THE SCHOOL FOR EXCELLENCE (TSFX)

UNIT 4 SPECIALIST MATHEMATICS 2010

WRITTEN EXAMINATION 2

Reading Time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOKLET

Structure of Booklet

Section	Number of questions	Number of questions to be answered	Number of marks
1	22	22	22
2	6	6	58

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Students are **NOT** permitted to bring into the examination room: notes of any kind, a calculator, blank sheets of paper and/or white out liquid/tape.

Students are **NOT** permitted to bring mobile phones and/or any electronic communication devices into the examination room.

All written responses must be in English.

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Please ensure that the paper size on your printer is selected as **A4** and that you select “**None**” under “Page Scaling”.

SECTION 1

Instructions for Section 1

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will not **be deducted for incorrect answers. You should attempt every question.**

No marks will be given if more than one answer is completed for any question.

QUESTION 1

The imaginary part of the complex number $z = 2cis\left(\frac{4\pi}{3}\right)$ is

A. $\frac{-1-\sqrt{3}}{2}i$

B. $-\frac{1}{2}i$

C. $-\frac{\sqrt{3}}{2}$

D. $-\sqrt{3}i$

E. $-\frac{\sqrt{3}}{2}i$

QUESTION 2

The position vectors of A, B and C are $2\underset{\sim}{j} + 2\underset{\sim}{k}$, $4\underset{\sim}{i} + 10\underset{\sim}{j} + 18\underset{\sim}{k}$ and $x\underset{\sim}{i} + 14\underset{\sim}{j} + 26\underset{\sim}{k}$ respectively. If A, B and C are collinear, the x value is

A. 2

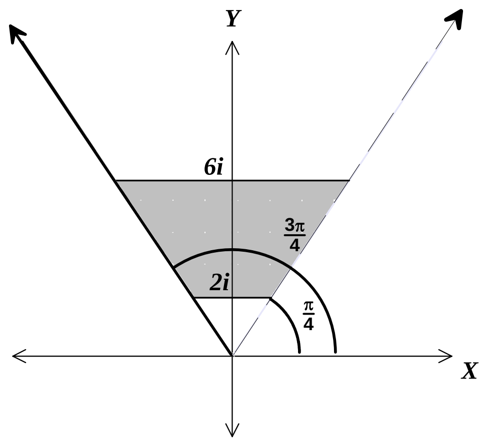
B. 3

C. 4

D. 0

E. 6

QUESTION 3



The best subset is used to describe the above sketch is

A. $\{z : 2 \le \text{Re}(z) \le 6, \frac{\pi}{4} < \text{Arg}\, z\}$

B. $\{z : 2 \le \text{Im}(z) \le 6, \frac{\pi}{4} < \text{Arg}\, z \le \frac{3\pi}{4}\}$

C. $\{z : 2 \le \text{Re}(z) \le 6, \text{Arg}\, z \le \frac{3\pi}{4}\}$

D. $\{z : 2i \le \text{Im}(z) \le 6i, \frac{\pi}{4} < \text{Arg}\, z \le \frac{3\pi}{4}\}$

E. $\{z : 2 \le \text{Im}(z) \le 6, \frac{\pi}{4} \le \text{Arg}\, z \le \frac{3\pi}{4}\}$

QUESTION 4

When $b = 0$, the ellipse with equation $a(x-b)^2 + y^2 = 36$ touches the hyperbola with equation $x^2 - y^2 = 9$, only twice. When $b \ne 0$, the ellipse touches the hyperbola no more than three times. The possible value of a and values of b are

A. $a = 9$ and $b \in R$

B. $a = 4$ and $b \in [-3,3]$

C. $a = 9$ and $b \in [-6,6]$

D. $a = 4$ and $b \in [-6,6]$

E. $a = 12$ and $b \in R$

QUESTION 5

The values of cube root of $\left(1+i\sqrt{3}\right)^4$ are

A. $16\text{cis}\left(\frac{4\pi}{9}\right)$, $16\text{cis}\left(\frac{10\pi}{9}\right)$, $16\text{cis}\left(\frac{16\pi}{9}\right)$

B. $2^{\frac{1}{3}}\text{cis}\left(\frac{4\pi}{3}\right)$

C. $16^{\frac{1}{3}}\text{cis}\left(\frac{4\pi}{9}\right)$, $16^{\frac{1}{3}}\text{cis}\left(\frac{10\pi}{9}\right)$, $16^{\frac{1}{3}}\text{cis}\left(\frac{16\pi}{9}\right)$

D. $2\text{cis}\left(\frac{4\pi}{9}\right)$

E. $16^{\frac{1}{3}}\text{cis}\left(\frac{4\pi}{3}\right)$, $16^{\frac{1}{3}}\text{cis}\left(\frac{10\pi}{3}\right)$, $16^{\frac{1}{3}}\text{cis}\left(\frac{16\pi}{3}\right)$

QUESTION 6

The area of enclosed by $y=\frac{1}{4+9x^2}$, where $x\in[-a,a]$ and the x-axis is 1 square unit.

The value of a is

A. R

B. $-\frac{\pi}{2}<a<\frac{\pi}{2}$

C. $\frac{2\tan(3+\pi)}{3}$

D. $\frac{4\tan(3+\pi)}{9}$

E. $\frac{2\tan(3)}{3}$

QUESTION 7

If $z = \cos(\theta) + i\sin(\theta)$, then $z^n - \frac{1}{z^n}$ is

A. $2^n \sin(\theta)$

B. $2\cos(\theta)$

C. $2i\sin(n\theta)$

D. $2\sin(n\theta)$

E. $2i\cos(n\theta)$

QUESTION 8

If $\int \frac{3x^2+9}{(2x-1)(x^2+2x+2)}dx = \frac{a}{2}\log|2x-1| - b\int g^{/}(x)\,dx$. Then the values of a, b and $g^{/}(x)$ respectively are

A. 1.5, 3 and $g^{/}(x) = \arctan(x+1)$

B. 3, 1 and $g^{/}(x) = \ln(x-1)$

C. 1, 3 and $g^{/}(x) = \arctan(x-1)$

D. 3, 3 and $g^{/}(x) = \ln(x+1)$

E. 1.5, 3 and $g^{/}(x) = \frac{1}{x^2+2x+2}$

QUESTION 9

$\sin(t+2t) =$

A. $4\cos^3(t) - 3\cos(t)$

B. $4\cos^3(t) + 3\cos(t)$

C. $3\sin(t) + 4\sin^3(t)$

D. $3\sin(t) - 4\sin^3(t)$

E. None of the above

QUESTION 10

A body is cooling in surroundings maintained at 10^0C. Its temperature $\theta\ ^0C$ after t minutes is given by $\frac{d\theta}{dt} = -k(\theta - 10)$ where k is a constant. If the temperature of the body is initially 70^0C and 10 minutes later is 40^0C. The body's temperature after a further 15 minutes is close to

A. 20.0^0C

B. 10.1^0C

C. 31.2^0C

D. 10.0^0C

E. 20.6^0C

QUESTION 11

The volume of the solid obtained by rotating the bounded region in the first quadrant define by $\left\{(x, y) : x \geq 0, 2x^2 \leq y \leq 2\right\}$ about the line $y = 2$ is given by

A. $V = \int_0^1 (4 - 8x^2 + 4x^4)\, dx$

B. $V = \pi \int_0^1 (4 - 8x^2 + 4x^4)\, dx$

C. $V = \pi \int_0^2 (4 - 8x^2 + 4x^4)\, dx$

D. $V = \pi \int_{-1}^1 (2x^2 - 2)^2\, dx$

E. $V = 2\pi \int_0^1 x(2x^2 - 2)^2\, dx$

QUESTION 12

The general solution of the differential equation $\frac{dQ}{dt} = 4(100 - Q)$ is

A. $Q = 100 + ke^{-2t}$

B. $Q = 100 - ke^{-4t}$

C. $Q = 100 - ke^{2t}$

D. $Q = 100 + ke^{4t}$

E. $Q = 100 - ke^{-t}$

QUESTION 13

If $A = 6\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}$ and $B = -6\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$, the scalar projection of *B* onto *A* is close to

A. 7.21

B. 6.92

C. -2.97

D. 1.12

E. -2.98

QUESTION 14

Given that $\int_{\pi/2}^{a} \left(\frac{\sin\theta}{1 - \cos\theta} \right) d\theta = \frac{1}{2}$ and $\frac{\pi}{2} < a < \pi$, which of the following is the exact value of a?

A. $\sin^{-1}\left(\frac{1}{\sqrt{e}} - 1 \right)$

B. $\sin^{-1}\left(\sqrt{e} - 1 \right)$

C. $\cos^{-1}\left(1 - \frac{1}{\sqrt{e}} \right)$

D. $\cos^{-1}\left(1 - \sqrt{e} \right)$

E. $\cos^{-1}\left(\sqrt{e} - 1 \right)$

QUESTION 15

Given the vector $\underset{\sim}{a} = 2\sin\theta\ \underset{\sim}{i} + (1-\sin\theta)\ \underset{\sim}{j}$. The value of the acute angle θ, so that $\underset{\sim}{a}$ is perpendicular to the line $x + y = 1$, is which of the following?

A. $\sin^{-1}(-1)$

B. $\sin^{-1}\left(\frac{1}{3}\right)$

C. $\sin^{-1}\left(\frac{1}{2}\right)$

D. $\tan^{-1}\left(\frac{1}{2}\right)$

E. $\tan^{-1}(1)$

QUESTION 16

The acceleration in ms^{-2} of a particle moving in a straight line at time t seconds, $t \geq 0$, is given by $a = -\frac{1}{2}v$. When $t = 0$, the velocity, v, is 40 ms^{-1}. Which of the following is a correct expression for v in terms of t?

A. $v = -\frac{t}{2} + 40$

B. $v = -\frac{t^2}{4} + 40$

C. $v = 40e^{-0.5t}$

D. $v = 40e^{2t}$

E. $v = 40e^{0.5t}$

QUESTION 17
A plane which is flying horizontally at an altitude of 2000 metres passes directly over an observation tower. If the plane flies with a constant speed of 150 *km/h*, the rate of change of the distance from the base of the observation tower to the plane at one minute later is close to

A. 149.99 *km/min*

B. 1.95 *km/min*

C. 2.50 *km/min*

D. 3.20 *km/min*

E. 1.28 *km/min*

QUESTION 18
The velocity $\dot{\underset{\sim}{r}}$ of a particle at time t is given by $\dot{\underset{\sim}{r}} = 2e^{05t}\underset{\sim}{i} + 2e^{-0.5t}\underset{\sim}{j}$. If $r = 0$ when $t = 0$, then $\underset{\sim}{r}$ at time t is equal to

A. $\left(e^{0.5t}\right)\underset{\sim}{i} - \left(e^{-0.5t}\right)\underset{\sim}{j}$

B. $\left(e^{0.5t} - 1\right)\underset{\sim}{i} - \left(e^{-0.5t} - 1\right)\underset{\sim}{j}$

C. $\left(4e^{0.5t}\right)\underset{\sim}{i} - \left(4e^{-0.5t}\right)\underset{\sim}{j}$

D. $4\left(e^{0.5t} - 1\right)\underset{\sim}{i} - 4\left(e^{-0.5t} - 1\right)\underset{\sim}{j}$

E. $4\left(e^{0.5t} + 1\right)\underset{\sim}{i} - 4\left(e^{-0.5t} + 1\right)\underset{\sim}{j}$

QUESTION 19
A mass of 3 kg is connected to a mass of 4 kg by a light inelastic string which passes over a smooth pulley as shown. The acceleration due to gravity has magnitude $g\ m/s^2$. If $a\ m/s^2$ is the magnitude of the acceleration of each mass, then a equals

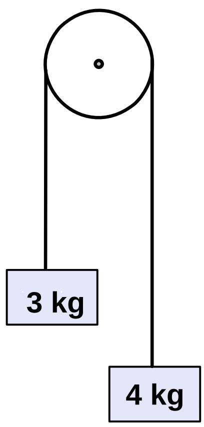
A. $\frac{g}{7}$

B. $\frac{g}{8}$

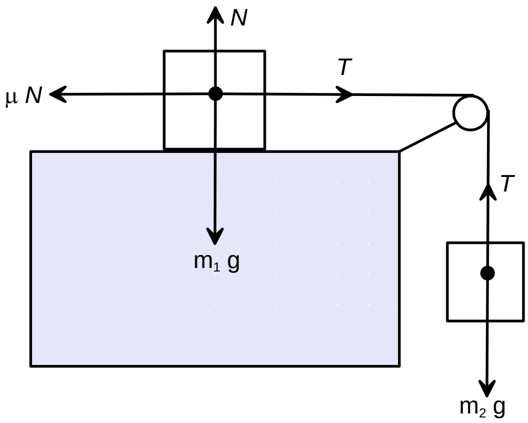
C. $\frac{g}{2}$

D. $\frac{7g}{8}$

E. $14g$



QUESTION 20
The accompanying diagram shows a particle of a mass m_1 on a rough, horizontal table, where the coefficient of friction between the particle and the table is μ. The particle is connected by a light, inelastic string which passes over a smooth pulley to a particle of mass m_2 and hangs vertically. The acceleration, a, of the system is given by



A. $\dfrac{(m_1 - \mu m_2)g}{m_1 + m_2}$

B. $\dfrac{(m_2 - \mu m_1)g}{m_1 - m_2}$

C. $\dfrac{(m_1 + \mu m_2)g}{m_1 + m_2}$

D. $\dfrac{(m_2 - \mu m_1)g}{m_1 + m_2}$

E. $\dfrac{(m_2 - \mu m_1)g}{m_2 - m_1}$

QUESTION 21
A railway carriage of mass M_1 kg moving at 7 $m s^{-1}$ and a second carriage of mass M_2 kg moving at 5 $m s^{-1}$ are travelling towards each other along the same track. The total momentum of the system is given by

A. $7M_1 + 5M_2$

B. $7M_2 + 5M_1$

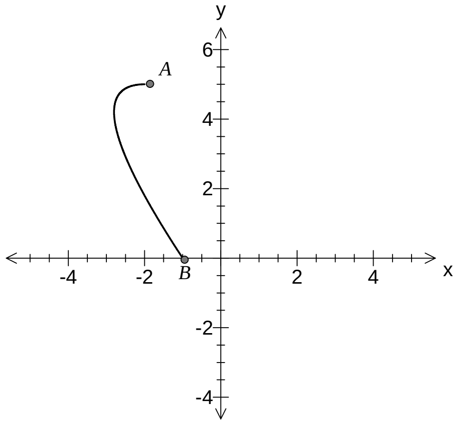
C. $7M_1 - 5M_2$

D. $5M_1 + 7M_2$

E. $\frac{1}{2}(7M_1 + 5M_2)$

QUESTION 22

The curve *AB* has parametric equations $x(t) = 5t^2 - 4t - 2$ and $y(t) = 5t^2 + 5$ where $0 \le t \le 1$. The curve *AB* is shown in the diagram below. The exact coordinates of the point on the curve *AB* that is the furthest to the left is



A (-0.8, 10.33)

B. (-2.8, 5.8)

C. (-2, 12.2)

D. (-1.36, 5.8)

E. (-2, 5.27)

SECTION 2

Instructions for Section 2

- Answer **all** questions in the spaces provided.
- A decimal approximation will not be accepted if an **exact** answer is required to a question.
- In questions where more than 1 mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- Take the **acceleration due to gravity** to have magnitude *g* m/s^2, where *g* = 9.8.

QUESTION 1

a. (i) Find the modulus of the complex number $(1+i)^k$ where *k* is a natural number.

1 mark

(ii) Hence show that $x^2 + y^2 = 2^k$ given that $x + iy = (1+i)^k$ where *x* and *y* are real numbers.

1 mark

b. (i) Express $1+i$ and $1-i$ in polar form.

1 mark

(ii) Find the value of $(1+i)^{2n}+(1-i)^{2n}$ in each of the cases $n=2$, $n=3$ and $n=4$.

3 marks

(iii) Hence or otherwise show that:

$$(1+i)^{2n}+(1-i)^{2n}=\begin{cases}2^{n+1} & \text{if } \frac{n}{2} \text{ is an even integer.}\\ 0 & \text{if } n \text{ is an odd integer.}\\ -2^{n+1} & \text{if } \frac{n}{2} \text{ is an odd integer.}\end{cases}$$

1 mark

c. (i) If $(\cos(\theta)+i\sin(\theta))^k=\cos(2\theta)+i\sin(2\theta)$, state the value of *k*.

1 mark

(ii) Show that $\dfrac{\cos(3\theta)+i\sin(3\theta)}{\cos(2\theta)+i\sin(2\theta)}=\cos(\theta)+i\sin(\theta)$.

1 mark

(iii) If $A=\cos(\theta)+\cos(2\theta)+\cos(3\theta)+\ldots\cos(2n\theta)$ and

$B=\sin(\theta)+\sin(2\theta)+\sin(3\theta)+\ldots\sin(2n\theta)$ where $\theta=\dfrac{\pi}{n}$ and n is a positive integer,

find an expression for $A+iB$.

1 mark

(iv) Given that the sum of a geometric sequence of *n* terms is $S_n = \dfrac{a(1-r^n)}{1-r}$ where a is the first term and *r* is the common ratio, show that $A = 0 = B$.

2 marks

Total 12 Marks

QUESTION 2

A particle moves in a straight line in a positive direction from a fixed point O. The velocity v m/s^{-1} at time t seconds, where $t \geq 0$, satisfies the differential equation

$$\frac{dv}{dt} = \frac{-v(1+v^2)}{50}$$

The particle starts from O with an initial velocity of 10 m/s^{-1}.

a. (i) Express as a definite integral the time taken for the velocity of the particle to decrease from 10 m/s^{-1} to 5 m/s^{-1}.

1 mark

(ii) Hence calculate, correct to four decimal places, the time taken for the velocity of the particle to decrease from 10 m/s^{-1} to 5 m/s^{-1}.

1 mark

b. (i) Show that when $v > 0$, the motion of the particle can also be described by the differential equation $\frac{dv}{dx} = \frac{-(1+v^2)}{50}$ where x meters is the displacement of the particle from O.

1 mark

(ii) Given that $v = 10$ when $x = 0$, solve the differential equation, expressing x in terms of v.

3 marks

(iii) Hence show that $v = \dfrac{10 - \tan\left(\dfrac{x}{50}\right)}{1 + 10\tan\left(\dfrac{x}{50}\right)}$.

1 mark

c. (i) Find the position of the particle relative to O where the particle stops.

1 mark

(ii) Explain whether or not the particle remains stationary at the position found in part (c) (i). If it does not remain stationary, find the direction in which it subsequently moves.

2 marks

Total 10 Marks

QUESTION 3

The function *f* is defined by $f(x) = x\sqrt{9 - x^2} + 2\arcsin\left(\frac{x}{3}\right)$.

a. State the largest possible domain *D* for *f*.

1 mark

b. (i) Write down a definite integral for the volume generated when the region bounded by the curve $y = f(x)$, the *x*-axis and the line $x = 2.8$ is rotated about the *x*-axis.

1 mark

(ii) Find, correct to one decimal place, the value of this volume.

1 mark

c. Use calculus to:

(i) find $f'(x)$ in simplest form.

1 mark

(ii) show that $f''(x) = \dfrac{x(2x^2 - 25)}{(9 - x^2)^{3/2}}$.

2 marks

d. Hence show that $\int_{-p}^{p} \frac{11-2x^2}{\sqrt{9-x^2}}\,dx = 2p\sqrt{9-p^2} + 4\arcsin\left(\frac{p}{3}\right)$ where $p \in D$.

1 mark

e. Find the value of *p* which maximises the value of the integral in part (d). Justify that it gives the maximum value.

2 marks

f. Show that $f(x)$ has a point of inflection at $x=0$, but **not** at $x=\pm\frac{5}{\sqrt{2}}$.

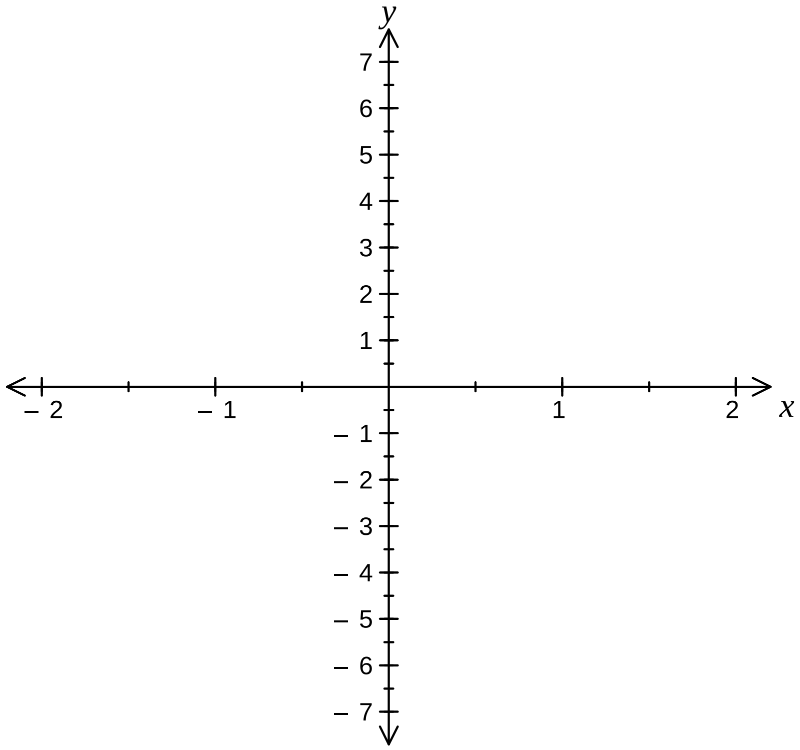
2 marks

Total 11 Marks

QUESTION 4

The function *f* is defined by $f(x) = \text{cosec}(x) + \tan(2x)$.

a. Sketch on the set of axes given below the graph of $y = f(x)$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, labelling all asymptotes with their equation and stating, correct to three decimal places, the coordinates of any axes intercepts and stationary points.



4 marks

b. Show that the roots of $f(x) = 0$ satisfy the equation $2\cos^3(x) - 2\cos^2(x) - 2\cos(x) + 1 = 0$.

2 marks

c. Show that the *x*-coordinates of all turning points on the graph of $y = f(x)$ satisfy the equation $4\cos^5(x) - 4\cos^3(x) + 2\cos^2(x) + \cos(x) - 2 = 0$.

2 marks

d. Show that $f(\pi - x) + f(\pi + x) = 0$.

2 marks

Total 10 Marks

QUESTION 5

A disused satellite has a position vector relative to the centre of the Earth given by

$$\underset{\sim}{r} = (2t+1)\underset{\sim}{i} + (t^2 - 11)\underset{\sim}{j} + (t^2 - 3)\underset{\sim}{k}$$

at time $t \in R$. All components are measured in kilometres.

A South Sea island has a position vector $-12\underset{\sim}{j} - 2\underset{\sim}{k}$ relative to the centre of the Earth.

NASA is worried that the satellite might crash too close to the island.

a. (i) Find the position vector of the satellite relative to the island when $t = -1$.

1 mark

(ii) Hence find the distance between the satellite and the island at $t = -1$.

1 mark

b. Show that the distance of the satellite from the island at time *t* is $\sqrt{2}(t^2 + 1)$.

2 marks

c. Find the closest distance that the satellite ever comes to the island.

1 mark

NASA plans to use a missile to divert the satellite. The missile is to be launched from the island to hit the satellite at time $t = -1$.

d. If the missile is launched at time $t = -2$ in a straight line, find (correct to the nearest degree) the angle its path needs to make to the line from the island to the satellite.

3 marks

Total 8 Marks

QUESTION 6

A truck is towing a large log of mass 1200 kg up a sloping road inclined at an angle of 8^0 to the horizontal. The log and truck are moving up the slope with a constant acceleration of 0.25 m/s^2. The coefficient of friction between the log and the road is 0.09.

a. Find, correct to the nearest newton, the magnitude of the tension in the tow rope.

3 marks

Later the truck is parked on the slope with its brakes applied. Let T newtons be the magnitude of the tension in the tow rope and let F newtons be the magnitude of the frictional force between the log and the road.

b. Express T in terms of F.

2 marks

c. Find T, correct to the nearest newton, if the coefficient of friction between the log and the road is

(i) 0.09

1 mark

(ii) 0.15

1 mark

Total 7 Marks

END OF QUESTION AND ANSWER BOOK