Section* | *Number of questions* | *Number of questions to be answered* | *Number of marks* |
|---|---|---|---|
| 1 | 22 | 22 | 22 |
| 2 | 5 | 5 | 58 |
| | | | Total 80 |

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved graphics calculator or approved CAS calculator and a scientific calculator.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

**Materials supplied**
- Question book of 22 pages.

**Instructions**
- Print your name in the space provided on the top of this page.
- All written responses must be in English.

**Students are NOT permitted to bring mobile phones and/or any other electronic devices into the examination room.**

## SECTION 1

**Instructions for Section 1**

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks are **not** deducted for incorrect answers.
If more than 1 answer is completed for any question, no mark will be given.
Take the **acceleration due to gravity**, to have magnitude g m/$s^2$, where g = 9.8

### Question 1

The equation of an asymptote of the hyperbola $\frac{(y-3)^2}{16}-\frac{(x+1)^2}{4}=1$ is

**A.** $x-2y+7=0$
**B.** $x-2y-1=0$
**C.** $2x+y-1=0$
**D.** $2x-y-5=0$
**E.** $2x+y+7=0$

### Question 2

A correct statement for the function $f(x)=\frac{x+a}{x^2-2ax-3a^2}$, where $a \in R^+$, is

**A.** $f(x)$ is differentiable for $x=3a$
**B.** $f(x)$ has two vertical asymptotes
**C.** $f(x)$ is continuous for $x=-a$
**D.** $\lim_{x \to -a} f(x) = -\frac{1}{4a}$
**E.** $f(x)$ has a local maximum for $x=a$

**SECTION 1-** continued

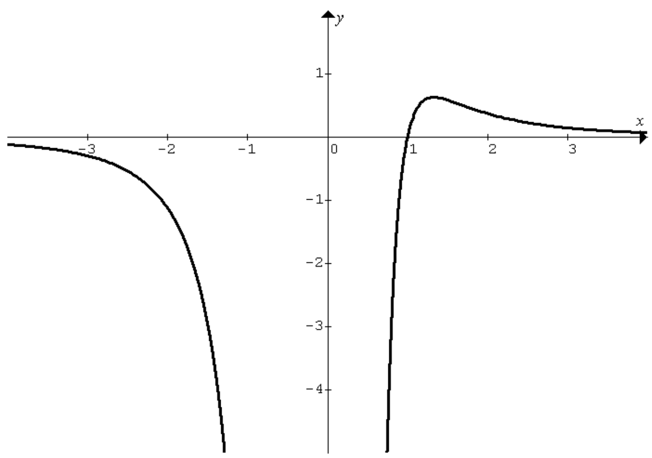
**Question 3**

If the domain of $\sin x$ is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the domain of $\cos x$ is restricted to $[0, \pi]$, then the implied domain and range of $y = \cos\left(\sin^{-1} 2x\right)$ are respectively

**A.** $\left[0, \frac{1}{2}\right]$, $[0,1]$
**B.** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $[0, \pi]$
**C.** $[-1,1]$, $[0,1]$
**D.** $[-\pi, \pi]$, $[0, \pi]$
**E.** $[0,2]$, $[0,1]$

**Question 4**

The graph below shows $y = f''(x)$.




Which one of the following statements **must** be incorrect?

**A.** $f(x)$ is discontinuous for $x = 0$
**B.** $f'(-2) = 0$
**C.** $f'(x) > 0$ for $x > 0$
**D.** $f(x)$ has a local maximum at $x = 1$.
**E.** $f(x)$ has a stationary point of inflexion for $x = 1$

**Question 5**

If a complex number $z$ lies in the first quadrant and $\text{Re}\, z$ is four times larger than $\text{Im}\, z$, then the ratio of $\text{Re}\, z^2 : \text{Im}\, z^2$ is

**A.** 51 : 20
**B.** 15 : 8
**C.** 57 : 20
**D.** 4875 : 1000
**E.** 16

**SECTION 1-** continued
**TURN OVER**

**Question 6**

If a non-zero complex number $z$ satisfies the condition $|z+1| = |z+i| = 1$, then $|z|$ and $Arg\ z$ are respectively

**A.** $\sqrt{2}$, $\frac{5\pi}{4}$

**B.** $\sqrt{2}$, $\frac{\pi}{4}$

**C.** $\sqrt{2}$, $-\frac{3\pi}{4}$

**D.** $1$, $-\frac{3\pi}{4}$

**E.** $2\sqrt{2}$, $-\frac{\pi}{4}$

**Question 7**

The equation $z^4 + az^3 + bz^2 - 32z + 56 = 0,\ a, b \in R$ has one solution $z = 2 - i\sqrt{3}$. The values of $a$ and $b$ are

**A.** $a = 4, b = 15$

**B.** $a = -154, b = 4$

**C.** $a = -4, b = -15$

**D.** $a = 15, b = 4$

**E.** $a = -4, b = 15$

**Question 8**

The position vectors of points $A$ and $B$ are $\overrightarrow{OA} = 3\mathbf{i} + \mathbf{j}$ and $\overrightarrow{OB} = 4\mathbf{i} + 3\mathbf{j}$. The shortest distance of point $B$ from the line $OA$ is

**A.** $\frac{\sqrt{10}}{2}$

**B.** $\sqrt{22}$

**C.** $\frac{\sqrt{90}}{2}$

**D.** $5$

**E.** $\frac{\sqrt{190}}{2}$

**SECTION 1-** continued

**Question 9**

If $|\mathbf{a}| = 4,\ |\mathbf{b}| = 3$ and the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is $\frac{2\pi}{3}$, then $|\mathbf{a}+\mathbf{b}|$ is

**A.** $7$
**B.** $\sqrt{47}$
**C.** $\sqrt{13}$
**D.** $13$
**E.** $47$

**Question 10**

The smallest value of $a$ for which a solution of the equation
$a\tan^2 x + b\cot^2 x = a,\ a, b \in R^+$ exists is

**A.** $a \in (-\infty, a] \cup [4b, \infty)$
**B.** $a = 0$
**C.** $a = b$
**D.** $a = 1$
**E.** $a = 4b$

**Question 11**

The angle which the tangent to the curve $y = \arcsin\frac{k}{x}$ at the point $\left(2k, \frac{\pi}{6}\right)$, where $k > 0$, makes with the $x$-axis is

**A.** $\frac{\pi}{3}$

**B.** $\arctan\left(\frac{\sqrt{3}}{k}\right)$

**C.** $\arctan\left(-\frac{\sqrt{3}}{3k}\right)$

**D.** $\arctan\left(\frac{\sqrt{3}}{3k}\right)$

**E.** $\arctan\left(-\frac{\sqrt{3}}{6k}\right)$

**SECTION 1-** continued
**TURN OVER**

**Question 12**

Which one of the following expressions is equivalent to $\tan\left(2\cos^{-1}\frac{2}{\sqrt{m^2+4}}\right)$?

**A.** $\frac{1}{\sqrt{m^2+4}}$

**B.** $\frac{4m}{4-m^2}$

**C.** $\frac{4m}{m^2-4}$

**D.** $\frac{4-m^2}{4m}$

**E.** $\frac{m}{\sqrt{m^2+4}}$

**Question 13**

Integral $I = \int \frac{\sin 2x}{1+\cos 2x}\,dx$ **cannot** be expressed as

**A.** $\int \tan x\,dx$

**B.** $-\frac{1}{2}\int \frac{1}{u}du,\ u = 1+\cos 2x$

**C.** $-\frac{1}{2}\int \frac{1}{1+u}du,\ u = \cos 2x$

**D.** $\frac{1}{2}\int \left(\frac{1}{1-u} - \frac{1}{1+u}\right)du,\ u = \sin x$

**E.** $\int \frac{(\sin x + \cos x)^2}{1+\sin 2x}\,dx$

**Question 14**

The area bounded by the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, the $x-$axis and the line $x = 2a$ is rotated about the $x-$axis. The volume of the resulting solid of revolution is

**A.** $\frac{2ab^2\pi}{3}$

**B.** $\frac{8ab^2\pi}{3}$

**C.** $\frac{4a^2b^2\pi}{3}$

**D.** $\frac{4ab^2\pi}{3}$

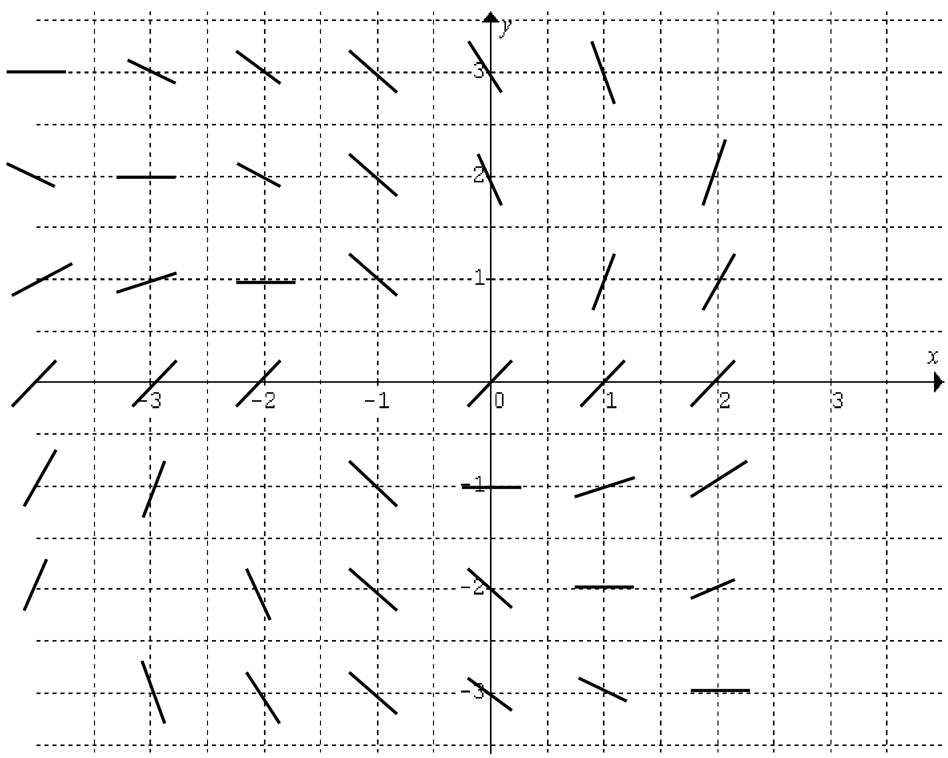
**E.** $\frac{8a^2b^2\pi}{3}$

**SECTION 1-** continued

**Question 15**
A block of ice melts at a rate proportional to the reciprocal of its mass. Initially the block weighs 100 kg, and two days later weighs 50 kg. The block will have melted completely after approximately

**A.** 2.67 days
**B.** 4 days
**C.** 1 day
**D.** 1.67 days
**E.** 2.33 days

**Question 16**




The family of solutions of a first order differential equation is shown above. The differential equation could be

**A.** $\frac{dy}{dx} = \frac{x^2 + y^2}{x - y}$

**B.** $\frac{dy}{dx} = \frac{x + y}{x - y}$

**C.** $\frac{dy}{dx} = \frac{x - y}{x + y}$

**D.** $\frac{dy}{dx} = \frac{xy}{x - y}$

**E.** $\frac{dy}{dx} = x^2 - y^2$

**SECTION 1-** continued
**TURN OVER**

**Question 17**

An approximation to the solution of the differential equation $y' - y = e^{-2x} - 1$ with $y(0) = 1$ is found by using Euler's method with $h = 0.2$. The value of $y_2$ is

**A.** $1.6 + 0.2e^{-0.4}$
**B.** 1.5586
**C.** 1.7346
**D.** $1.24 + 0.2e^{-0.4}$
**E.** 0.6

**Question 18**

A ball is projected vertically upwards with initial velocity $5g$ ms$^{-1}$ from the base of the Eureka building. At the same time another ball falls from rest from a 196 m high window. When they meet, the speed of the falling ball is four times the speed of the rising ball. The distance the falling ball has travelled when the balls meet is

**A.** 49 m
**B.** 117 6m
**C.** 78.4 m
**D.** 100 m
**E.** 96 m

**Question 19**

A particle moves in a line and its position at time $t$ is given by $x(t) = 2 + 30t - \frac{1}{1+10t}, t \geq 0$.

The particle's acceleration after 2 seconds and its terminal velocity are respectively

**A.** 1.784, 32
**B.** $-0.022$, 40
**C.** $-0.022$, 30
**D.** 1.784, 40
**E.** 0.022, 20

**Question 20**

The position of a particle at time $t$ seconds is given by $r(t) = (t^2 + 1)\mathbf{i} + e^{-2t}\mathbf{j},\ t \geq 0$. The particle's speed is 4.2 ms$^{-1}$ after

**A.** 2.1 seconds
**B.** 1.8 seconds
**C.** 2.2 seconds
**D.** 1.2 and 2.2 seconds
**E.** $-$ 0.7 and 2.1

**SECTION 1-** continued

**Question 21**

A particle of mass 5 kg is on a rough horizontal platform and is connected by means of an inelastic string passing over a smooth pulley at the platform's edge to a particle of mass 2 kg. The system is released from rest and the first particle moves a distance of 1.5 metres in 2 seconds. The coefficient of friction is closest to

**A.** 0.26
**B.** 0.27
**C.** 0.28
**D.** 0.29
**E.** 0.31

**Question 22**

A particle moves in a straight line with an initial velocity 1 $ms^{-1}$ and the acceleration, $a$ $ms^{-2}$, given by $a = \frac{1+v^2}{v}$. The particle's position when the velocity is 3 $ms^{-1}$ is

**A.** 2.42 metres
**B.** 1.54 metres
**C.** 2.12 metres
**D.** 1.51 metres
**E.** 2.54 metres

**END OF SECTION 1**
**TURN OVER**

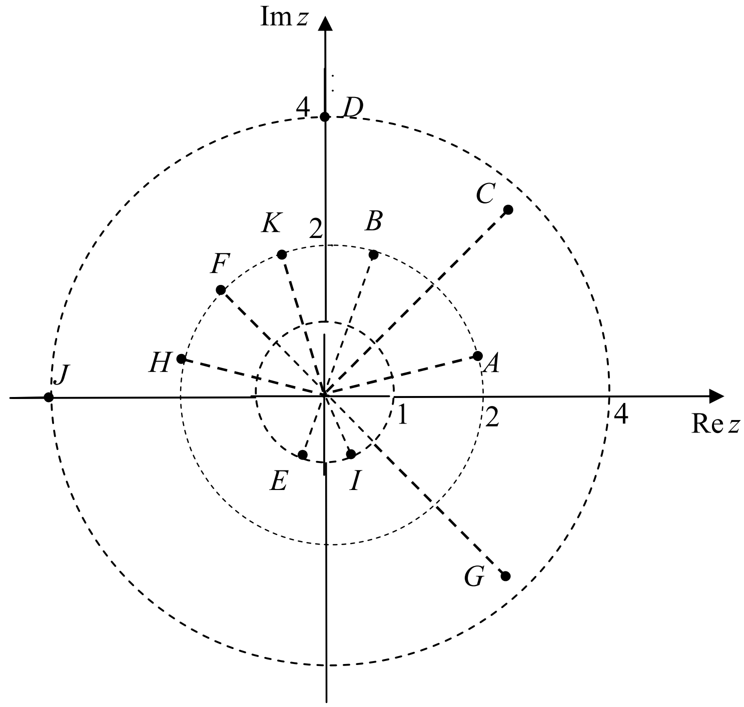
## SECTION 2

**Instructions for Section 2**

Answer **all** questions.
A decimal approximation will not be accepted if the question specifically asks for an **exact** answer is required.
For questions worth more than one mark, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams are **not** drawn to scale.
Take the **acceleration due to gravity**, to have magnitude $g$ m/s$^2$ , where $g = 9.8$

### Question 1

In the diagram below, complex numbers $u = 2cis\left(\frac{\pi}{12}\right)$ and $v = 2cis\left(\frac{5\pi}{12}\right)$ are represented by points $A$ and $B$.




**a.** Some of the points $C$ to $K$ represent certain complex numbers. Label each complex number below with the most appropriate point:

$\frac{u}{v}$ ________ $uv$ ________ $\overline{u}+\overline{v}$ ________ $\frac{16}{v^3}$ ________

______________________________________________________________________

______________________________________________________________________

______________________________________________________________________

4 marks

**SECTION 2- Question 1-** continued

**b.** Let $x+\frac{1}{x}=2\cos\alpha$, $\alpha\in[0,\pi]$

**i** Show that $x=\cos\alpha+i\sin\alpha$.

**ii** Prove that $x^n-\frac{1}{x^n}=2i\sin(n\alpha)$

**iii** **Hence,** calculate $\left(\sqrt{3}+i\right)^7-\left(\sqrt{3}-i\right)^7$ giving your answer in Cartesian form.

2 + 2 + 3 = 7 marks
Total 11 marks

**SECTION 2-** continued
**TURN OVER**

**Question 2**

Let the position vectors of points A, B and C be, respectively $\mathbf{a} = 2p\mathbf{i} + \mathbf{j} + (1-p)\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j}$ and $\mathbf{c} = 5\mathbf{i} - \mathbf{j} + 8\mathbf{k}$.

**a.** Find the value of $p$ such that vector **a** makes equal angles with vectors **b** and **c**.

3 marks

**b.** Find the value of $p$ such that vectors **a**, **b** and **c** are linearly dependent.

3 marks

**SECTION 2- Question 2-** continued

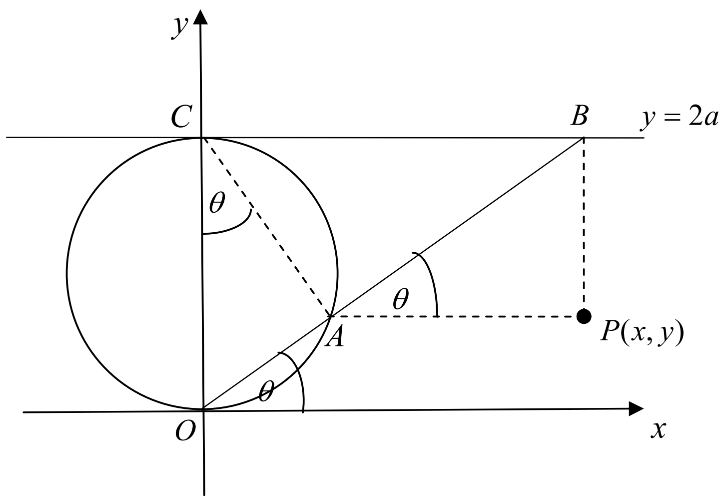
**c.** Find the position vector of the point D if points $O$, $C$ and $D$ are collinear and $\left|\overrightarrow{BD}\right| = \left|\overrightarrow{OB}\right|$.

3 marks
Total 9 marks

**SECTION 2-** continued
**TURN OVER**

**Question 3**

In the diagram below, the line $y = \tan\theta\, x$, where $0 < \theta < \pi$, cuts the circle $x^2 + (y+a)^2 = a^2$ at point *A* and is extended until it meets the line $y = 2a$ at point *B*. The horizontal line through point *A* intersects the vertical line through point *B* at point *P(x, y)*.




**a.** Show that the coordinates of point *P* can be written as $x = 2a\cot\theta,\ y = 2a\sin^2\theta$.

2 marks

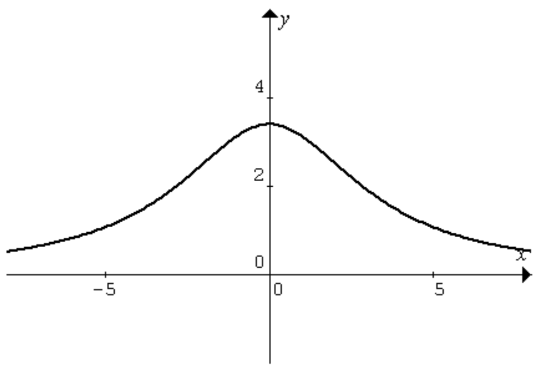
**SECTION 2- Question 3-** continued

The locus of point $P$ is known as The Witch of Maria Agnesi.

**b.** Show that the Cartesian equation of the curve is $y = \dfrac{8a^3}{x^2 + 4a^2}$.

2 marks

The graph of The Witch of Maria Agnesi is given below.




**c.** Find the coordinates of the points of inflexion in terms of $a$.

3 marks

**SECTION 2- Question 3-** continued

**TURN OVER**

**d.** **i** Show that the area bounded by the curve, the lines $x = \pm\frac{2}{\sqrt{3}}a$ and the $x$-axis is $\frac{4a^2\pi}{3}$.

**ii** The region bounded by the curve and the lines $y = a$ and $y = 2a$ revolves around the $y$-axis. Show that the volume of this solid of revolution is $V = 4a^3\pi\left(2\log_e 2 - 1\right)$.

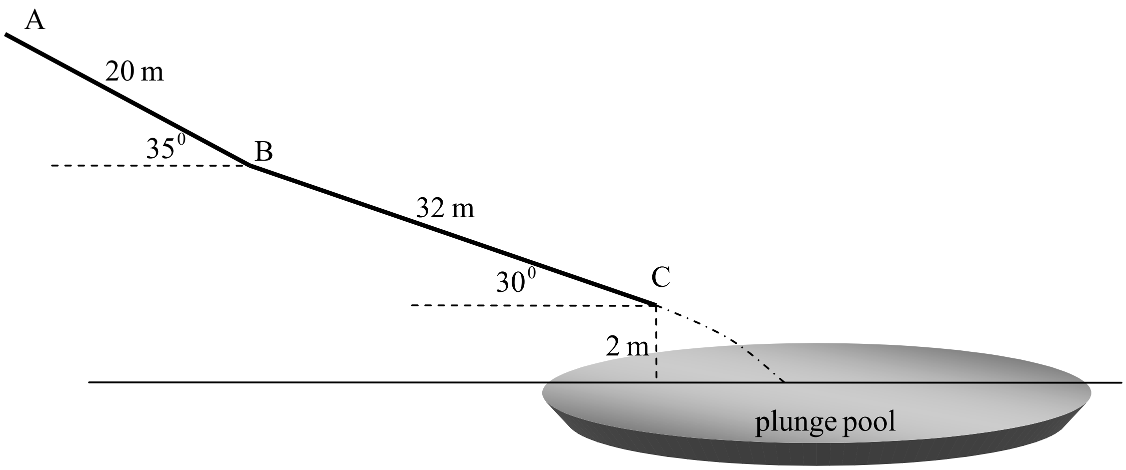
**iii** Find the non-zero value(s) of *a,* accurate to three decimal places, such that the area from part **i** is numerically equal to the volume from part **ii.**

2 + 2 + 1 = 5 marks
Total 12 marks

**SECTION 2-** continued

**Question 4**

The Big Banana Slip at Fun Park has an inflatable waterslide for children. It consists of two sections. The upper section (AB) is 20 metres long and it is inclined $35^0$ to the horizontal. The lower section (BC) is 32 metres long and the angle of inclination is decreased to $30^0$. The slide finishes 2 metres above the plunge pool (point C). A constant stream of water is pumped at the top of the slide to reduce friction.




**a.** Adam has a mass of 40 kg. He enters the waterslide at point A and reaches the end of the upper section (point B) with the speed of $6 ms^{-1}$ **.**

**i** Show that Adam's acceleration along this section is $\frac{9}{10} ms^{-2}$.

______________________________________________

______________________________________________

______________________________________________

**ii** Calculate, correct to three decimal places, the coefficient of friction between Adam's body and the waterslide.

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______________________________________________

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1 + 2 = 3 marks

**SECTION 2- Question 4-** continued
**TURN OVER**

**b.** To increase the safety of children by reducing the speed before they slide into the plunge pool, rollers are attached to the lower section of the waterslide. They cause riders to decelerate at $0.5\text{ms}^{-2}$. How long does it take to slide along the lower section?

2 marks

**c.** At the bottom of the waterslide (at point C), Adam's position can be described as $\mathbf{r}(0) = 2\mathbf{j}$ and the acceleration is now $\ddot{\mathbf{r}}(t) = -g\mathbf{j}$. Show that:

**i** $\dot{\mathbf{r}}(0) = \sqrt{3}\mathbf{i} - \mathbf{j}$.

**ii** $\mathbf{r}(t) = \sqrt{3}\,t\,\mathbf{i} - \left(\dfrac{gt^2}{2} + t - 2\right)\mathbf{j}$.

1 + 2 = 3 marks

**SECTION 2- Question 4-** continued

**d.** Find, correct to two decimal places:

**i** the speed at which Adam hits the water as he plunges into the pool.

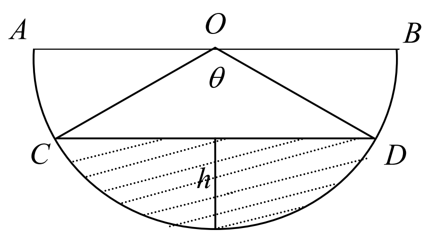
**ii** the horizontal distance (from the bottom of the slide) at which Adam hits the water.

3 + 1 = 4marks
Total 12 marks

**SECTION 2-** continued
**TURN OVER**

**Question 5**
An aquarium displays fish in a big glass tank which has the length of 8 metres and a semi-circular cross-section of radius 1.5 metres. The cross-section of the tank is shown below.

AB is the diameter of the semi-circle, $\angle COD = \theta$ and $h$ is the depth of water.




**a.** **i** Show that the volume of water in the tank is given by $V = 9(\theta - \sin\theta)$.

1 mark

**ii** Show that $h = \frac{3}{2}\left(1 - \cos\frac{\theta}{2}\right)$.

1 mark

**SECTION 2- Question 5-** continued

**b.** Water is pumped into the tank at a rate of $0.2\,\text{m}^3$ per minute. Find, correct to three decimal places, the rate at which the water level is rising when $h = 0.8\,\text{m}$.

4 marks

**c.** A year later, the management of the aquarium decided to add new species of tropical fish that require a certain concentration of sea-salt in the water. At that time the tank contained 20 000 litres of pure water.
Before the new species are placed in the tank, brine containing 0.05 kg of sea-salt per litre of water is pumped into the tank through a pipe at a rate of 100 litres per minute. Through a second pipe, brine containing 0.04kg of sea-salt per litre of water is pumped into the tank at a rate of 50 litres per minute. To keep the volume constant, the solution is drained from the tank at a rate 150 litres per minute.

**i** Show that $\dfrac{dx}{dt} = 7 - \dfrac{3x}{400}$, where $x$ is the amount of salt in the tank after $t$ hours.

**SECTION 2- Question 5 –part c-**continued
**TURN OVER**

**ii** Hence, show that $x = \frac{2800}{3}\left(1 - e^{-\frac{3t}{400}}\right)$.

1 + 3 = 4 marks

**d.** The optimal concentration is 0.03kg of salt per every litre of water.

**i** For how long, correct to two decimal places, should the mixing process continue before this concentration is achieved?

**ii** What is the limiting concentration of salt in kilograms per litre? Give your answer correct to three decimal places.

2 + 2 = 4 marks
Total 14 marks

**END OF QUESTION AND ANSWER BOOK**