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SPECIALIST MATHS TRIAL EXAMINATION 1 SOLUTIONS 2011

Question 1

a. If $z+3$ is a factor then $z=-3$ is a solution. So $f(-3)=0$.

$$(-3)^3+(-3)^2-4\times(-3)+a=0$$
$$-27+9+12+a=0$$
$$a=6 \text{ as required}$$

(1 mark)

b.

$$\begin{array}{r|l} & z^2-2z+2 \\ z+3 & z^3+z^2-4z+6 \\ & \underline{z^3+3z^2} \\ & -2z^2-4z \\ & \underline{-2z^2-6z} \\ & 2z+6 \\ & \underline{2z+6} \end{array}$$

$$f(z)=(z+3)(z^2-2z+2)$$ **(1 mark)**

$$=(z+3)\{(z^2-2z+1)-1+2\}$$
$$=(z+3)\{(z-1)^2+1\}$$ **(1 mark)**
$$=(z+3)\{(z-1)^2-i^2\}$$
$$=(z+3)(z-1-i)(z-1+i)$$

For $f(z)=0$,
$z=-3$ or $z=1\pm i$

(1 mark)

Question 2

a. $$kx^2-2x^2y+y^3-6x=2,\ k\in R$$

$$2kx-4xy-2x^2\frac{dy}{dx}+3y^2\frac{dy}{dx}-6=0$$ **(1 mark)**

$$(-2x^2+3y^2)\frac{dy}{dx}=4xy-2kx+6$$

$$\frac{dy}{dx}=\frac{4xy-2kx+6}{-2x^2+3y^2}$$

(1 mark)

b. Since the curve passes through the point $(1,0)$, we have

$$k-0+0-6=2$$

$$k=8$$ **(1 mark)**

So at $(1,0)$, $$\frac{dy}{dx}=\frac{0-16+6}{-2+0}=5$$

At $(1,0)$, the gradient is 5.

(1 mark)

Question 3

Method 1

$$|z-1-i|\le|z| \quad \text{where} \quad z=x+yi$$

$$|x+yi-1-i|\le|x+yi|$$

$$\sqrt{(x-1)^2+(y-1)^2}\le\sqrt{x^2+y^2}$$

$$x^2-2x+1+y^2-2y+1\le x^2+y^2$$

$$-2x-2y+2\le 0$$

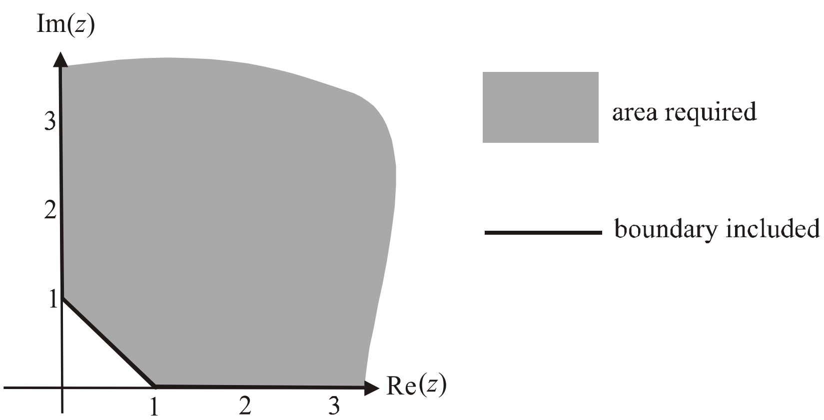
$$-2y\le 2x-2$$

$$y\ge -x+1$$

(1 mark)

Method 2

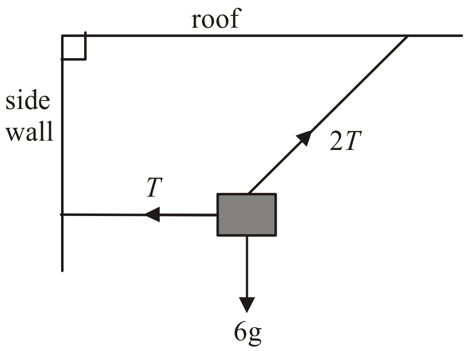
The equation $|z-1-i|=|z|$ describes the set of points that are equidistant from $0+0i$ and $1+i$ This set of points forms a straight line that passes through $(0,1)$ and $(1,0)$. The inequation $|z-1-i|\le|z|$ describes the set of points closer to $1+i$ than $0+0i$. This describes a half plane with the straight line passing through $(0,1)$ and $(1,0)$ as its boundary. **(1 mark)**



(1 mark) – correct area **(1 mark)** – correct boundaries including corner points

Question 4

a. Let T be the tension in the horizontal wire.



(1 mark)

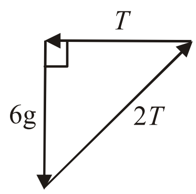
b.

$(2T)^2 = T^2 + 36g^2$ **(1 mark)**

$3T^2 = 36g^2$

$T^2 = 12g^2$

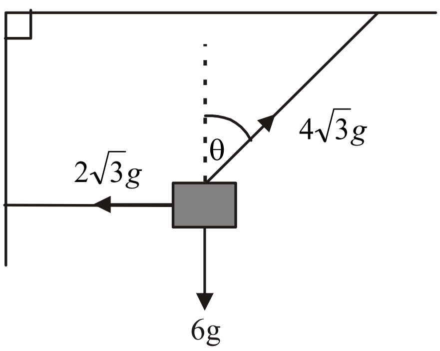
$T = 2\sqrt{3}g$



Tension in horizontal wire is $2\sqrt{3}$ g newtons.
Tension in other wire is $4\sqrt{3}$ g newtons.

(1 mark)

c.



$4\sqrt{3}\text{g}\sin\theta = 2\sqrt{3}\text{g}$ OR $4\sqrt{3}\text{g}\cos\theta = 6\text{g}$

$\sin\theta = \dfrac{2\sqrt{3}\text{g}}{4\sqrt{3}\text{g}} = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{6}$

$\cos\theta = \dfrac{6\text{g}}{4\sqrt{3}\text{g}} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$

$\theta = \dfrac{\pi}{6}$

(1 mark)

Question 5

a. For $g(x)=\arctan(x)$, $d_g=R$

For $f(x)=2\arctan\left(\frac{2x+1}{3}\right)-1$, $d_f=R$ **(1 mark)**

For $g(x)=\arctan(x)$, $r_g=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

For $f(x)=2\arctan\left(\frac{2x+1}{3}\right)-1$, $r_f=\left(\left(2\times-\frac{\pi}{2}\right)-1,\ \left(2\times\frac{\pi}{2}\right)-1\right)$

$=(-\pi-1,\ \pi-1)$

(1 mark)

b.

$$f(x)=2\arctan\left(\frac{2x+1}{3}\right)-1$$

Let $y=2\arctan\left(\frac{2x+1}{3}\right)-1$

$y=2\arctan\left(\frac{u}{3}\right)-1$ where $u=2x+1$

$\frac{dy}{du}=2\times\frac{3}{9+u^2}$ and $\frac{du}{dx}=2$

$$=\frac{6}{u^2+9}$$

$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ (Chain rule) **(1 mark)** – use of the chain rule

$$=\frac{6}{u^2+9}\times 2$$

$$=\frac{12}{(2x+1)^2+9}$$

$$=\frac{12}{4x^2+4x+10}$$

$$\frac{dy}{dx}=\frac{12}{2(2x^2+2x+5)}$$

So $f'(x)=\frac{6}{2x^2+2x+5}$

(1 mark) – correct answer

Question 6

$$a=\sqrt{9-v^2}$$
$$\frac{dv}{dt}=\sqrt{9-v^2}$$
$$\frac{dt}{dv}=\frac{1}{\sqrt{9-v^2}}$$
$$t=\int\frac{1}{\sqrt{9-v^2}}dv$$
$$t=\arcsin\left(\frac{v}{3}\right)+c$$

(1 mark)

When $t=\frac{\pi}{3},\ v=\frac{3}{2}$

$$\frac{\pi}{3}=\arcsin\left(\frac{1}{2}\right)+c$$
$$c=\frac{\pi}{3}-\frac{\pi}{6}$$
$$=\frac{\pi}{6}$$

(1 mark)

$$t=\arcsin\left(\frac{v}{3}\right)+\frac{\pi}{6}$$
$$t-\frac{\pi}{6}=\arcsin\left(\frac{v}{3}\right)$$
$$\frac{v}{3}=\sin\left(t-\frac{\pi}{6}\right)$$
$$v=3\sin\left(t-\frac{\pi}{6}\right)$$

(1 mark)

Question 7

a.

$$\text{Let } \frac{1}{x^2-2x-3} \equiv \frac{A}{(x-3)} + \frac{B}{(x+1)}$$
$$\equiv \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$
$$\text{True iff } 1 \equiv A(x+1)+B(x-3)$$

Put $x=-1 \quad 1=-4B, \quad B=-\frac{1}{4}$

Put $x=3 \quad 1=4A, \quad A=\frac{1}{4}$

$$\int \frac{1}{x^2-2x-3}dx = \int\left(\frac{1}{4(x-3)} - \frac{1}{4(x+1)}\right)dx$$

(1 mark)

$$= \frac{1}{4}\log_e|x-3| - \frac{1}{4}\log_e|x+1| + c$$
$$= \frac{1}{4}\log_e\left|\frac{x-3}{x+1}\right| + c$$

(1 mark)

b. Method 1

$$\int_0^1 \frac{x}{2-x}\,dx$$
$$= \int_2^1 (2-u)u^{-1} \times -1\frac{du}{dx}dx$$
$$= -\int_2^1 (2u^{-1}-1)du$$
$$= \int_1^2 (2u^{-1}-1)du$$
$$= \left[2\ln|u| - u\right]_1^2$$
$$= \{(2\ln(2)-2)-(2\ln(1)-1)\}$$
$$= 2\ln(2)-1$$

let $u=2-x$

$\frac{du}{dx}=-1$

$x=2-u$

$x=1, \; u=1$

$x=0, \; u=2$

(1 mark) – correct integrand
(1 mark) – correct terminals

(1 mark)

Method 2

$$\int_0^1 \frac{x}{2-x}dx$$

$$\begin{array}{r} -1 \\ 2-x\,\overline{)\,x+0} \\ \underline{x-2} \\ 2 \end{array}$$

$$= \int_0^1 \left(-1+\frac{2}{2-x}\right)dx$$

(1 mark)

$$= \left[-x-2\ln|2-x|\right]_0^1$$

(1 mark)

$$= \{(-1-2\ln(1))-(0-2\ln(2))\}$$
$$= -1+2\ln(2)$$

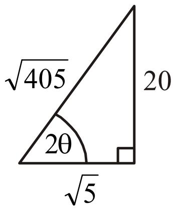
(1 mark)

Question 8

<u>Method 1</u>

$$\cot(2\theta)=\frac{\sqrt{5}}{20},\quad 0<\theta<\frac{\pi}{4}$$

$$\tan(2\theta)=\frac{20}{\sqrt{5}}$$



$$\cos(2\theta)=\frac{\sqrt{5}}{\sqrt{405}}$$

$$=\frac{1}{9}$$

(1 mark)

$$\cos(2\theta)=2\cos^2(\theta)-1\quad \text{(formula sheet)}$$

(1 mark)

$$2\cos^2(\theta)=1+\frac{1}{9}$$

$$\cos^2(\theta)=\frac{5}{9}$$

$$\cos(\theta)=+\frac{\sqrt{5}}{3}\quad \text{since}\quad 0<\theta<\frac{\pi}{4}$$

(1 mark)

<u>Method 2</u>

$$\cot(2\theta)=\frac{\sqrt{5}}{20},\quad 0<\theta<\frac{\pi}{4}$$

$$\tan(2\theta)=\frac{20}{\sqrt{5}}$$

$$1+\tan^2(2\theta)=\sec^2(2\theta)\quad \text{(formula sheet)}$$

$$1+\frac{400}{5}=\sec^2(2\theta)$$

(1 mark)

$$\sec(2\theta)=+\sqrt{81}\quad \text{since}\quad 0<\theta<\frac{\pi}{4}$$

$$\sec(2\theta)=9\qquad 0<2\theta<\frac{\pi}{2}$$

$$\frac{1}{\cos(2\theta)}=9$$

(1 mark)

$$\cos(2\theta)=\frac{1}{9}$$

$$\cos(2\theta)=2\cos^2(\theta)-1\quad \text{(formula sheet)}$$

$$2\cos^2(\theta)=1+\frac{1}{9}$$

$$\cos^2(\theta)=\frac{5}{9}$$

$$\cos(\theta)=+\frac{\sqrt{5}}{3}\quad \text{since}\quad 0<\theta<\frac{\pi}{4}$$

(1 mark)

Question 9

a. $\underset{\sim}{r} = \arccos\left(\frac{t}{5}\right)\underset{\sim}{i} + 2t\,\underset{\sim}{j} \qquad t \in [0, 5]$

$x = \arccos\left(\frac{t}{5}\right)$ and $y = 2t$ **(1 mark)**

$\frac{t}{5} = \cos(x) \qquad t = \frac{y}{2}$

$t = 5\cos(x)$

$\frac{y}{2} = 5\cos(x)$

$y = 10\cos(x)$

(1 mark)

b. To find the endpoints,

when $t = 0$, $x = \arccos(0) = \frac{\pi}{2}$, $y = 10\cos\left(\frac{\pi}{2}\right)$

$= 0$

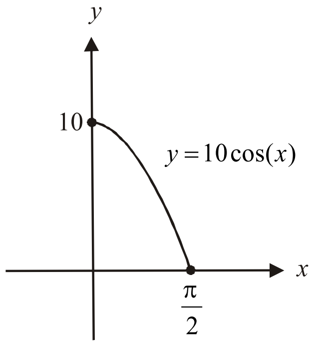
One endoint is $\left(\frac{\pi}{2},\ 0\right)$.

when $t = 5$, $x = \arccos(1) = 0$, $y = 10\cos(0)$

$= 10$

$(0, 10)$

The other endpoint is (0,10).



(1 mark) – correct shape
(1 mark) – correct endpoints

c. $\underset{\sim}{r}(t) = \arccos\left(\frac{t}{5}\right)\underset{\sim}{i} + 2t\,\underset{\sim}{j}$

$\underset{\sim}{v}(t) = \frac{-1}{\sqrt{25 - t^2}}\underset{\sim}{i} + 2\,\underset{\sim}{j}$ **(1 mark)**

$|\underset{\sim}{v}(t)| = \sqrt{\frac{1}{25 - t^2} + 4}$

$|\underset{\sim}{v}(2)| = \sqrt{\frac{1}{21} + 4}$

$= \sqrt{\frac{85}{21}}$

(1 mark)

Question 10

<u>x-intercept</u> occurs when $y = 0$

$$\frac{x+1}{x^2+1} = 0$$
$$x = -1$$

$$\text{Area} = \int_{-1}^{0} \frac{x+1}{x^2+1}\,dx$$ **(1 mark)**

$$= \int_{-1}^{0} \frac{x}{x^2+1}\,dx + \int_{-1}^{0} \frac{1}{x^2+1}\,dx$$
$$= \int_{2}^{1} \frac{1}{2}\frac{du}{dx} u^{-1}\,dx + \left[\arctan(x)\right]_{-1}^{0}$$

where $u = x^2 + 1$

$$\frac{du}{dx} = 2x$$
$$x = 0, \quad u = 1$$
$$x = -1, \quad u = 2$$

$$= \frac{1}{2}\int_{2}^{1} u^{-1}\,du + \{\arctan(0) - \arctan(-1)\}$$

S	A
T	C

$$= \frac{1}{2}\left[\log_e |u|\right]_{2}^{1} + \left(0 - \frac{-\pi}{4}\right)$$
$$= \frac{1}{2}\{\log_e(1) - \log_e(2)\} + \frac{\pi}{4}$$
$$= \frac{-1}{2}\log_e(2) + \frac{\pi}{4}$$
$$= -\log_e\left(2^{\frac{1}{2}}\right) + \frac{\pi}{4}$$
$$= \frac{\pi}{4} - \log_e\left(\sqrt{2}\right)$$

(1 mark) – correct integration of $\frac{x}{x^2+1}$

(1 mark) correct integration of $\frac{1}{x^2+1}$

(1 mark) – correct answer in required form