2010 Specialist Maths Trial Exam 2 Solutions

Free download from www.itute.com

Section 1

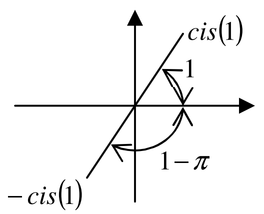
1	2	3	4	5	6	7	8	9	10	11
C	C	C	B	B	A	A	E	C	C	C

12	13	14	15	16	17	18	19	20	21	22
D	E	E	B	A	E	C	A	D	D	B

Q1 $z^2 - z + 1 = 0$, $z = \dfrac{-^{-}1 \pm \sqrt{\left(^{-}1\right)^2 - 4(1)(1)}}{2(1)}$

$= \dfrac{1 \pm \sqrt{^{-}3}}{2} = \dfrac{1 \pm \sqrt{3}i}{2} = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2} i$ C

Q2 $z = -cis(1)$



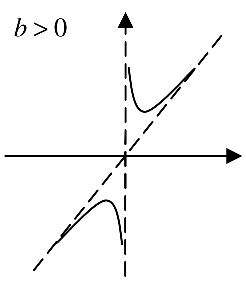
$|z| = 1$, $Arg(z) = 1 - \pi$ C

Q3 The polynomial has three linear factors, not solutions or roots. C

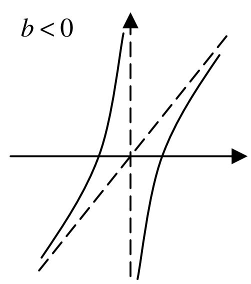
Q4 Any complex number on the straight line is equidistant from $z = 0$ and $z = a + ai$, i.e. $|z - a - ai| = |z|$.

$\therefore |z - a - ai| - |z| = 0$ B

Q5 $b > 0$



$b < 0$



B

Q6 $4x^2 + px + q^2$ cannot be a perfect square for $y = \dfrac{2}{4x^2 + px + q^2}$ to have a stationary point, i.e.

$\Delta = p^2 - 16q^2 \neq 0$. $\therefore p^2 \neq 16q^2$, $p \neq \pm 4q$.

Only choice A ($p < -4q$) satisfies this requirement. A

Q7 $\sin\left(a + \cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)\right) = \sin\left(a + \dfrac{5\pi}{6}\right)$

$= \sin a \cos\left(\dfrac{5\pi}{6}\right) + \cos a \sin\left(\dfrac{5\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}\sin a + \dfrac{1}{2}\cos a$ A

Q8 $f(x) = \sin^{-1}\left(\dfrac{x}{a} + b\right) + c$, where $a, b, c \in (-\infty, 0)$, i.e. $a < 0$, $b < 0$ and $c < 0$.

$-1 \le \dfrac{x}{a} + b \le 1$, $-1 - b \le \dfrac{x}{a} \le 1 - b$, $a(-1 - b) \ge x \ge a(1 - b)$,

i.e. $a(1 - b) \le x \le -a(1 + b)$ E

Q9 $\tan^{-1} b = 0.2$, $\therefore b = \tan 0.2 > 0$

$b = \tan 0.2 = \tan 2(0.1) = \dfrac{2\tan 0.1}{1 - \tan^2 0.1}$

$\therefore b(1 - \tan^2 0.1) = 2\tan 0.1$

Simplify to: $b\tan^2 0.1 + 2\tan 0.1 - b = 0$

$\therefore \tan 0.1 = \dfrac{\sqrt{1 + b^2} - 1}{b}$ but not $\dfrac{-\sqrt{1 + b^2} - 1}{b}$ because $\tan 0.1 > 0$. C

Q10 The range of $\cos^{-1} x$ is $[0, \pi]$.

$\cos^{-1}(\cos\theta) = \cos^{-1}(\cos(-\theta)) = \cos^{-1}(\cos(2\pi - \theta)) = 2\pi - \theta$ C

Q11 $3\tilde{i} - 4\tilde{j}$ cannot be expressed as a linear combination of $-2\tilde{i} + 3\tilde{k}$ and $\tilde{j} - 2\tilde{k}$ C

Q12 $\overrightarrow{AO}.\overrightarrow{BO} = \dfrac{1}{2}$, $\therefore \cos\angle AOB = \dfrac{1}{2}$, $\angle AOB = \dfrac{\pi}{3}$,

$\angle APB = \dfrac{1}{2}\angle AOB = \dfrac{\pi}{6}$, $\therefore \cos\angle APB = \dfrac{\sqrt{3}}{2}$.

$\overrightarrow{AP}.\overrightarrow{BP} = \sqrt{3}$, $\left|\overrightarrow{AP}\right|\left|\overrightarrow{BP}\right|\cos\angle APB = \sqrt{3}$, $\therefore \left|\overrightarrow{AP}\right|\left|\overrightarrow{BP}\right| = 2$ D

Q13 $\overrightarrow{OQ} = \dfrac{3\overrightarrow{OP} + 2\overrightarrow{OR}}{2 + 3} = \dfrac{(-3,0,6) + (4,2,-4)}{5} = \dfrac{(1,2,2)}{5}$

$\left|\overrightarrow{OQ}\right| = \dfrac{1}{5}\sqrt{1^2 + 2^2 + 2^2} = \dfrac{3}{5}$ E

Q14 $\left|\dfrac{\sqrt{3}}{2}\tilde{i} - \dfrac{1}{\sqrt{2}}\tilde{j} - \dfrac{1}{2}\tilde{k}\right| = \sqrt{\dfrac{3}{4} + \dfrac{1}{2} + \dfrac{1}{4}} = \dfrac{\sqrt{3}}{\sqrt{2}}$

Angle with the *x*-axis $= \cos^{-1}\left(\dfrac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{\sqrt{2}}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = 45°$

Angle with the *y*-axis $= \cos^{-1}\left(\dfrac{-\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{\sqrt{2}}}\right) = \cos^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$

Angle with the *z*-axis $= \cos^{-1}\left(\dfrac{-\frac{1}{2}}{\frac{\sqrt{3}}{\sqrt{2}}}\right) = \cos^{-1}\left(-\dfrac{1}{\sqrt{6}}\right)$ E

Q15 $\tilde{v} = \cos^{-1}(t)\left(2\tilde{i} - 3\tilde{j} + \tilde{k}\right)$. $2\tilde{i} - 3\tilde{j} + \tilde{k}$ is a constant vector, $\therefore$ the particle moves in the same direction, i.e. in a straight line. B

Q16 $\frac{dy}{dh}=\sqrt{y(2-y)}\frac{dx}{dh}$, $\frac{dy}{dx}=\sqrt{y(2-y)}$, $\frac{dx}{dy}=\frac{1}{\sqrt{y(2-y)}}$

$\therefore \frac{dx}{dy}=\frac{1}{\sqrt{1-(y-1)^2}}$, $x=\int\frac{1}{\sqrt{1-(y-1)^2}}dy=\sin^{-1}(y-1)+c$

$\therefore y=\sin(x-c)+1$ A

Q17 At $t=8$, $s={}^{-}2$, $x=2+{}^{-}2=0$ E

Q18 The particle starts from rest and travels in the negative direction with increasing speed, then slows down to a stop at $t=7$. During $7<t\le 9$ the particle travels in the positive direction with increasing speed. In the first 7 s the particle has the greatest displacement (area between the curve and the t-axis) from its initial position. C

Q19 $\int_0^{\frac{\pi}{2}}\cos^3 x\sqrt{1-\sin x}\,dx$

Let $u=1-\sin x$, $\sin x=1-u$, $\frac{du}{dx}=-\cos x$ When $x=0$, $u=1$; when $x=\frac{\pi}{2}$, $u=0$.

$=\int_0^{\frac{\pi}{2}}\cos^2 x\sqrt{1-\sin x}\cos x\,dx$

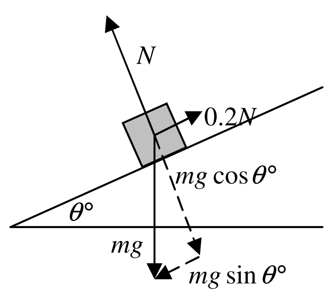
$=\int_0^{\frac{\pi}{2}}(1-\sin^2 x)\sqrt{1-\sin x}\cos x\,dx$

$=\int_1^0 -\left(1-(1-u)^2\right)\sqrt{u}\,du$

$=\int_0^1 (2u-u^2)\sqrt{u}\,du$

$=\int_0^1\left(2u^{\frac{3}{2}}-u^{\frac{5}{2}}\right)du$ A

Q20



$N-mg\cos\theta°=0$

$mg\sin\theta°-0.2N=0$

$\therefore \sin\theta°=0.2\cos\theta°$, $\tan\theta°=0.2$, $\theta°=\tan^{-1}(0.2)\approx 11°$ D

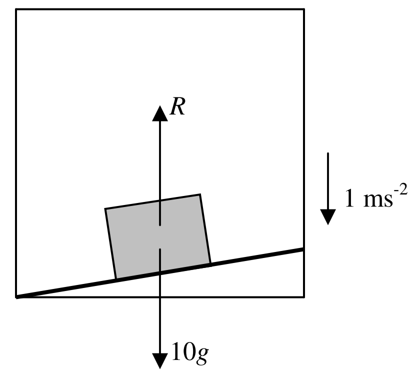
Q21 $v=-\sqrt{100-x}$

The negative sign indicates that the particle moves in the negative direction.

$v^2=100-x$, $a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-\frac{1}{2}$

The acceleration is constant and it is in the negative direction, i.e. in the same direction as the velocity. $\therefore$ the particle speeds up and maintains the same direction of motion. D

Q22



$10g-R=10\times 1$, $98-R=10$

$\therefore R=88$ N B

Section 2

Q1a $\tan 75°=\tan(45°+30°)=\frac{\tan 45°+\tan 30°}{1-\tan 45°\tan 30°}$

$=\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}=2+\sqrt{3}$

$\therefore y=\pm\left(2+\sqrt{3}\right)x$

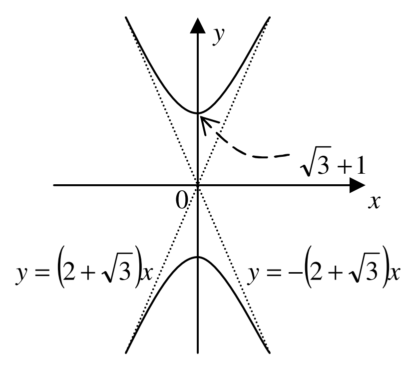
Q1b $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$

When $x=0$, $y=\pm b=\pm\left(\sqrt{3}+1\right)$. $\therefore b=\sqrt{3}+1$

Gradient of asymptote $=\frac{b}{a}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$. $\therefore a=\sqrt{3}-1$

$\therefore \frac{x^2}{\left(\sqrt{3}-1\right)^2}-\frac{y^2}{\left(\sqrt{3}+1\right)^2}=-1$

Q1c



Q1di On the positive-gradient asymptote: $x=a$,

$y=a\tan 75°=\left(2+\sqrt{3}\right)a$ $\therefore \overrightarrow{OA}=a\tilde{i}+\left(2+\sqrt{3}\right)a\tilde{j}$

On the negative-gradient asymptote: $x=b$,

$y=-b\tan 75°=-\left(2+\sqrt{3}\right)b$ $\therefore \overrightarrow{OB}=b\tilde{i}-\left(2+\sqrt{3}\right)b\tilde{j}$

Q1dii $\overrightarrow{OM}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OB}\right)=\left(\frac{a+b}{2}\right)\tilde{i}+\left(2+\sqrt{3}\right)\left(\frac{a-b}{2}\right)\tilde{j}$

Q1diii $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = (a-b)\tilde{i} + \left(2+\sqrt{3}\right)(a+b)\tilde{j}$

$\therefore \left|\overrightarrow{BA}\right|^2 = (a-b)^2 + \left(2+\sqrt{3}\right)^2 (a+b)^2 = 4$

Q1div $\overrightarrow{OM} = \left(\frac{a+b}{2}\right)\tilde{i} + \left(2+\sqrt{3}\right)\left(\frac{a-b}{2}\right)\tilde{j}$

Coordinates of point *M* are $x = \frac{a+b}{2}$ and $y = \left(2+\sqrt{3}\right)\left(\frac{a-b}{2}\right)$.

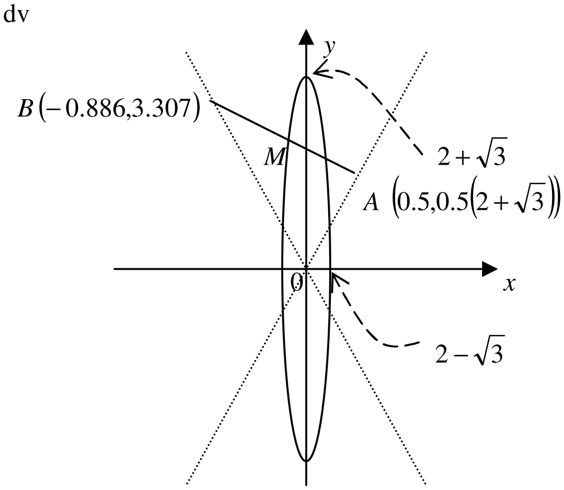
$\therefore a+b = 2x$ and $a-b = \frac{2y}{2+\sqrt{3}}$

Substitute into $(a-b)^2 + \left(2+\sqrt{3}\right)^2 (a+b)^2 = 4$

$$\left(\frac{2y}{2+\sqrt{3}}\right)^2 + \left(2+\sqrt{3}\right)^2 (2x)^2 = 4$$

Simplify: $\frac{x^2}{\left(2-\sqrt{3}\right)^2} + \frac{y^2}{\left(2+\sqrt{3}\right)^2} = 1$, an ellipse.

Q1dv



Q1dvi $(a-b)^2 + \left(2+\sqrt{3}\right)^2 (a+b)^2 = 4$

When $a = 0.5$, $b = -0.886$ or 0.020 by calculator.

A: When $x = a = 0.5$, $y = \left(2+\sqrt{3}\right)0.5$

B: When $x = b \approx -0.886$, $y = -\left(2+\sqrt{3}\right)(-0.886) \approx 3.307$

Q2ai $f(z) = z^3 - 6iz^2 - 12z + 7i$

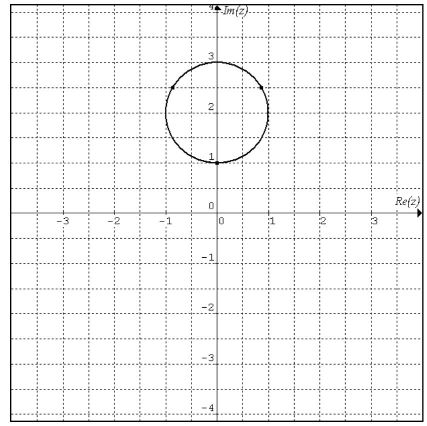
$= (z-a)^3 + b = z^3 - 3az^2 + 3a^2 z - a^3 + b$

$\therefore 3a = 6i$ and $-a^3 + b = 7i$

$\therefore a = 2i$ and $b = -i$

Hence $f(z) = (z-2i)^3 - i$

Q2aii, iii



Q2aiv $f(z) = 0$, $(z-2i)^3 - i = 0$, $(z-2i)^3 = i$

$\therefore (z-2i)^3 = cis\left(\frac{\pi}{2} + 2n\pi\right)$, $z - 2i = cis\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right)$

$n = 0$, $z - 2i = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $\therefore z = \frac{\sqrt{3}}{2} + \frac{5}{2}i$

$n = -1$, $z - 2i = -i$, $\therefore z = i$

$n = 1$, $z - 2i = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$, $\therefore z = -\frac{\sqrt{3}}{2} + \frac{5}{2}i$

Q2b $-iz^3 - 6z^2 + 12iz + 7 = 0$

$-iz^3 - 6z^2 + 12iz + 7 = 0$

$-i\left(z^3 - 6iz^2 - 12z + 7i\right) = 0$

$\therefore z^3 - 6iz^2 - 12z + 7i = 0$

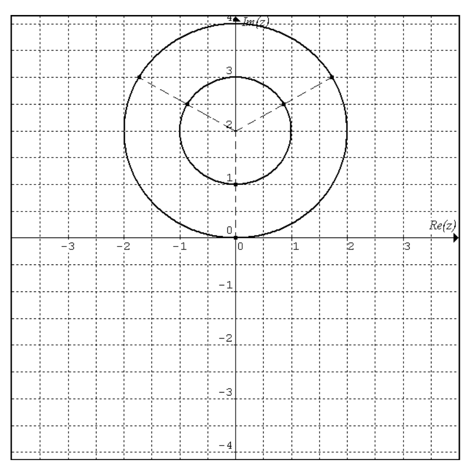
$\therefore z = \pm\frac{\sqrt{3}}{2} + \frac{5}{2}i$ or $z = i$

Q2c $g(z) = f(z) - 7i = (z-2i)^3 - 8i = 0$

$\therefore (z-2i)^3 = 8i = 8cis\left(\frac{\pi}{2} + 2n\pi\right)$

$\therefore z - 2i = 2cis\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right)$

$\therefore$ the roots of $g(z) = 0$ are on a circle of centre $(0,2)$ and radius of 2 units. For $f(z) = 0$ the roots are on a circle of the same centre $(0,2)$ and radius of 1 unit.

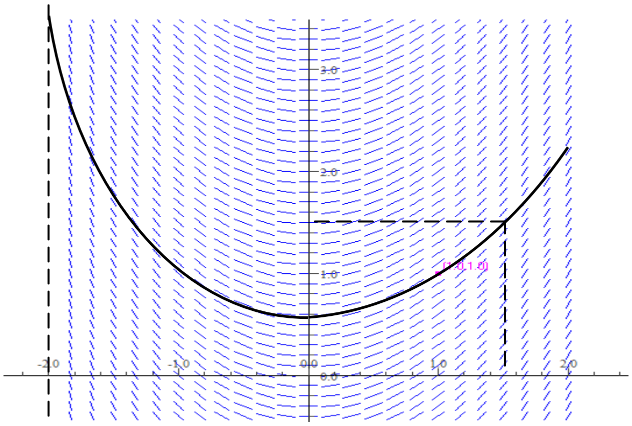


Q3a $-1 \le \frac{x}{2} \le 1$ and $\frac{x+2}{2} > 0$

$\therefore -2 \le x \le 2$ and $x > -2$

$\therefore x \in (-2,2]$

Q3bi



Q3bii When $x = 1.5$, $y \approx 1.5$

Q3c $\frac{dy}{dx} = \sin^{-1}\left(\frac{x}{2}\right) + \log_e\left(\frac{x+2}{2}\right)$

$x = 1$, $y = 1$, $\frac{dy}{dx} = 0.92906$

$x = 1.25$, $y = 1 + 0.25 \times 0.92906 = 1.23227$, $\frac{dy}{dx} = 1.16064$

$x = 1.5$, $y = 1.23227 + 0.25 \times 1.16064 \approx 1.52$

Q3di $\int_1^{1.5} \left(\sin^{-1}\left(\frac{x}{2}\right) + \log_e\left(\frac{x+2}{2}\right) \right) dx$

Q3dii By calculator $\int_1^{1.5} \left(\sin^{-1}\left(\frac{x}{2}\right) + \log_e\left(\frac{x+2}{2}\right) \right) dx \approx 0.58$

$\therefore y \approx 1 + 0.58 = 1.58$

Q4a $f(x) = \frac{x}{\sqrt{p^2 - x^2}} + q$

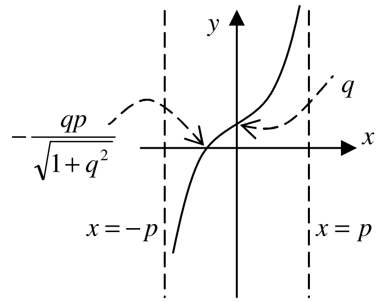
Asymptotes: $p^2 - x^2 = 0$, $x = \pm p$

y-intercept: $x = 0$, $y = q$

x-intercept: $y = 0$, $\frac{x}{\sqrt{p^2 - x^2}} + q = 0$, $x = -q\sqrt{p^2 - x^2}$

$x^2 = q^2(p^2 - x^2)$, $x^2 = q^2p^2 - q^2x^2$, $x^2(1+q^2) = q^2p^2$

$x = \pm \frac{qp}{\sqrt{1+q^2}}$



Q4b $p = \sqrt{3}$, $q = \sqrt{3}$, $\therefore x = -\frac{qp}{\sqrt{1+q^2}} = -\frac{3}{2}$

Area $= \int_{-\frac{3}{2}}^{\frac{3}{2}} \left(\frac{x}{\sqrt{3-x^2}} + \sqrt{3} \right) dx = \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{x}{\sqrt{3-x^2}} dx + \int_{-\frac{3}{2}}^{\frac{3}{2}} \sqrt{3} dx$

$= \int_{\frac{3}{4}}^{\frac{3}{4}} \left(-\frac{1}{2\sqrt{u}} \right) du + \left[\sqrt{3}x\right]_{-\frac{3}{2}}^{\frac{3}{2}}$ (where $u = 3 - x^2$)

$= 0 + 3\sqrt{3} = 3\sqrt{3}$

Q4ci $V = \int_{-\frac{3}{2}}^{\frac{3}{2}} \pi \left(\frac{x}{\sqrt{3-x^2}} + \sqrt{3} \right)^2 dx$

Q4cii By calculator $V \approx 33.2$

Q4d $V = \int_{-\frac{3}{2}}^{h} \pi \left(\frac{x}{\sqrt{3-x^2}} + \sqrt{3} \right)^2 dx$

$= \pi \int_{-\frac{3}{2}}^{h} \left(\frac{x^2}{3-x^2} + \frac{2\sqrt{3}x}{\sqrt{3-x^2}} + 3 \right) dx$

$= \pi \int_{-\frac{3}{2}}^{h} \left(\frac{3}{3-x^2} - 1 + \frac{2\sqrt{3}x}{\sqrt{3-x^2}} + 3 \right) dx$

$= \pi \int_{-\frac{3}{2}}^{h} \left(\frac{3}{(\sqrt{3})^2 - x^2} + 2 + \frac{2\sqrt{3}x}{\sqrt{3-x^2}} \right) dx$

$= \pi \int_{-\frac{3}{2}}^{h} \left(\frac{\sqrt{3}}{2} \left(\frac{1}{\sqrt{3}+x} + \frac{1}{\sqrt{3}-x} \right) + 2 + \frac{2\sqrt{3}x}{\sqrt{3-x^2}} \right) dx$

$= \frac{\pi\sqrt{3}}{2} \int_{-\frac{3}{2}}^{h} \left(\frac{1}{\sqrt{3}+x} + \frac{1}{\sqrt{3}-x} \right) dx + 2\pi \int_{-\frac{3}{2}}^{h} dx + 2\sqrt{3}\pi \int_{-\frac{3}{2}}^{h} \frac{x}{\sqrt{3-x^2}} dx$

$= \frac{\pi\sqrt{3}}{2} \int_{-\frac{3}{2}}^{h} \left(\frac{1}{\sqrt{3}+x} + \frac{1}{\sqrt{3}-x} \right) dx + 2\pi \int_{-\frac{3}{2}}^{h} dx - \pi\sqrt{3} \int_{\frac{3}{4}}^{3-h^2} \frac{1}{\sqrt{u}} du$

$= \frac{\pi\sqrt{3}}{2} \left[\log_e \frac{\sqrt{3}+x}{\sqrt{3}-x} \right]_{-\frac{3}{2}}^{h} + \pi[2x]_{-\frac{3}{2}}^{h} - \pi\sqrt{3}\left[2\sqrt{u}\right]_{\frac{3}{4}}^{3-h^2}$

$= \pi \left(\frac{\sqrt{3}}{2} \log_e \frac{(\sqrt{3}+\frac{3}{2})(\sqrt{3}+h)}{(\sqrt{3}-\frac{3}{2})(\sqrt{3}-h)} + 2h - 2\sqrt{3}\sqrt{3-h^2} + 6 \right)$

Q4e $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$$0.5 = \pi \left(\frac{h}{\sqrt{3-h^2}} + \sqrt{3} \right)^2 \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{1}{2\pi \left(\frac{h}{\sqrt{3-h^2}} + \sqrt{3} \right)^2}$$

When $h = 0$, $\frac{dh}{dt} = \frac{1}{6\pi}$ cm per second.

Q5a $v = \frac{1}{2}\tan^{-1}(5t + 2.5) + \frac{\pi}{2}$, $a = \frac{dv}{dt} = \frac{5}{2\left(1 + (5t + 2.5)^2\right)}$

At $t = 0$, $v = 2.166\ \text{ms}^{-1}$, $a = 0.345\ \text{ms}^{-2}$.

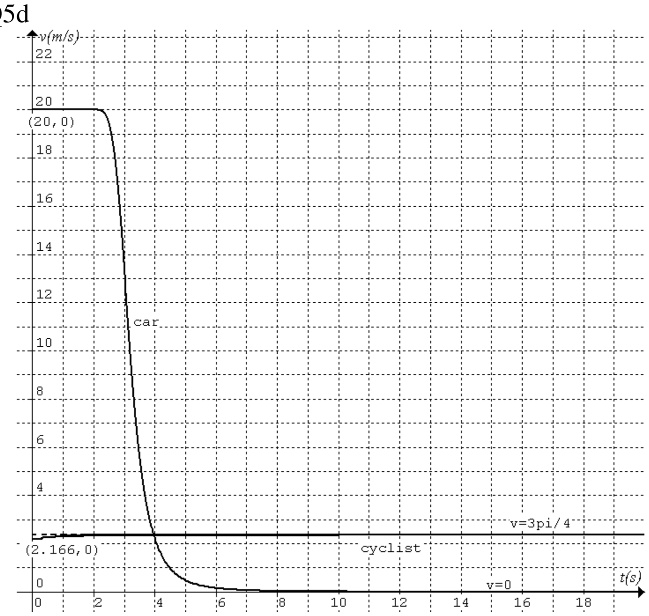
Q5b Resultant force $R = ma = 85 \times 0.345 \approx 29$ N

Q5c Cyclist: Momentum $p = mv = 85 \times 2.166 \approx 184.11$ kg ms^{-1}

Car: Momentum $p = mv = 1200 \times 20 = 24000$ kg ms^{-1}

Difference $= 24000 - 184.11 \approx 23816$ kg ms^{-1}

Q5d



Cyclist: $v = \frac{1}{2}\tan^{-1}(5t + 2.5) + \frac{\pi}{2}$

As $t \to \infty$, $\tan^{-1}(5t + 2.5) \to \frac{\pi}{2}$, $\therefore v \to \frac{3\pi}{4}$.

Asymptote is $v = \frac{3\pi}{4}$

Car: As $t \to \infty$, $v = \frac{20}{1 + 0.5(t-2)^4} \to 0$.

Asymptote is $v = 0$

Q5e At $t = 10$ s, distance by car = area under graph (0 to 10 s) $= 20 \times 2 + 26.39 = 66.39$ m; distance by cyclist = 23.26 m.

$\therefore$ car is ahead by $66.39 - 23.26 \approx 43$ m

Q5f At $t = 10$ s, car speed ≈ 0 and cyclist speed $\approx \frac{3\pi}{4} \approx 2.356$.

Let T be the time when the car and the cyclist are next to each other.

$(T - 10) \times 2.356 \approx 43$, $\therefore T \approx 28$ s

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors