2010 VCAA Specialist Mathematics Exam 1 Solutions

Q1 $f(z)=z^3+9z^2+28z+20$ and $f(-1)=0$, .: $z+1$ is a factor of $f(z)$.

$f(z)=(z+1)(z^2+pz+20)$.

Expand and compare coefficients, $p+1=9$, .: $p=8$.

.: $f(z)=(z+1)(z^2+8z+20)=(z+1)((z+4)^2+2^2)$

$=(z+1)(z+4-2i)(z+4+2i)$

Q2a $a=\frac{R}{m}$, $\frac{dv}{dt}=\frac{v-4}{2}$ and $v=0$ at $t=0$.

Note: The body starts from rest, .: the acceleration is in the negative direction and the body moves in the negative direction, i.e. its velocity is negative for $t>0$.

Q2b $\frac{dv}{dt}=\frac{v-4}{2}$, $\frac{dt}{dv}=\frac{2}{v-4}$, $t=\int\frac{2}{v-4}dv=2\log_e|v-4|+c$.

$v=0$ at $t=0$, .: $c=-2\log_e 4$ and

$t=2\log_e|v-4|-2\log_e 4=2\log_e\left|\frac{v-4}{4}\right|$, .: $\left|\frac{v-4}{4}\right|=e^{\frac{t}{2}}$

Since the velocity is negative for $t>0$, $\frac{v-4}{4}=-e^{\frac{t}{2}}$.

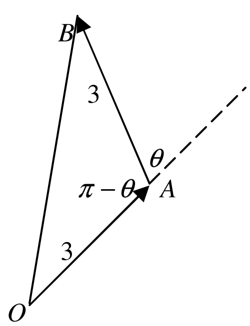
Hence $v=4\left(1-e^{\frac{t}{2}}\right)$.

Q3a $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\left(-\tilde{i}+3\tilde{j}+4\tilde{k}\right)-\left(\tilde{i}+2\tilde{j}+2\tilde{k}\right)$

$=-2\tilde{i}+\tilde{j}+2\tilde{k}$

Q3b $\overrightarrow{OA}.\overrightarrow{AB}=\left|\overrightarrow{OA}\right|\left|\overrightarrow{AB}\right|\cos\theta$, .: $4=9\cos\theta$, $\cos\theta=\frac{4}{9}$.

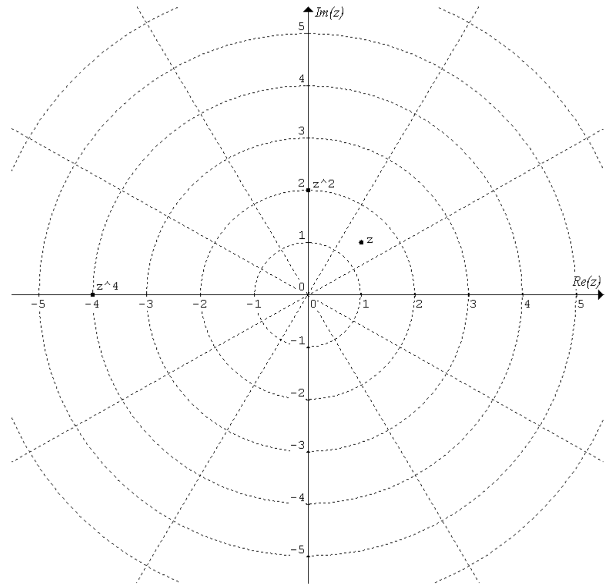
Q3c



$\cos\theta=\frac{4}{9}$, $\sin\theta=\sqrt{1-\cos^2\theta}=\frac{\sqrt{65}}{9}$

Area of $\Delta OAB=\frac{1}{2}\times3\times3\times\sin(\pi-\theta)=\frac{9}{2}\sin\theta=\frac{\sqrt{65}}{2}$ unit^2

Q4



Q5 $f(x)=\tan^{-1}(2x)$, $f'(x)=\frac{2}{1+4x^2}$, $f''(x)=-\frac{16x}{(1+4x^2)^2}$,

$f''\left(\frac{\pi}{2}\right)=-\frac{16\times\frac{\pi}{2}}{\left(1+4\left(\frac{\pi}{2}\right)^2\right)^2}=-\frac{8\pi}{(1+\pi^2)^2}$

Q6 Let $u=\cos(2x)$, $\frac{du}{dx}=-2\sin(2x)$, .: $\sin(2x)=-\frac{1}{2}\frac{du}{dx}$.

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\cos^2(2x)\sin(2x)dx=\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}-\frac{1}{2}u^2\frac{du}{dx}dx=\int_{-1}^{0}-\frac{1}{2}u^2du$

$=\left[-\frac{u^3}{6}\right]_{-1}^{0}=-\frac{1}{6}$

Q7 $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{2}{1-x^2}\right)=\frac{4x}{(1-x^2)^2}$, $-1<x<1$.

.: $\frac{dy}{dx}=\frac{2}{1-x^2}+c$.

Given $\frac{dy}{dx}=3$ when $x=0$, .: $c=1$ and $\frac{dy}{dx}=\frac{2}{1-x^2}+1$.

$y=\int\left(\frac{2}{1-x^2}+1\right)dx=\int\left(\frac{1}{1+x}+\frac{1}{1-x}+1\right)dx$ (partial fractions)

$=\log_e(1+x)-\log_e(1-x)+x+C$ $(-1<x<1)$

$=\log_e\left(\frac{1+x}{1-x}\right)+x+C$

Given $y=4$ when $x=0$, .: $C=4$ and $y=\log_e\left(\frac{1+x}{1-x}\right)+x+4$.

Q8a $y = -t\cos(t) = 0$, .: $t = 0$

or $\cos(t) = 0$, i.e. $t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

Second time at $t = \frac{3\pi}{2}$.

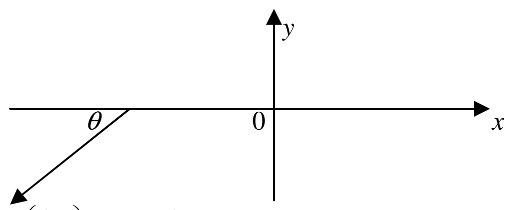
Q8b $\tilde{r}(t) = t\sin(t)\tilde{i} - t\cos(t)\tilde{j}$, $t \geq 0$.

$\tilde{v}(t) = \frac{d\tilde{r}}{dt} = (\sin(t) + t\cos(t))\tilde{i} - (\cos(t) - t\sin(t))\tilde{j}$

.: $\tilde{v}\left(\frac{3\pi}{2}\right) = -\tilde{i} - \frac{3\pi}{2}\tilde{j}$

Speed $= |\tilde{v}| = \sqrt{(-1)^2 + \left(-\frac{3\pi}{2}\right)^2} = \frac{\sqrt{4 + 9\pi^2}}{2}$

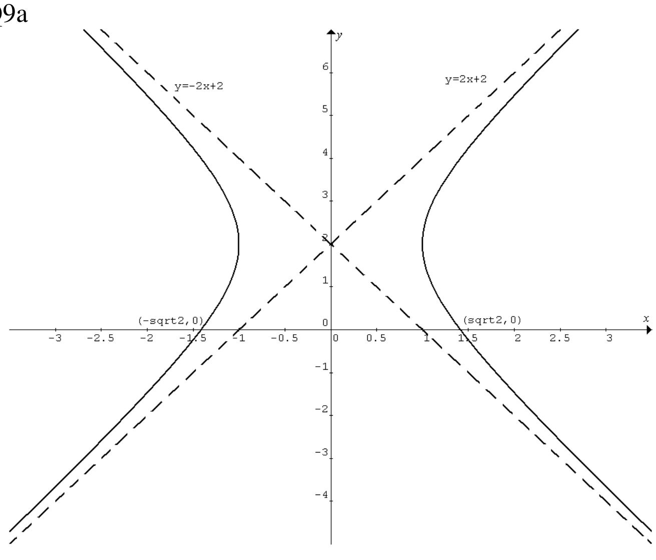
Q8c



$\tilde{v}\left(\frac{3\pi}{2}\right) = -\tilde{i} - \frac{3\pi}{2}\tilde{j}$

$\tan\theta = \frac{3\pi}{2}$

Q9a



Q9b $x^2 - \frac{(y-2)^2}{4} = 1$

When $x = 2$, $y = 2 - 2\sqrt{3}$ ($y < 0$)

Implicit differentiation: $2x - \frac{1}{2}(y-2)\frac{dy}{dx} = 0$, .: $\frac{dy}{dx} = \frac{4x}{y-2}$.

At $\left(2, 2-2\sqrt{3}\right)$, $\frac{dy}{dx} = \frac{4 \times 2}{\left(2 - 2\sqrt{3}\right) - 2} = -\frac{4\sqrt{3}}{3}$.

Q10 x-intercepts: $\left(x^2 - 1\right)\sqrt{x+1} = 0$, .: $x^2 - 1 = 0$, $x = \pm 1$.

Let $u = x + 1$, .: $x = u - 1$

.: $x^2 - 1 = (u-1)^2 - 1 = u^2 - 2u$ and $\frac{du}{dx} = 1$.

Area $= -\int_{-1}^{1}\left(x^2 - 1\right)\sqrt{x+1}\,dx = -\int_{-1}^{1}\left(u^2 - 2u\right)\sqrt{u}\,\frac{du}{dx}\,dx$

$= -\int_{0}^{2}\left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}}\right)du$

$= -\left[\frac{2u^{\frac{7}{2}}}{7} - \frac{4u^{\frac{5}{2}}}{5}\right]_0^2$

$= -\frac{2^{\frac{9}{2}}}{7} + \frac{2^{\frac{9}{2}}}{5}$

$= 2^{\frac{9}{2}}\left(-\frac{1}{7} + \frac{1}{5}\right)$

$= 2^{\frac{9}{2}}\left(\frac{2}{35}\right) = \frac{2^{\frac{11}{2}}}{35} = \frac{2^5\sqrt{2}}{35} = \frac{32\sqrt{2}}{35}$

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