SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1

2010 Trial Examination

SOLUTIONS

Question 1

a. Vectors $\mathbf{a} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = m\mathbf{i} + n\mathbf{j} - 2\mathbf{k}$ are perpendicular, therefore their scalar product must be zero.

$(\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \bullet (m\mathbf{i} + n\mathbf{j} - 2\mathbf{k}) = 0 \Rightarrow m - n - 8 = 0$

$m = n + 8 \qquad (1)$ M1

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (m-1)\mathbf{i} + (n+1)\mathbf{j} - 6\mathbf{k}$

$\left|\overrightarrow{AB}\right| = \sqrt{(m-1)^2 + (n+1)^2 + 36} = 2\sqrt{14} \qquad (2)$ M1

Substituting (1) and (2) and then simplifying gives

$n^2 + 8n + 15 = 0$, so $n = -3$ or $n = -5$ A1

Therefore m=5 or m=3. A1

So the possible values of m and n are $n = -5, m = 3$ or $n = -3, m = 5$.

b. The angle between $\mathbf{a} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{AB} = 4\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$ is determined by finding the angle between $\overrightarrow{AB}$ and $\overrightarrow{AO}$.

$\overrightarrow{AO} \bullet \overrightarrow{AB} = -4 - 2 + 24 = 18$ M1

$\cos\theta = \dfrac{18}{2\sqrt{14} \times \sqrt{18}} = \dfrac{3}{2\sqrt{7}} \Rightarrow p = 3, q = 2$ A1

Question 2

a. If $z = i$ is a solution, then $p(i) = 0$

$p(i) = i^3 - ci^2 + 3i^2 + 1 - i$

$-i + c - 3 + 1 - i = 0$ M1

$c = 2 + 2i$

b. Method 1

By dividing polynomials:

$z^3 - (2+2i)z^2 + 3iz + 1 - i = (z - i)\left(z^2 - (2+i)z + (1+i)\right)$ A1

The quadratic equation $z^2 - (2+i)z + (1+i) = 0$ can be solved by using the quadratic formula.

$$z = \frac{(2+i) \pm \sqrt{(2+i)^2 - 4(1+i)}}{2}$$

$$= \frac{2 + i \pm \sqrt{-1}}{2}$$ M1

$z = 1$ or $z = 1 + i$ A2

Method 2

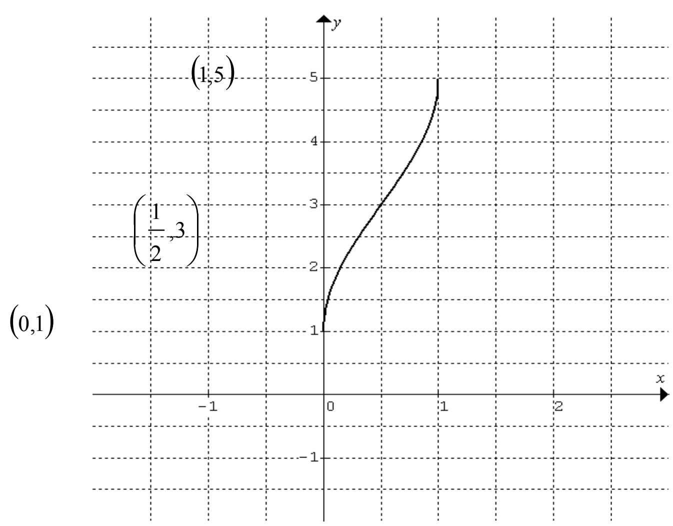
By inspection, $p(1) = 1^3 - (2+2i) + 3i + 1 - i = 0$, therefore $z - 1$ is a factor of

$z^3 - (2+2i)z^2 + 3iz + 1 - i$. M2

$(z-1)(z-i) = z^2 - (1+i)z + i$ is the quadratic factor and $z - (1+i)$ is the third linear factor.

The required two solutions are $z = 1$ and $z = 1 + i$ A2

Question 3



Correct graph A1

Domain: $-1 \le 2x - 1 \le 1 \Rightarrow 0 \le x \le 1$ Correct domain and range A1

Range: $\left[\frac{4}{\pi} \times \frac{-\pi}{2} + 3, \frac{4}{\pi} \times \frac{\pi}{2} + 3\right] = [1,5]$

Inflection point $\left(\frac{1}{2}, 3\right)$ A1

Question 4

If $\tan\alpha = \frac{1}{12}$, $\tan\beta = \frac{2}{5}$ and $\tan\gamma = \frac{1}{3}$, where α, β and γ are acute angles, show that $\alpha + \beta + \gamma = \frac{\pi}{4}$.

$$\tan(\alpha+\beta+\gamma) = \frac{\tan(\alpha+\beta)+\tan\gamma}{1-\tan(\alpha+\beta)\tan\gamma}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{1}{12}+\frac{2}{5}}{1-\frac{2}{60}} = \frac{\frac{29}{60}}{\frac{58}{60}} = \frac{1}{2}$$ M1A1

$$\tan(\alpha+\beta+\gamma) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}} = 1$$ M1

$$\alpha+\beta+\gamma = \tan^{-1}(1) = \frac{\pi}{4}$$

Note that since all α, β and γ are all less than $\frac{\pi}{4}$ (since their tangents are less than 1) it is impossible for $\alpha+\beta+\gamma$ to be in the third quadrant. We can therefore disregard the possibility that $\alpha+\beta+\gamma = \frac{5\pi}{4}$.

Question 5

a. $y^2 = xy - \log_e y$

$$2y\frac{dy}{dx} = y + x\frac{dy}{dx} - \frac{1}{y}\frac{dy}{dx}$$ M1

$$2y\frac{dy}{dx} - x\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} = y$$

$$\frac{dy}{dx}\left(2y - x + \frac{1}{y}\right) = y \Rightarrow \frac{dy}{dx} = \frac{y^2}{2y^2 - xy + 1}$$ A1

b. $xy = y^2 + \log_e y \Rightarrow x = y + \frac{\log_e y}{y}$ A1

$$\frac{dx}{dy} = 1 + \frac{\frac{1}{y} \times y - \log_e y}{y^2} = \frac{y^2 + 1 - \log_e y}{y^2}.$$ A1

Alternative approach: differentiate implicitly with respect to *y* to get:

$$2y = x + y\frac{dx}{dy} - \frac{1}{y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{2y^2 - xy + 1}{y^2}.$$

The result in part c then follows immediately from this.

c. Substituting $x = y + \frac{\log_e y}{y}$ into $\frac{dy}{dx} = \frac{y^2}{2y^2 - xy + 1}$ gives

$$\frac{dy}{dx} = \frac{y^2}{2y^2 - y^2 - \log_e y + 1} = \frac{y^2}{y^2 - \log_e y + 1} = \frac{1}{\frac{dx}{dy}}$$ M1

Question 6

$$V = \pi\int_2^4 \left(\frac{x+1}{\sqrt{x^2-1}}\right)^2 dx$$ A1

$$V = \pi\int_2^4 \frac{(x+1)^2}{x^2-1}dx$$
$$= \pi\int_2^4 \frac{x+1}{x-1}dx$$
$$= \pi\int_2^4 \left(1+\frac{2}{x-1}\right)dx$$ M1
$$= \pi\left[x+2\ln|x-1|\right]_2^4$$
$$= \pi\left(4+2\ln 3-2-2\ln 1\right)$$
$$= 2\pi\left(1+\ln 3\right) \text{ units}^3$$ A1

Question 7

a. Let $y = x\sin^2 x$. Using the product rule:

$$\frac{dy}{dx} = \sin^2 x + 2x\sin x\cos x$$ A1

b. $x\sin^2 x = \int(\sin^2 x + 2x\sin x\cos x)dx$ M1

$x\sin^2 x = \int \sin^2 x dx + \int x\sin(2x)dx$

Using double angle formula $\sin^2 x = \frac{1-\cos(2x)}{2}$, we have

$$\int x\sin(2x)dx = x\sin^2 x - \int\frac{1-\cos(2x)}{2}dx$$
$$= x\sin^2 x - \frac{x}{2} + \frac{1}{4}\sin(2x)$$ M1

$$\int_0^{\frac{3\pi}{2}} x\sin(2x)dx = \left[x\sin^2 x - \frac{x}{2} + \frac{1}{4}\sin(2x)\right]_0^{\frac{3\pi}{2}} = \frac{3\pi}{4}$$ M1A1

Question 8

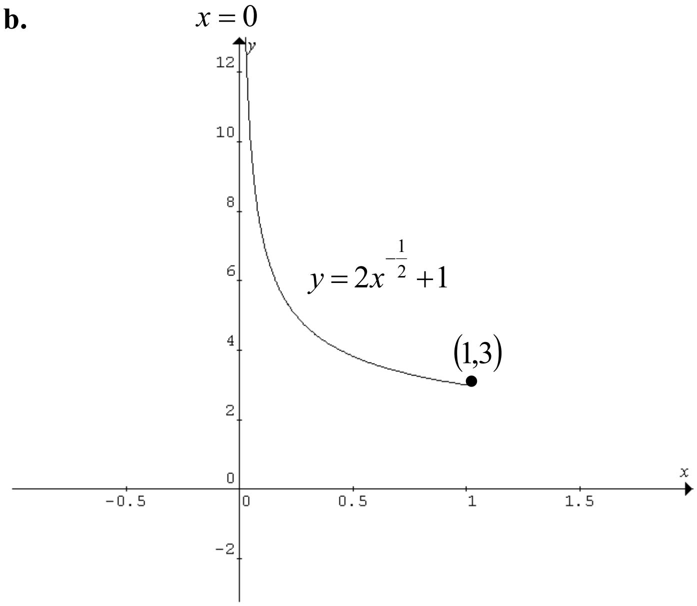
a. From $\mathbf{r}(t) = e^{-2t}\mathbf{i} + (2e^t + 1)\mathbf{j},\ t \geq 0,$ we have $x = e^{-2t},\quad y = 2e^t + 1$

$$e^t = x^{-\frac{1}{2}},\quad y = 2x^{-\frac{1}{2}} + 1$$ A1

As $t \geq 0$, the domain is $x \in (0,1]$ and the range is $y \in [3, +\infty)$ M2

(from their Cartesian equation)

b.



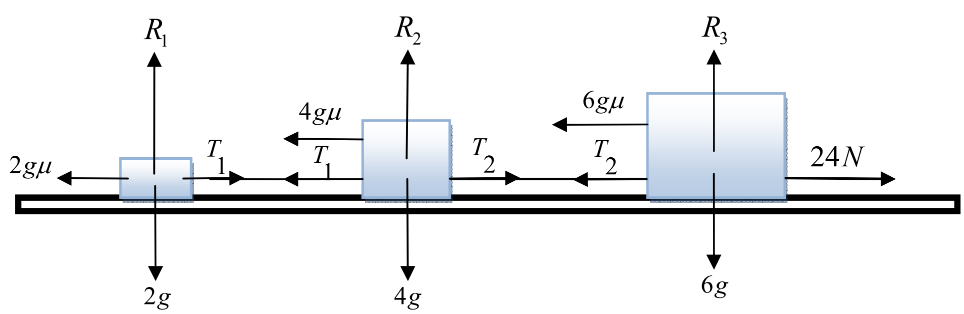
correct shape M1

asymptotic behavior and end point shown M1

(from their Cartesian equation)

Question 9

a.



All forces correct A2

(one mark off for any errors)

$$24 - \frac{1}{10} \times 12g = 12a \Rightarrow a = 2 - \frac{g}{10}$$ A1

b. $T_1 - 2g\mu = 2a \Rightarrow T_1 = 4N$ M1

$T_2 - 6g\mu = 6a \Rightarrow T_2 = 12N$ M1