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SPECIALIST MATHS TRIAL EXAMINATION 2 SOLUTIONS 2010

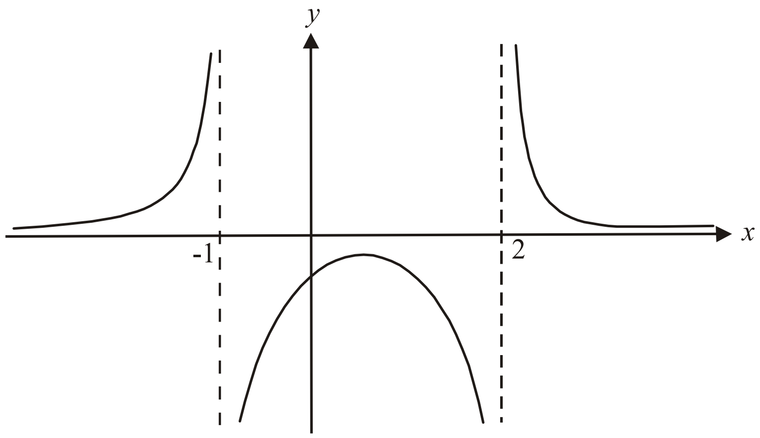
SECTION 1 - Multiple-choice answers

1.	**D**	**7.**	**E**	**13.**	**B**	**19.**	**D**
2.	**B**	**8.**	**C**	**14.**	**D**	**20.**	**C**
3.	**E**	**9.**	**C**	**15.**	**A**	**21.**	**C**
4.	**A**	**10.**	**C**	**16.**	**E**	**22.**	**D**
5.	**B**	**11.**	**D**	**17.**	**B**		
6.	**E**	**12.**	**B**	**18.**	**A**		

SECTION 1 - Multiple-choice solutions

Question 1

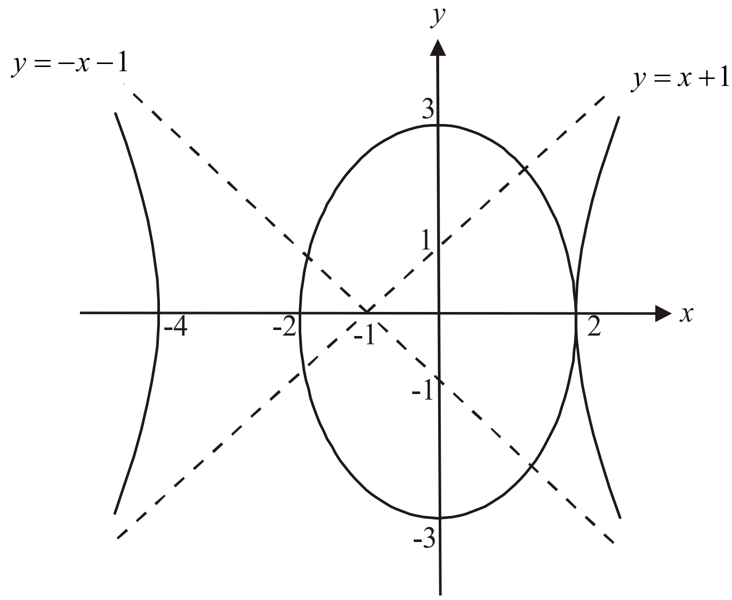
$$y = \frac{1}{x^2 - x - 2}$$
$$= \frac{1}{(x-2)(x+1)}$$



The graph has three asymptotes; $y = 0,\ x = -1$ and $x = 2$ and one local maximum.
The answer is D.

Question 2

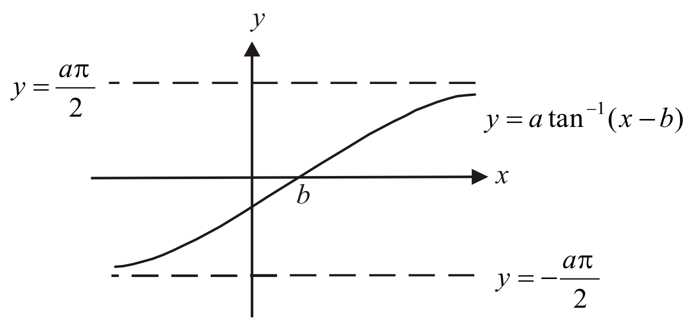
Do a quick sketch.



The only point common to both graphs is $(2,0)$.
The answer is B.

Question 3

The graph of $y = a\tan^{-1}(x-b)$ is shown.



If $c = \frac{a\pi}{2}$, the lower asymptote occurs at $y = 0$ and the graph of $y = a\tan^{-1}(x-b)+c$ cannot have any x-intercepts.

If $c > \frac{a\pi}{2}$, the lower asymptote occurs above $y = 0$, i.e. above the x-axis and the graph of $y = a\tan^{-1}(x-b)+c$ cannot have any x-intercepts.

So we require $c \geq \frac{a\pi}{2}$
The answer is E.

Question 4

Method 1

$\cot^2(x) + 1 = \text{cosec}^2(x)$ (formula sheet)

$2 + 1 = \text{cosec}^2(x)$

$\text{cosec}(x) = -\sqrt{3}$ (4th quadrant)

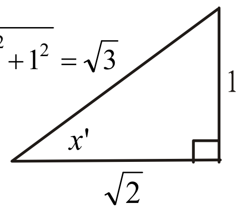
$\frac{1}{\sin(x)} = -\sqrt{3}$

$\sin(x) = -\frac{1}{\sqrt{3}}$

The answer is A.

Method 2

$\sqrt{\left(\sqrt{2}\right)^2 + 1^2} = \sqrt{3}$



$\sin(x') = \frac{1}{\sqrt{3}}$

$\sin(x) = \frac{-1}{\sqrt{3}}$ (4th quadrant)

Question 5

Method 1 – using CAS

$\frac{z_1}{z_2} = \frac{1+\sqrt{3}i}{1+i}$

$= \sqrt{2}\text{cis}\left(\frac{\pi}{12}\right)$

The answer is B.

Method 2 – by hand

$z_1 = 1 + \sqrt{3}i$

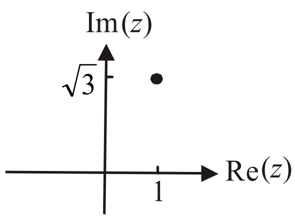
$r = \sqrt{1+3}$

$= 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$= \frac{\pi}{3}$

$z_1 = 2\text{cis}\left(\frac{\pi}{3}\right)$



$z_2 = 1 + i$

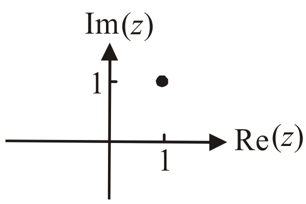
$r = \sqrt{1+1}$

$= \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right)$

$= \frac{\pi}{4}$

$z_2 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$



$\frac{z_1}{z_2} = \frac{2\text{cis}\left(\frac{\pi}{3}\right)}{\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)}$

$= \sqrt{2}\text{cis}\left(\frac{\pi}{12}\right)$

The answer is B.

Question 6

There will be 6 solutions so option A is eliminated.

Solutions to the equation $z^6 = a,\ a \in C$ will be spaced $\frac{2\pi}{6} = \frac{\pi}{3}$ apart around a circle with centre at $O(0,0)$.

Only option E offers this.

The answer is E.

Question 7

$\text{Arg}(z)$ is the principal valued argument of z. That is, $-\pi < \text{Arg}(z) \le \pi$.

The region shown is described by $\left\{ z : 0 \le \text{Arg}(z) \le \frac{\pi}{3} \right\}$.

The answer is E.

Question 8

At $x = 0, \frac{dy}{dx} = 0$.
This eliminates options B and D.
For $0 < x < \frac{\pi}{2},\ \frac{dy}{dx} > 0$.
This eliminates option A.
At $x = \frac{\pi}{2}, \frac{dy}{dx} = 0.$
This eliminates option E.
The answer is C.

Question 9

$$u = \sin(x) \qquad \text{if } x = 0, \quad u = 0$$

$$\frac{du}{dx} = \cos(x) \qquad \text{if } x = \frac{\pi}{2}, \quad u = 1$$

$$\int_0^{\frac{\pi}{2}} \sin^2(x)\cos^3(x)dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2(x)\cos^2(x)\cos(x)dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2(x)(1 - \sin^2(x))\cos(x)dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2(x) - \sin^4(x))\cos(x)dx$$

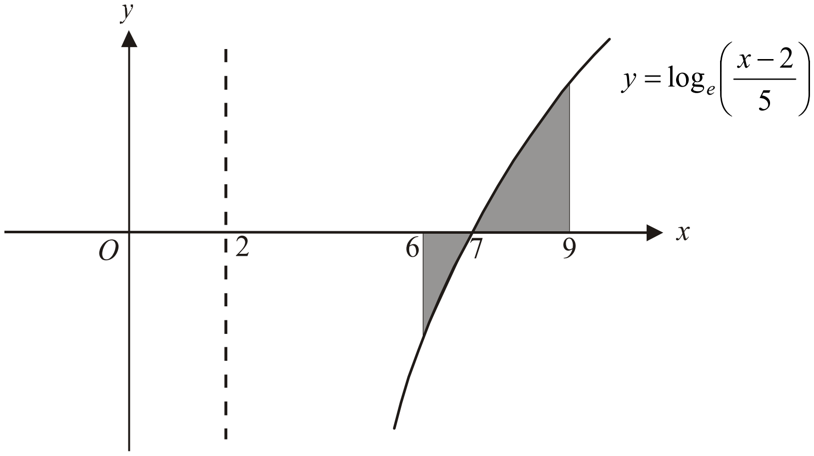
$$= \int_0^1 (u^2 - u^4) \times \frac{du}{dx}dx$$

$$= \int_0^1 (u^2 - u^4)du$$

The answer is C.

Question 10

Do a quick sketch.



$$\text{Area} = -\int_6^7 \log_e\left(\frac{x-2}{5}\right)dx + \int_7^9 \log_e\left(\frac{x-2}{5}\right)dx$$
$$= 0.4627...$$

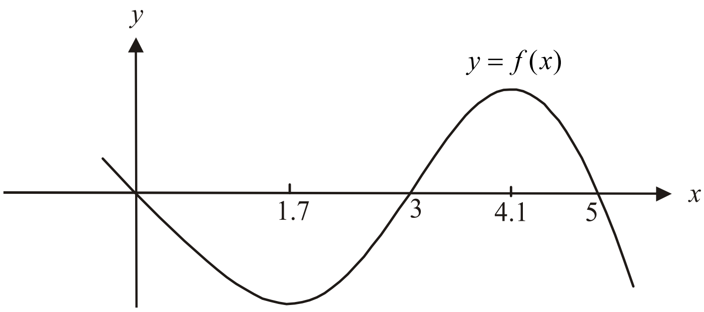
The closest answer is 0.46.
The answer is C.

Question 11

$$\frac{dT}{dt} = -k(T-25), \qquad T(0)=62$$

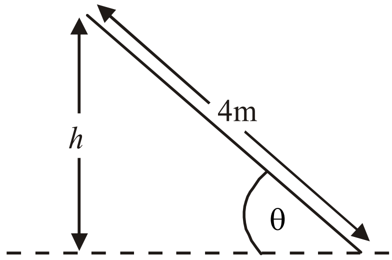
The answer is D.

Question 12



The graph of $y=f(x)$ is the gradient function of $y=F(x)$.
The graph of the antiderivative function $y=F(x)$ will have three stationary points at the points where $x=0, x=3$ and $x=5$. Options C and D are incorrect.
At $x=0$ there is a local maximum because for $x<0$, $f(x)>0$ and for $x>0$, $f(x)<0$.
The same is true for the point where $x=5$. At $x=3$ there will be a local minimum. Option B is correct – there are no stationary points of inflection.
At $x=1.7$ and $x=4.1$ there will be a point of inflection.
The answer is B.

Question 13



$$\frac{dh}{dt} = \frac{dh}{d\theta}\frac{d\theta}{dt} \quad \text{(chain rule)}$$
$$= 4\cos(\theta) \times 0.05$$
$$= 0.2\cos(\theta)$$

$$\sin(\theta) = \frac{h}{4}$$
$$h = 4\sin(\theta)$$
$$\frac{dh}{d\theta} = 4\cos(\theta)$$

When $\theta = \frac{\pi}{3}$,

$$\frac{dh}{dt} = 0.2 \times \cos\left(\frac{\pi}{3}\right)$$
$$= 0.2 \times \frac{1}{2}$$
$$= \frac{1}{10} \text{ m/sec}$$

The answer is B.

Question 14

$$\underset{\sim}{a}(t) = 2\underset{\sim}{i} + 4t\underset{\sim}{j} - \sqrt{t}\,\underset{\sim}{k}$$

$$\underset{\sim}{v}(t) = 2t\underset{\sim}{i} + 2t^2\underset{\sim}{j} - \frac{2t^{\frac{3}{2}}}{3}\underset{\sim}{k} + \underset{\sim}{c}$$ where $\underset{\sim}{c}$ is a constant vector

$$\underset{\sim}{v}(0) = \underset{\sim}{i} - \underset{\sim}{j}$$

$$\underset{\sim}{i} - \underset{\sim}{j} = 0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k} + \underset{\sim}{c}$$

$$\underset{\sim}{c} = \underset{\sim}{i} - \underset{\sim}{j}$$

$$\underset{\sim}{v}(t) = (2t+1)\underset{\sim}{i} + (2t^2 - 1)\underset{\sim}{j} - \frac{2t^{\frac{3}{2}}}{3}\underset{\sim}{k}$$

$$\underset{\sim}{v}(1) = 3\underset{\sim}{i} + \underset{\sim}{j} - \frac{2}{3}\underset{\sim}{k}$$

The answer is D.

Question 15

$\underset{\sim}{u} = \underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$\underset{\sim}{v} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

$\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{6}}(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$

The vector resolute of $\underset{\sim}{u}$ perpendicular to $\underset{\sim}{v}$ is given by

$$\underset{\sim}{u} - (\underset{\sim}{u} \cdot \hat{\underset{\sim}{v}})\hat{\underset{\sim}{v}}$$

$$= \underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k} - \frac{1}{\sqrt{6}}(1 \times 2 - 2 \times 1 - 3 \times -1) \times \frac{1}{\sqrt{6}}(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$$

$$= \underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k} - \frac{1}{2}(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$$

$$= \underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k} - \underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}$$

$$= -\frac{5}{2}(\underset{\sim}{j} + \underset{\sim}{k})$$

The answer is A.

Question 16

Since $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent,

$$\alpha(\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}) + \beta(-\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}) = \underset{\sim}{i} + m\underset{\sim}{k}$$

$$\alpha - \beta = 1$$

$$\alpha = 1 + \beta \quad -(1)$$

and $\alpha + 2\beta = 0$

$$\alpha = -2\beta \quad -(2)$$

In (1) $-2\beta = 1 + \beta$

$$\beta = -\frac{1}{3}$$

In (2) $\alpha = \frac{2}{3}$

Also $2\alpha - 2\beta = m$

$$\frac{4}{3} + \frac{2}{3} = m$$

$$m = 2$$

The answer is E.

Note: the often used definition for linear dependence is $\alpha\underset{\sim}{a} + \beta\underset{\sim}{b} + \delta\underset{\sim}{c} = \underset{\sim}{0} \quad -(A)$ or similar.

This can be rearranged to give $\alpha\underset{\sim}{a} + \beta\underset{\sim}{b} = -\delta\underset{\sim}{c}$

$$-\frac{\alpha}{\delta}\underset{\sim}{a} - \frac{\beta}{\delta}\underset{\sim}{b} = \underset{\sim}{c} \quad -(B)$$

Since α, β and δ are just parameters (or constants), so are $-\frac{\alpha}{\delta}$ and $-\frac{\beta}{\delta}$. Working with equation (B) is simpler than working with equation (A) because it contains 2 parameters not 3 and is therefore an easier starting point.

Question 17

For options A, B and C, $(\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k})\bullet(\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k})=0$

$\left|\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}\right|=\sqrt{3}$

So a vector perpendicular to $\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}$ with a magnitude of 3 is

$$3\times\frac{1}{\sqrt{3}}(\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k})$$
$$=\sqrt{3}(\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k})$$

So option B is correct.
The answer is B.

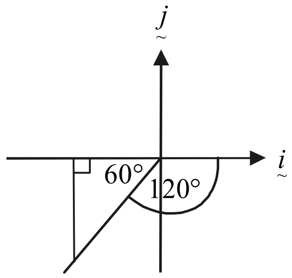
Question 18

The components in the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ directions should both be negative.

This eliminates options B and E.
$\tan(60^\circ)=\sqrt{3}$

For option A, $\frac{-\sqrt{3}}{-1}=\sqrt{3}$

Note for option C, $\frac{-1}{-\sqrt{3}}\neq\sqrt{3}$ and for option D $\frac{-\sqrt{3}}{-\sqrt{3}}\neq\sqrt{3}$.



The answer is A.

Question 19

Initially $mv=20$.
At $t=10$, $\quad m\times 20=80$
$$m=4\text{kg}$$
So initially $u=5\text{ms}^{-1}$

At $t=10$,
$$\begin{aligned}s&=\frac{1}{2}(u+v)t\\&=\frac{1}{2}(5+20)\times 10\\&=\frac{250}{2}\\&=125\end{aligned}$$

The answer is D.

Question 20

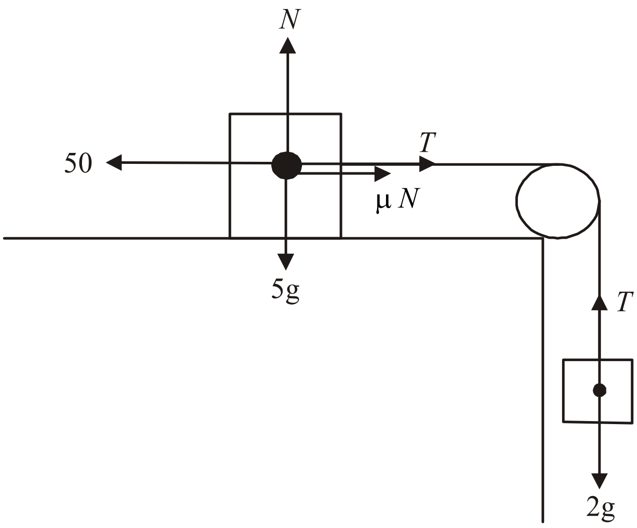
The bicycle and its rider travelled in a straight line starting at O and finishing at a point some distance from O. That distance is given by the area enclosed by the graph and the x-axis. The speed was constant and greatest between $t = 2$ and $t = 5$. The speed was variable between $t = 0$ and $t = 2$ and between $t = 5$ and $t = 6$.
The answer is C.

Question 21

The 2kg mass will; depending on the value of μ, either be in limiting equilibrium or accelerating down the plane inclined at 60° in which case the 1kg mass will be accelerating up the plane inclined at 30°. Even if the system is in equilibrium, the 2kg mass will be on the verge of moving down the plane and the 1kg mass will be on the verge of moving up the plane.
The friction force for the 2kg mass will therefore be directed up the plane inclined at 60° and the friction force for the 1kg mass will be directed down the plane inclined at 30°.
The forces due to gravity are given by g and 2g not 1kg and 2kg.
The answer is C.

Question 22

Draw in the forces.



The 5kg mass accelerates to the left and the 2kg mass accelerates upwards.

Around the 2kg mass
$T - 2g = 2a$

$T = 2a + 2g$

Around the 5kg mass
Vertically: $N = 5g$
Horizontally: $50 - \mu N - T = 5a$

$50 - 0.2 \times 5g - 2a - 2g = 5a$

$50 - 3g = 7a$

The answer is D.

SECTION 2

Question 1

a. **i.** $f(x)=\frac{30}{\pi}\text{arsin}\left(\frac{x}{2}\right)$

$-1\le\frac{x}{2}\le 1$

$-2\le x\le 2$

$d_f=[-2,2]$

(1 mark)

ii. The range of $y=\text{arsin}\left(\frac{x}{2}\right)$ is $\frac{-\pi}{2}\le y\le\frac{\pi}{2}$

So the range of $y=\frac{30}{\pi}\text{arsin}\left(\frac{x}{2}\right)$ is

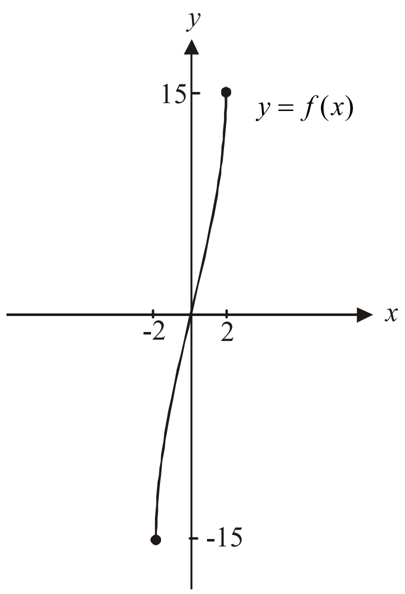
$\frac{-\pi}{2}\times\frac{30}{\pi}\le y\le\frac{\pi}{2}\times\frac{30}{\pi}$

$-15\le y\le 15$

$r_f=[-15, 15]$

(1 mark)

b.



(1 mark)

c. $f''(x)=\frac{30x}{\pi(4-x^2)^{\frac{3}{2}}}$

Solve $f''(x)=0$

$x=0$

(1 mark)

$f''(-1)=-1.8377...<0$

$f''(1)=1.8377...>0$

Since $f''(x)=0$ and the sign of $f''(x)$ changes around the point where $x=0$, there is a point of inflection at $x=0$.

(1 mark)

d. The rotation is about the y-axis and $r_g = [0,15]$ so

$$V = \pi \int_0^{15} x^2 \, dy$$ **(1 mark)**

Now, $g(x) = \frac{30}{\pi}\text{arsin}\left(\frac{x}{2}\right)$

so, $y = \frac{30}{\pi}\text{arsin}\left(\frac{x}{2}\right)$

$$\frac{\pi y}{30} = \text{arsin}\left(\frac{x}{2}\right)$$

$$\frac{x}{2} = \sin\left(\frac{\pi y}{30}\right)$$

$$x = 2\sin\left(\frac{\pi y}{30}\right)$$ **(1 mark)**

So, $V = 4\pi \int_0^{15} \sin^2\left(\frac{\pi y}{30}\right) dy$ as required.

e. From **d.**, $V = 4\pi \int_0^{15} \sin^2\left(\frac{\pi y}{30}\right) dy$

Now $\cos(2x) = 1 - 2\sin^2(x)$ (from formula sheet)

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

So, $V = 4\pi \int_0^{15} \left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi y}{30}\right)\right) dy$

$$= 2\pi \int_0^{15} \left(1 - \cos\left(\frac{\pi y}{15}\right)\right) dy$$

So, $a = 2,\ b = 1$ and $c = \frac{\pi}{15}$

(1 mark)

f. Using CAS find $4\pi \int_0^{15} \sin^2\left(\frac{\pi y}{30}\right) dy$.

Volume is $30\pi\,\text{cm}^3$.

(1 mark)

g. Solve $4\pi \int_0^{h} \sin^2\left(\frac{\pi y}{30}\right) dy = 15\pi$ for h.

(1 mark)

$h = 11.03$ (correct to 2 decimal places)
So the height of the frozen orange juice is 11.03cm (correct to 2 decimal places).

(1 mark)
Total 11 marks

Question 2

a. $\underset{\sim}{r}(t) = \sec\left(\frac{\pi t}{12}\right)\underset{\sim}{i} + 2\tan\left(\frac{\pi t}{12}\right)\underset{\sim}{j}$

$\underset{\sim}{r}(0) = \sec(0)\underset{\sim}{i} + 2\tan(0)\underset{\sim}{j}$

$= \underset{\sim}{i} + 0\underset{\sim}{j}$

(1 mark)

Initially the boat is 1km due east of the surf life saving club.

(1 mark)

b. $\underset{\sim}{r}(t) = \sec\left(\frac{\pi t}{12}\right)\underset{\sim}{i} + 2\tan\left(\frac{\pi t}{12}\right)\underset{\sim}{j}$

$x = \sec\left(\frac{\pi t}{12}\right)$ and $y = 2\tan\left(\frac{\pi t}{12}\right)$

$x^2 = \sec^2\left(\frac{\pi t}{12}\right)$ $y^2 = 4\tan^2\left(\frac{\pi t}{12}\right)$

$\frac{y^2}{4} = \tan^2\left(\frac{\pi t}{12}\right)$

Now $1 + \tan^2(x) = \sec^2(x)$ (formula sheet)

$1 + \frac{y^2}{4} = x^2$

$x^2 - \frac{y^2}{4} = 1$

as required

(1 mark) for $x = \sec\ldots \quad y = 2\tan\ldots$
(1 mark) – use of identity

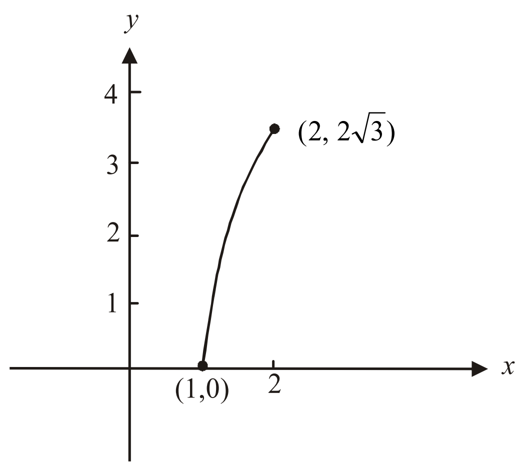
c.

When $t = 4$,

$\underset{\sim}{r}(4) = \sec\left(\frac{\pi}{3}\right)\underset{\sim}{i} + 2\tan\left(\frac{\pi}{3}\right)\underset{\sim}{j}$

$= 2\underset{\sim}{i} + 2\sqrt{3}\,\underset{\sim}{j}$

So one endpoint occurs at $\left(2, 2\sqrt{3}\right)$.
From part **a.**, the other occurs at (1,0).



(1 mark) – correct endpoints
(1 mark) – correct shape

d. Solve $2\tan\left(\frac{\pi t}{12}\right)=2$

$$\tan\left(\frac{\pi t}{12}\right)=1$$

$$\frac{\pi t}{12}=\frac{\pi}{4}$$

$$t=3\text{ hours}$$

(1 mark)

e. $\underset{\sim}{r}(t)=\sec\left(\frac{\pi t}{12}\right)\underset{\sim}{i}+2\tan\left(\frac{\pi t}{12}\right)\underset{\sim}{j}$

$$\underset{\sim}{v}(t)=\frac{\pi\sin\left(\frac{\pi t}{12}\right)}{12\cos^2\left(\frac{\pi t}{12}\right)}\underset{\sim}{i}+\frac{\pi}{6\cos^2\left(\frac{\pi t}{12}\right)}\underset{\sim}{j}$$

(1 mark)

$$|\underset{\sim}{v}(t)|=\sqrt{\frac{\pi^2\sin^2\left(\frac{\pi t}{12}\right)}{144\cos^4\left(\frac{\pi t}{12}\right)}+\frac{\pi^2}{36\cos^4\left(\frac{\pi t}{12}\right)}}$$

(1 mark)

$$|\underset{\sim}{v}(2)|=\sqrt{\frac{\pi^2}{4}\div\frac{144\times 9}{16}+\frac{\pi^2\times 16}{36\times 9}}$$

$$=\sqrt{\frac{\pi^2}{324}+\frac{16\pi^2}{324}}$$

$$=\frac{\sqrt{17}\pi}{18}\text{ km/hr}$$

(1 mark)

f. We need to find the angle between the vectors $\underset{\sim}{v}(2)$ and $\underset{\sim}{j}$.

From part **e.**

$$\underset{\sim}{v}(t)=\frac{\pi\sin\left(\frac{\pi t}{12}\right)}{12\cos^2\left(\frac{\pi t}{12}\right)}\underset{\sim}{i}+\frac{\pi}{6\cos^2\left(\frac{\pi t}{12}\right)}\underset{\sim}{j}$$

$$\begin{aligned}\underset{\sim}{v}(2)&=\frac{\pi\sin\left(\frac{\pi}{6}\right)}{12\cos^2\left(\frac{\pi}{6}\right)}\underset{\sim}{i}+\frac{\pi}{6\cos^2\left(\frac{\pi}{6}\right)}\underset{\sim}{j}\\&=\left(\frac{\pi}{2}\div(12\times\frac{3}{4})\right)\underset{\sim}{i}+\frac{\pi}{6\times\frac{3}{4}}\underset{\sim}{j}\\&=\frac{\pi}{18}\underset{\sim}{i}+\frac{2\pi}{9}\underset{\sim}{j}\end{aligned}$$

(1 mark)

$$\underset{\sim}{v}(2)\bullet\underset{\sim}{j}=|\underset{\sim}{v}(2)||\underset{\sim}{j}|\cos(\theta)$$

$$\frac{2\pi}{9}=\frac{\sqrt{17}\pi}{18}\cos(\theta) \qquad \text{from part } \textbf{e.}$$

$$\cos(\theta)=\frac{4}{\sqrt{17}}$$

$$\theta=14.04^\circ \quad \text{(correct to 2 decimal places)}$$

(1 mark)
Total 12 marks

Question 3

a. $y = \dfrac{x^2 + x - 1}{\sqrt{1-x}}$

$$\frac{dy}{dx} = \frac{\sqrt{1-x}(2x+1) - (x^2 + x - 1) \times \frac{1}{2}(1-x)^{-\frac{1}{2}} \times -1}{1-x}$$

$$= \left(\sqrt{1-x}(2x+1) + \frac{x^2 + x - 1}{2\sqrt{1-x}}\right) \div (1-x)$$

$$= \frac{2(1-x)(2x+1) + x^2 + x - 1}{2\sqrt{1-x}} \times \frac{1}{1-x}$$

$$= \frac{2(1 + x - 2x^2) + x^2 + x - 1}{2(1-x)^{\frac{3}{2}}}$$

$$= \frac{-3x^2 + 3x + 1}{2(1-x)^{\frac{3}{2}}}$$

as required

(1 mark)

b. Using CAS, $\dfrac{d^2y}{dx^2} = \dfrac{0.75x^2 - 2.25x + 2.25}{(x-1)^2\sqrt{1-x}}$

For a point of inflection, $\dfrac{d^2y}{dx^2} = 0$

So $0.75x^2 - 2.25x + 2.25 = 0$

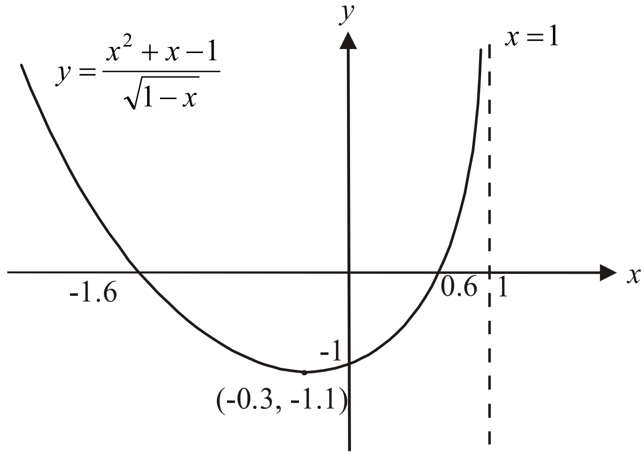
but
$$\begin{aligned} \Delta &= (-2.25)^2 - 4 \times 0.75 \times 2.25 \\ &= -1.6875 \\ &< 0 \end{aligned}$$

(1 mark)

So no solution exists and therefore no point of inflection exists.

(Note: $\dfrac{d^2y}{dx^2} = 0$ is a necessary condition but not a sufficient condition that a point of inflection exists. That is, you would have to show that $\dfrac{d^2y}{dx^2}$ changes sign to either side of a point where $\dfrac{d^2y}{dx^2} = 0$ as well as showing that $\dfrac{d^2y}{dx^2} = 0$.)

c.



(1 mark) – correct shape including asymptote
(1 mark) – correct intercepts and turning point

Note that the x-intercepts and the turning point coordinates are expressed correct to 1 decimal place.

d. The solution which has its turning point located at $(a,0)$ is given by

$$y=\frac{x^2+x-1}{\sqrt{1-x}}+1.1$$ **(1 mark)**

(Since the question directs that information from part **c.** be used, the value of 1.1 is expressed correct to 1 decimal place because that is what was asked for in part **c.**.)
This solution is obtained by translating the graph of $y=f(x)$ upwards by 1.1 units.

e. **i.**
$$\frac{dy}{dx}=\frac{-3x^2+3x+1}{2(1-x)^{\frac{3}{2}}}$$

$x_{n+1}=x_n+h, \quad y_{n+1}=y_n+h\,f(x_n)$

$x_0=0, \;\; y_0=1$

$$x_1=0.1, \;\; y_1=1+0.1\times\left(\frac{1}{2}\right)$$
$$=1.05$$ **(1 mark)**

$$x_2=0.2, \;\; y_2=1.05+0.1\times(0.743721)$$
$$=1.12437...$$

So $y=1.124$ (correct to 3 decimal places) is the approximate value found using Eulers method.

(1 mark)

ii.
$$\frac{dy}{dx}=\frac{-3x^2+3x+1}{2(1-x)^{\frac{3}{2}}}$$

$$y=\frac{x^2+x-1}{\sqrt{1-x}}+c$$

When $x=0, \; y=1$

$$1=\frac{-1}{\sqrt{1}}+c$$

$c=2$

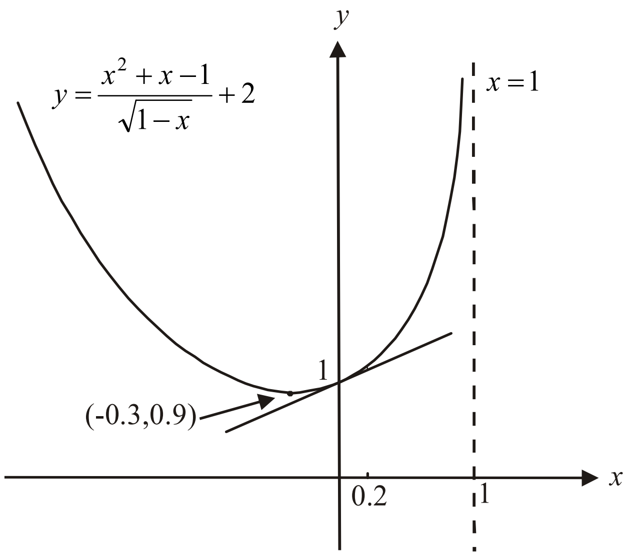
$$y=\frac{x^2+x-1}{\sqrt{1-x}}+2$$

When $x=0.2$

$y=1.150$ (correct to 3 decimal places) **(1 mark)**

f. From the graph in part **c.** the turning point is located at $(-0.3,-1.1)$ and there are no points of inflection. Therefore the gradient of the graph for $x > -0.3$ is positive and increasing. For this solution the graph is the same shape but is translated 2 units upwards.
At $x = 0$, a tangent is drawn.
At $x = 0.1$ the gradient of the tangent will be calculated and this gradient will be used to draw the line segment from $x = 0.1$ to $x = 0.2$. It will still be below the value lying on the curve at $x = 0.2$.

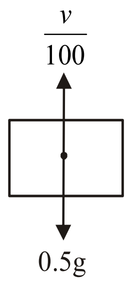
(1 mark)



Total 9 marks

Question 4

a.



(1 mark)

b.

$$\underset{\sim}{R} = m\underset{\sim}{a}$$

$$0.5g - \frac{v}{100} = 0.5a$$

$$a = g - \frac{2v}{100}$$

$$a = g - \frac{v}{50} \text{ as required}$$

(1 mark)

c.

$$a = g - \frac{v}{50} \qquad \text{(using "Hence")}$$

$$\frac{dv}{dt} = \frac{50g - v}{50}$$

$$\frac{dt}{dv} = \frac{50}{50g - v}$$

$$t = \int \frac{50}{50g - v} dv$$

$$t = -50\log_e(50g - v) + c \qquad \text{since } v < 50g \text{ (as } a > 0\text{), } 50g - v > 0$$

(1 mark)

$$t = 0,\ v = 0 \qquad 0 = -50\log_e(50g) + c$$

$$c = 50\log_e(50g)$$

$$t = -50\log_e(50g - v) + 50\log_e(50g)$$

$$t = 50\log_e\left(\frac{50g}{50g - v}\right)$$

(1 mark)

$$\frac{t}{50} = \log_e\left(\frac{50g}{50g - v}\right)$$

$$e^{\frac{t}{50}} = \frac{50g}{50g - v}$$

$$50g - v = \frac{50g}{e^{\frac{t}{50}}}$$

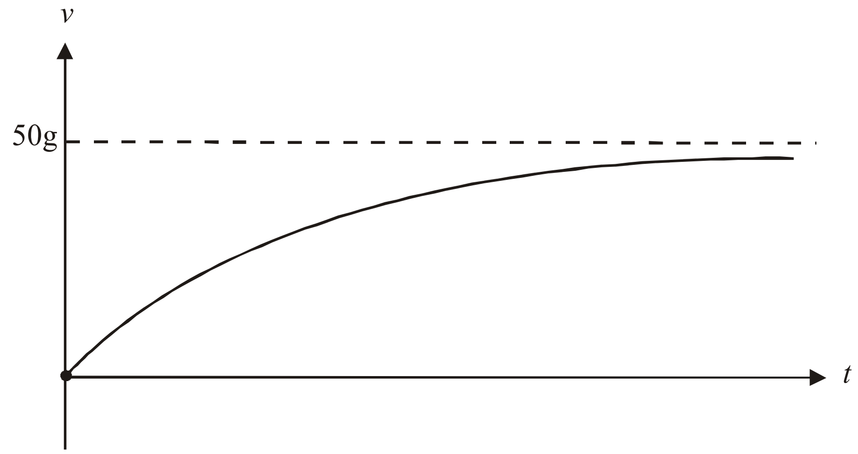
$$-v = 50ge^{-\frac{t}{50}} - 50g$$

$$v = 50g - 50ge^{-\frac{t}{50}}$$

$$v = 50g(1 - e^{-\frac{t}{50}}) \text{ as required}$$

(1 mark)

d. **i.** $v(t) = 50g(1 - e^{-\frac{t}{50}})$



(1 mark)

ii. The limiting (terminal) velocity is 50g ms^{-1}.

(1 mark)

e. Solve $\int_0^m v(t) = 40$ for m. Note that the area under the graph equals distance travelled.

(1 mark)

Since $t \geq 0$, $t = 2.88$ secs (correct to 2 decimal places)

(1 mark)

f. Solve $v(t) = 50$ for t

$t = 5.38153...$

(1 mark)

Find $\int_0^{5.38153} v(t)dt$

$= 136.95$ metres (correct to 2 decimal places)

(1 mark)

g. 3 seconds after the platform starts its motion, the object has been falling for five seconds. In 5 seconds the distance it has covered is given by

$$\int_0^5 v(t)dt = 118.517$$

(1 mark)

The platform moves with constant acceleration. After 3 seconds the distance it has covered is given by

$$s = ut + \frac{1}{2}at^2$$

$$s = 0 + \frac{1}{2} \times 1 \times 9$$

$$s = 4.5\text{m}$$

So 3 seconds after the platform begins its motion, it is still $118.517 - 4.5 = 114.02$m above the object. (correct to 2 decimal places).

(1 mark)

Total 13 marks

Question 5

a.

$$\begin{aligned} |z-1| &= 1 \quad z \in C \\ |x+iy-1| &= 1 \\ \sqrt{(x-1)^2 + y^2} &= 1 \\ (x-1)^2 + y^2 &= 1 \quad \text{as required} \end{aligned}$$

(1 mark)

b. The Cartesian equation of $\operatorname{Im}(z) = -\frac{1}{\sqrt{2}}$ is $y = -\frac{1}{\sqrt{2}}$.

Sub into $(x-1)^2 + y^2 = 1$

$$(x-1)^2 + \frac{1}{2} = 1$$

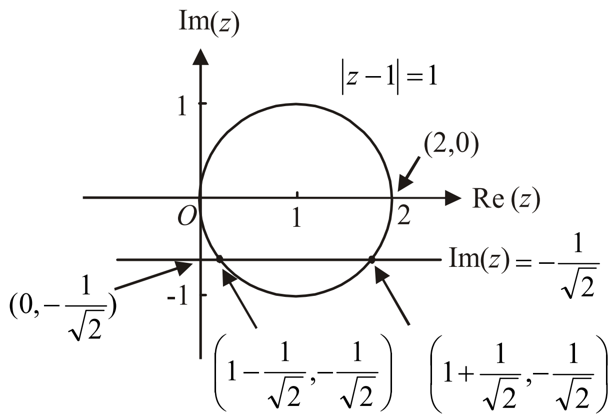
(1 mark)

$$\begin{aligned} (x-1)^2 &= \frac{1}{2} \\ x-1 &= \pm\frac{1}{\sqrt{2}} \\ x &= 1 \pm \frac{1}{\sqrt{2}} \end{aligned}$$

The other point of intersection is $\left(1-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

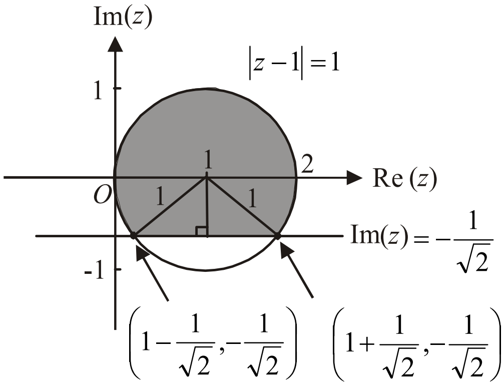
(1 mark)

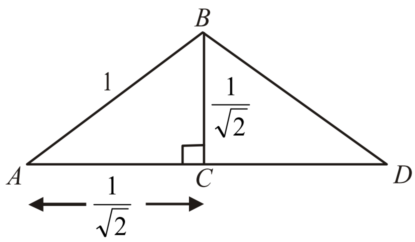
c.



(1 mark) – correct circle
(1 mark) – correct straight line with points of intersection shown.

d. Find the area given by $\{z : |z-1| \le 1\} \cap \left\{z : \text{Im}(z) \ge -\frac{1}{\sqrt{2}}\right\}$





So ΔABC is isosceles so $\angle ABC = 45^\circ$ so $\angle ABD = 90^\circ$

(1 mark)

Method 1

$$\begin{aligned}\text{Area required } &= \frac{1}{2} \times AD \times BC + \frac{3}{4} \times \pi \times r^2\\ &= \frac{1}{2} \times \frac{2}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{3}{4} \times \pi\\ &= \frac{1}{2} + \frac{3\pi}{4}\\ &= \frac{2+3\pi}{4} \text{ square units}\end{aligned}$$

(1 mark)

Method 2

Area required is a major segment with $\theta = \frac{3\pi}{2}$.

$$\begin{aligned}\text{Area} &= \frac{1}{2} r^2 (\theta - \sin(\theta))\\ &= \frac{1}{2} \times 1 \times \left(\frac{3\pi}{2} - \sin(\frac{3\pi}{2})\right)\\ &= \frac{3\pi}{4} + \frac{1}{2}\\ &= \frac{3\pi+2}{4} \text{ square units}\end{aligned}$$

(1 mark)

e. $$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)} \quad \text{(formula sheet)}$$

$$\tan\left(2 \times \frac{-\pi}{8}\right) = \frac{2\tan\left(\frac{-\pi}{8}\right)}{1-\tan^2\left(\frac{-\pi}{8}\right)}$$

$$-1 = \frac{2\tan\left(\frac{-\pi}{8}\right)}{1-\tan^2\left(\frac{-\pi}{8}\right)} \quad \text{since } \tan\left(\frac{-\pi}{4}\right) = -1$$

$$\tan^2\left(\frac{-\pi}{8}\right) - 1 = 2\tan\left(\frac{-\pi}{8}\right)$$

$$\tan^2\left(\frac{-\pi}{8}\right) - 2\tan\left(\frac{-\pi}{8}\right) - 1 = 0$$

Let $x = \tan\left(\frac{-\pi}{8}\right)$ **(1 mark)**

$$x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{4 - 4 \times 1 \times -1}}{2}$$

$$= \frac{2 \pm \sqrt{8}}{2}$$

$$= 1 \pm \sqrt{2}$$

$\tan\left(\frac{-\pi}{8}\right) = 1 \pm \sqrt{2}$ **(1 mark)**

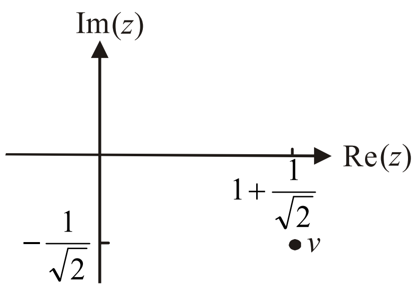
but $\frac{-\pi}{8}$ is in the 4th quadrant **(1 mark)**

so, $\tan\left(\frac{-\pi}{8}\right)$ is negative.

So $\tan\left(\frac{-\pi}{8}\right) = 1 - \sqrt{2}$ as required.

f. **i.** $V\left(1 + \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$$v = \left(1 + \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}i$$



(1 mark)

ii. $v = \left(1 + \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}i$

$$\begin{aligned}\text{Arg}(v) &= \tan^{-1}\left(\frac{y}{x}\right)\\ &= \tan^{-1}\left(-\frac{1}{\sqrt{2}} \div \left(1 + \frac{1}{\sqrt{2}}\right)\right)\\ &= \tan^{-1}\left(\frac{-1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}+1}\right)\\ &= \tan^{-1}\left(\frac{-1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}\right)\\ &= \tan^{-1}\left(\frac{1-\sqrt{2}}{2-1}\right)\\ &= \tan^{-1}(1-\sqrt{2})\end{aligned}$$

(1 mark)

From part **e.**,

$$\tan\left(\frac{-\pi}{8}\right) = 1 - \sqrt{2}$$

So $\tan^{-1}(1-\sqrt{2}) = \frac{-\pi}{8}$

So $\text{Arg}(v) = \frac{-\pi}{8}$ as required.

(1 mark)

(Check: $-\pi < \text{Arg}(z) \le \pi$ from the formula sheet; that is we required a principal valued argument of z.)

Total 13 marks