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Specialist Mathematics

2011

Trial Examination 2

SECTION 1 Multiple-choice questions

Instructions for Section 1

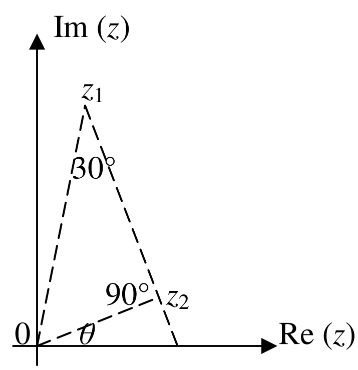
Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1 Given $z_1^2 + 3z_2^2 = 0$, z_1 and z_2 are complex numbers, which one of the following statements is **NOT** true?

A. $|z_1 - z_2| = 2|z_2|$
B. $z_1 = i\sqrt{3}z_2$
C. $z_1 = -i\sqrt{3}z_2$
D. $\sqrt{3}|z_1 - z_2| = 2|z_2|$
E. $\sqrt{3}|z_1 - z_2| = 2|z_1|$

Question 2 Refer to the diagram on the right. $0 < \theta < \frac{\pi}{2}$, z_1 and z_2 are complex numbers.
$Arg(z_1 - z_2) =$

A. $\frac{\pi}{2} - \theta$
B. $\frac{\pi}{2} + \theta$
C. $\frac{\pi}{3} + \theta$
D. $\frac{\pi}{3} - \theta$
E. $\frac{\pi}{3}$



Question 3 Polynomial P defined by $P(z) = z^3 + 3iz^2 - 3z - i$ has

A. no real solutions
B. a pair of conjugate roots
C. only one unique linear factor
D. three complex roots
E. two real factors and a complex factor

Question 4 In the complex plane the set of complex numbers defined by $\{z : \text{Im}(z) = |z - i|\}$ is

A. a straight line
B. a hyperbola
C. a parabola
D. a circle
E. an ellipse

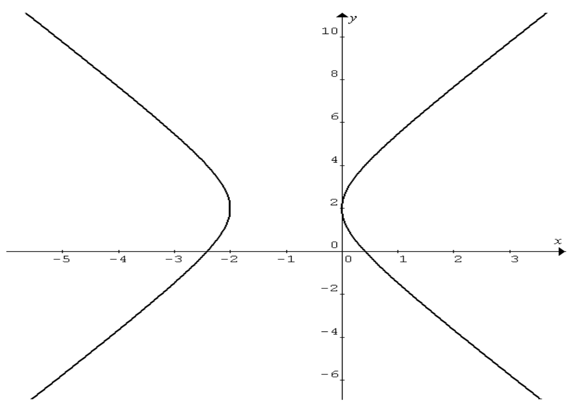
Question 5 The graph of $y = x - a - \frac{b}{x-a}$, where $a \in R$ and $b \in R \setminus \{0\}$,

A. always has two stationary points
B. always has two asymptotes
C. has R as its domain
D. has $R \setminus \{0\}$ as its range
E. has $R \setminus \{a\}$ as its range

Question 6 The graph of $y = \frac{1}{a + bx + 4ax^2}$, where $a > 1$, has only one asymptote when

A. $-1 \le b \le 4$
B. $-5 < b < 5$
C. $b \le -4a$ or $b \ge 4a$
D. $b < -4a$ or $b > 4a$
E. $b \le -4\sqrt{a}$ or $b \ge 4\sqrt{a}$

Question 7



The equations of the asymptotes of the hyperbola shown above are

A. $y = \pm 2(x+2) + 1$

B. $y - 2 = \pm \frac{1}{2}(x+1)$

C. $y + 2 = \pm \frac{1}{2}(x-1)$

D. $y + 2 = \pm 2(x-1)$

E. $y = -2x$, $y = 2x + 4$

Question 8 Given $a^2 = 3$, $b^2 = \dfrac{3}{4}$ and both $\tan^{-1}(a)$ and $\sin^{-1}(b)$ are in the open interval $\left(-\dfrac{\pi}{2}, 0\right)$, $\sec\left(\tan^{-1}(a) + \sin^{-1}(b)\right) =$

A. -1

B. 1

C. $-\dfrac{3}{2}$

D. $\dfrac{3}{2}$

E. -2

Question 9 The range of the function *f* defined by $f(x) = \cos^{-1}\left(\dfrac{x}{a} + b\right) + c$, where $a, b, c \in (-1, 0)$, is

A. $[c,\ c - a\pi]$
B. $(c,\ c - a\pi)$
C. $[-c,\ a\pi - c]$
D. $(-c,\ a\pi - c)$
E. $[a\pi - c,\ -c]$

Question 10 The solution(s) to the equation $\tan^{-1}(x - a + 1) = \tan^{-1}(x - a) + \dfrac{\pi}{4}$ is/are

A. $x = a - 1$ or $x = a$
B. $x = a - 1$ or $x = a + 1$
C. $x = a + 1$ or $x = a$
D. $x = \pm a$
E. $x = \pm(a - 1)$

Question 11 The position vectors of points *P* and *Q* are $\underset{\sim}{i} - \underset{\sim}{j}$ and $-\underset{\sim}{j} + \underset{\sim}{k}$ respectively. The measure of $\angle OPQ$ is

A. 120°
B. 105°
C. 90°
D. 60°
E. 45°

Question 12 $2\tilde{i} + p\tilde{j} + 3\tilde{k}$, $-\tilde{i} + 3\tilde{j} + q\tilde{k}$ and $\tilde{i} - \tilde{j} + \tilde{k}$ are linearly dependent if

A. $q = -\dfrac{p+2}{p}$

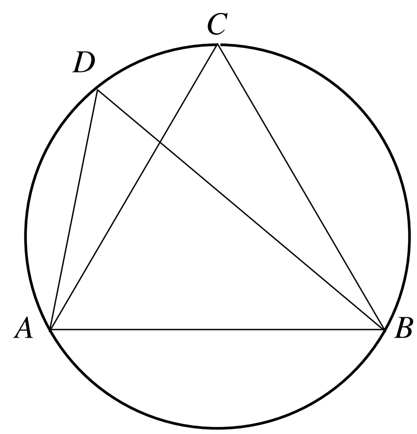
B. $q = \dfrac{p+2}{p}$

C. $q = -\dfrac{p}{p+2}$

D. $q = \dfrac{p}{p+2}$

E. $p = \dfrac{q-2}{q+2}$

Question 13 A, B, C and D are different points on the circumference of a circle.
Which one of the following statements is true?



A. $\dfrac{\overrightarrow{AC}.\overrightarrow{BC}}{|\overrightarrow{AC}||\overrightarrow{BC}|} = \dfrac{\overrightarrow{AD}.\overrightarrow{BD}}{|\overrightarrow{AD}||\overrightarrow{BD}|}$

B. $\dfrac{\overrightarrow{AC}.\overrightarrow{BC}}{|\overrightarrow{AC}||\overrightarrow{BC}|} > \dfrac{\overrightarrow{AD}.\overrightarrow{BD}}{|\overrightarrow{AD}||\overrightarrow{BD}|}$

C. $\dfrac{\overrightarrow{AC}.\overrightarrow{BC}}{|\overrightarrow{AC}||\overrightarrow{BC}|} < \dfrac{\overrightarrow{AD}.\overrightarrow{BD}}{|\overrightarrow{AD}||\overrightarrow{BD}|}$

D. $\dfrac{\overrightarrow{AC}.\overrightarrow{BC}}{|\overrightarrow{AC}||\overrightarrow{BC}|} \geq \dfrac{\overrightarrow{AD}.\overrightarrow{BD}}{|\overrightarrow{AD}||\overrightarrow{BD}|}$

E. $\dfrac{\overrightarrow{AC}.\overrightarrow{BC}}{|\overrightarrow{AC}||\overrightarrow{BC}|} \leq \dfrac{\overrightarrow{AD}.\overrightarrow{BD}}{|\overrightarrow{AD}||\overrightarrow{BD}|}$

Question 14 Given $\tilde{a} = \cos\left(\dfrac{\pi}{n}\right)\tilde{i} + \sin\left(\dfrac{\pi}{n}\right)\tilde{j}$ and $\tilde{b} = \cos\left(\dfrac{2\pi}{n}\right)\tilde{i} + \sin\left(\dfrac{2\pi}{n}\right)\tilde{j}$, the angle between vector $\tilde{a} + \tilde{b}$ and $\tilde{i}$ is

A. $\dfrac{2\pi}{n}$

B. $\dfrac{3\pi}{2n}$

C. $\dfrac{4\pi}{3n}$

D. $\dfrac{5\pi}{4n}$

E. $\dfrac{6\pi}{5n}$

Question 15 The velocity of a particle moving in a straight line is given by $v(t)=2\sin^{-1}\left(\frac{t}{10}-1\right)+\pi$ for $0\le t\le 10$. The average velocity of the particle in the interval $0\le t\le 10$ is closest to

A. $\frac{3}{2}$

B. $\frac{\pi}{2}$

C. $\frac{3\pi}{5}$

D. 2

E. $\frac{2\pi}{3}$

Question 16 The value of $\int_0^1 \frac{1-2x-x^2}{\sqrt{1-x^2}}\,dx$ is closest to

A. $\frac{2500\pi}{10000}-2$

B. $\frac{2501\pi}{10000}-2$

C. $\frac{2502\pi}{10000}-2$

D. $\frac{2503\pi}{10000}-2$

E. $\frac{2504\pi}{10000}-2$

Question 17 Which one of the following statements is **NOT** true about a particle with position vector $\tilde{r}=2\cos^{-1}(t)\tilde{i}-2\cos^{-1}(t)\tilde{j}+\cos^{-1}(t)\tilde{k}$, $0\le t\le 1$?

A. The initial distance of the particle from the origin is $\frac{3\pi}{2}$

B. The particle moves in a straight line

C. The particle moves with constant acceleration

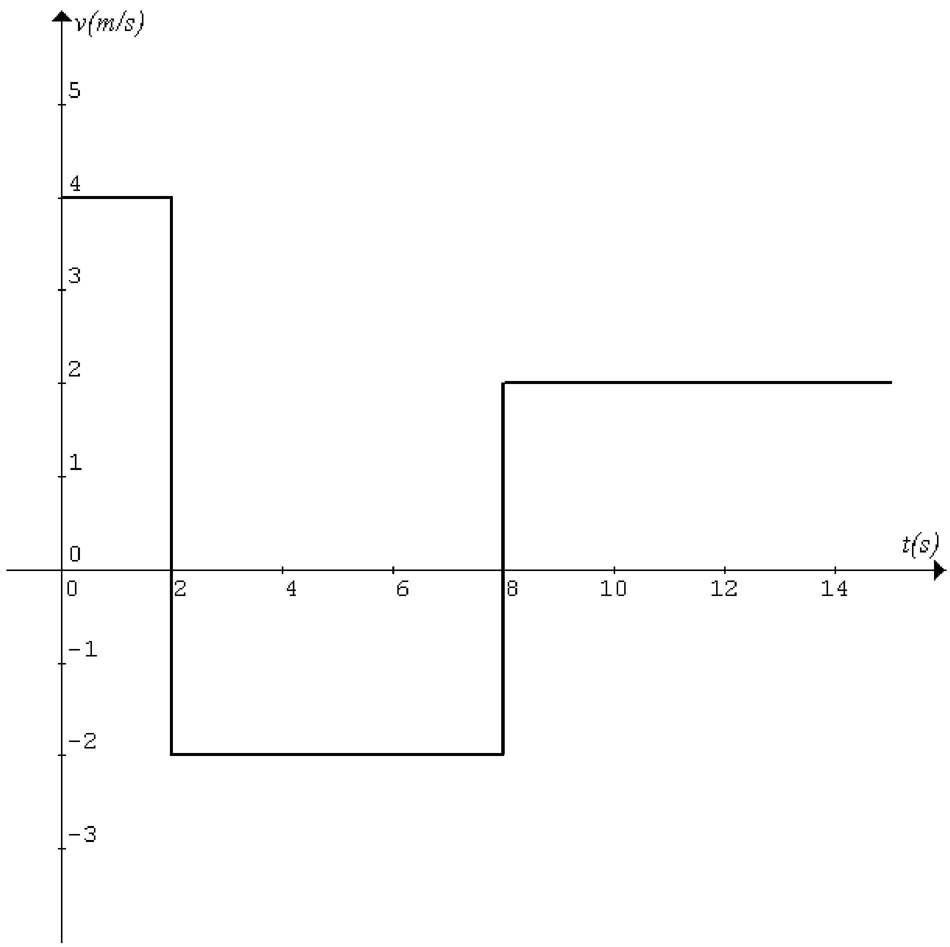
D. The initial speed of the particle is 3

E. The speed of the particle increases with time t

Question 18 A particle moves in a straight line with an acceleration of 2 ms^{-2} east. At $t = 0$ it is 5 m east of a reference point and has a velocity of 10 ms^{-1} west.
The time in seconds when it is 16 m east of the reference point is closest to

A. $t = 10.5$
B. $t = 11.0$
C. $t = 11.5$
D. $t = 12.0$
E. $t = 12.5$

Question 19 The velocity-time graph of a particle moving in a straight line is shown below.



The particle **returns** to its initial position at

A. $t = 2$
B. $t = 8$
C. $t = 12$
D. $t = 2$ and $t = 8$
E. $t = 6$ and $t = 10$

Question 20 A particle is on a plane inclined at an angle of $35°$ with the horizontal. The coefficient of friction is $\mu = 0.4$. Force of gravity and force due to friction are the only forces on the particle. Which one of the following statements **CANNOT** be true?

A. The particle moves at constant velocity.
B. Initially the particle has acceleration opposite to its motion.
C. The particle has a constant acceleration.
D. The acceleration of the particle is in the direction of its motion.
E. The resultant (net) force is along the inclined plane.

Question 21 A particle moves along the x-axis. Its velocity is given by $v^2 = x - 2$ where x is the position of the particle at time t. Which one of the following statements is **NOT** true?

A. The particle has a constant acceleration.
B. The particle has a positive acceleration.
C. The particle always moves along the **positive** x-axis.
D. The particle always moves away from the origin.
E. The magnitude of the particle's acceleration is 0.5.

Question 22 Three 2-kg bricks are stacked on top of each other on the floor of a lift. The lift moves downward with an upward acceleration of 2.2 ms^{-2}. The reaction force of the bottom brick on the middle brick is closest to

A. 48 N
B. 46 N
C. 44 N
D. 34 N
E. 24 N

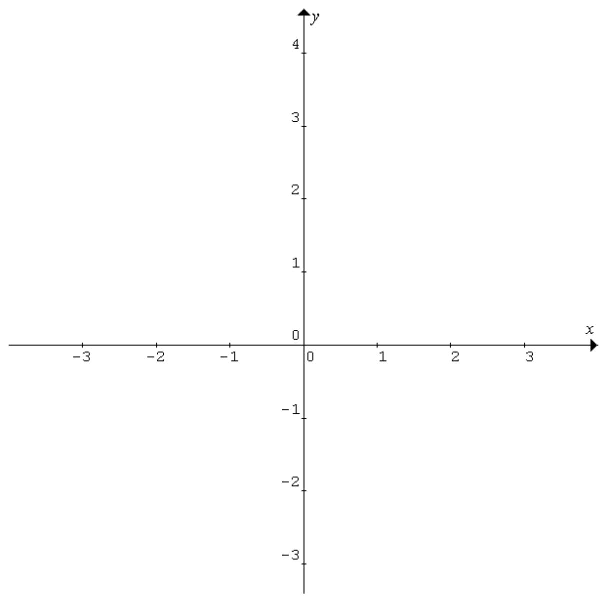
SECTION 2 Extended-answer questions

Instructions for Section 2

Answer **all** questions.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1 The locus of point $P(x, y)$ is given by the cartesian equation $4x^2 + 9y^2 = 36$.

a. Sketch the locus of point P, showing the coordinates of the axis intercepts. 2 marks



b. Find the exact value(s) of $\frac{dy}{dx}$ at $y = 1$. 2 marks

c. Find the exact coordinates of the points where the curve $x^2 + 3y = c$, $c \in R$, **touches** the locus of point P.

3 marks

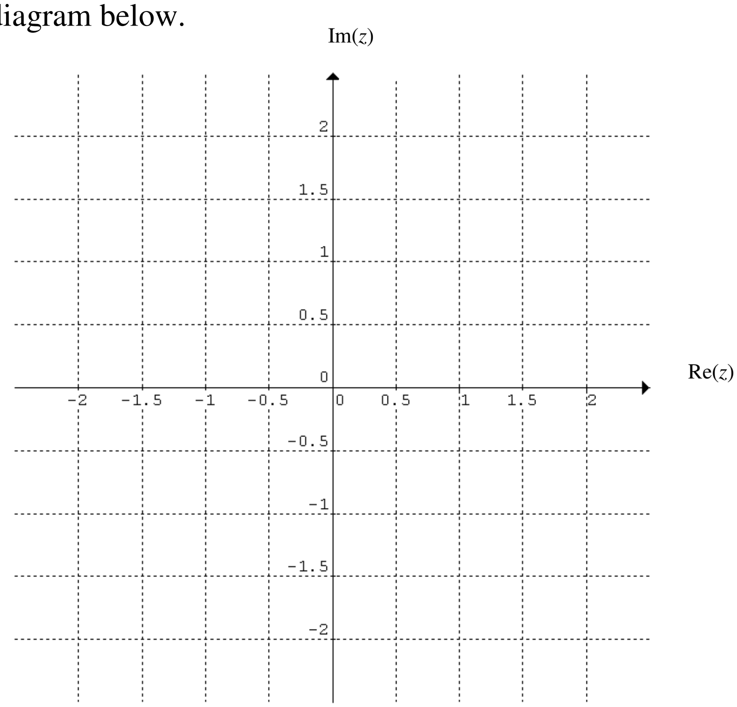
d i. Determine the minimum value of x^2+3y where (x, y) are the coordinates of point P. 1 mark

d ii. Determine the maximum value of x^2+3y where (x, y) are the coordinates of point P. 2 marks

Question 2 Let S be a set of complex numbers defined by $|z-1|=1$.

a i. Find the cartesian equation of the locus defined by $|z-1|=1$. 2 marks

a ii. Sketch S on the argand diagram below. 1 mark



b i. Given that $z \in S$ and $z = rcis\theta$, show that $r = 2\cos\theta$. 1 mark

b ii. Given that $z \in S$ and $z = rcis\theta$, find $\dfrac{1}{z}$ in terms of θ. 1 mark

b iii. Given that $z \in S$ and $z = rcis\theta$, find $\left|\dfrac{z-r}{z+r}\right|$ and $Arg\left(\dfrac{z-r}{z+r}\right)$ in terms of θ if necessary. 3 marks

Given z_1, z_2 and $z_3 \in S$, use the results in part b. to show that

c i. $\dfrac{z_1-r}{z_1+r}$, $\dfrac{z_2-r}{z_2+r}$ and $\dfrac{z_3-r}{z_3+r}$ are collinear. 2 marks

c ii. $\dfrac{1}{z_1}$, $\dfrac{1}{z_2}$ and $\dfrac{1}{z_3}$ are collinear. 2 marks

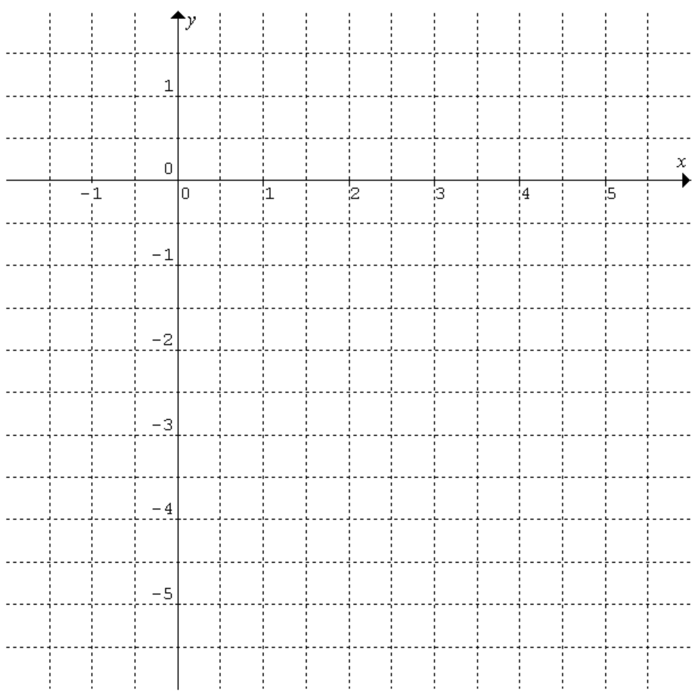
Question 3 The position of an aeroplane at time $t \geq 0$ is given by $\tilde{r}(t) = \sqrt{\frac{1+(t-1)^2}{2}}\,\tilde{i} - (t-1)\tilde{j} + \frac{\sqrt{2}|t-2|}{4}\tilde{k}$,

where $\tilde{i}$ and $\tilde{j}$ point to the east and north respectively, and $\tilde{k}$ points vertically upward. The airfield controller is located at the origin which is at the ground level of a very large horizontal airfield. The aeroplane was first spotted by the controller at $t = 0$.

a. Determine the cartesian equation of the locus of the shadow cast on the ground when the sun was directly above the aeroplane.

2 marks

b Sketch the locus of the **shadow** on the ground for $t \geq 0$. 2 marks



c. Find the exact time when the aeroplane was closest to the controller. 2 marks

d i. Find the exact speed of the aeroplane relative to the controller when it was first spotted. 2 marks

d ii. At what angle (nearest degree) did the flight path of the aeroplane make with the ground at the time when it was first spotted?

2 marks

e. Determine the true bearing (nearest degree) of the aeroplane's eventual destination from the controller.

2 marks

f. While the aeroplane was in flight pilot P looked at the rectangular roof $ABCD$ of a distant hangar and pointed out to the copilot that $\left|\overrightarrow{PA}\right|^2 + \left|\overrightarrow{PC}\right|^2 = \left|\overrightarrow{PB}\right|^2 + \left|\overrightarrow{PD}\right|^2$. Show that the pilot was correct irrespective of the position of the aeroplane. AC is a diagonal of $ABCD$.

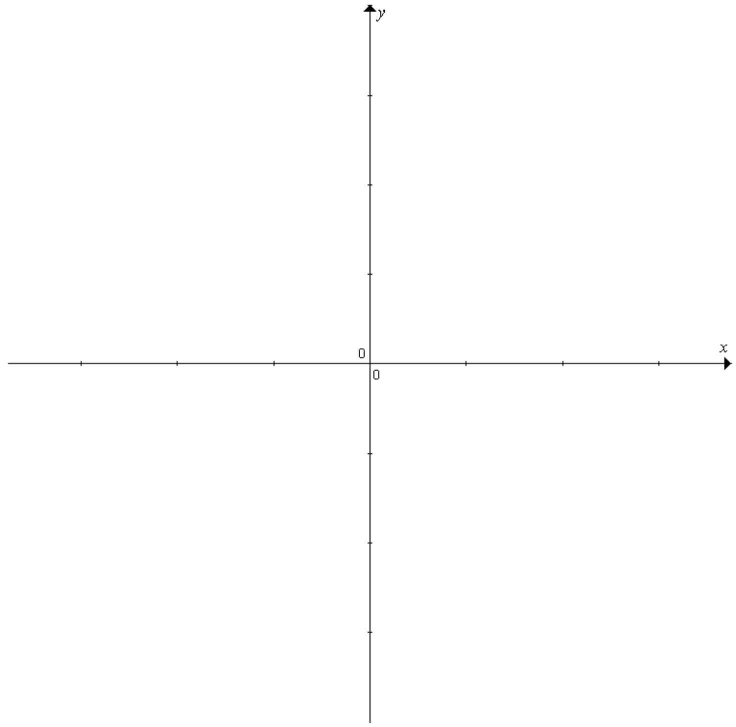
2 marks

Question 4

Consider function f defined by $f(x)=\dfrac{p}{\sqrt{p^2-x^2}}-1$, where $p\in R^+$.

a. Sketch the graph of f on the axes below. Label the axis-intercept(s) and asymptote(s) in terms of p.

2 marks



Now let $p=2$.

b. Without using CAS or calculator show that the area of the region bounded by the graph of f and $y=2$ is $8\sqrt{2}-4\sin^{-1}\left(\dfrac{2\sqrt{2}}{3}\right)$.

3 marks

c i. The region specified in part **b.** is rotated about the y-axis to form a solid of revolution with volume V. Write a definite integral for V.

2 marks

c ii. Show that the volume of the solid of revolution is $\frac{16\pi}{3}$.

1 mark

A container with the internal wall in the shape of the solid described in **c i.** is filled with water at a rate of $\frac{\pi}{3}$ cm^3 per second. All linear measurements are in cm.

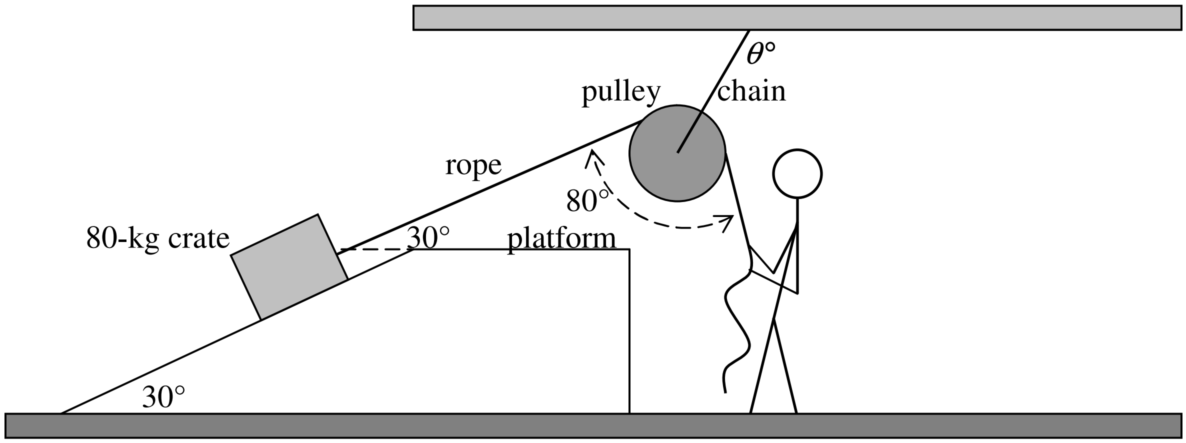
d. Find the exact time in seconds required to fill the container.

1 mark

e. Find the exact rate of increase (cm per second) in the depth of water in the container when the depth is 1 cm.

2 marks

Question 5 A pulley is suspended with a chain securely fastened to the ceiling. A person uses a long rope and the pulley to pull a 80-kg crate **at a constant speed of 0.2 m s^{-1}** up an inclined plane sloping at 30° with the horizontal. The two sections of the long rope make an angle of 80°. The coefficient of friction between the crate and the inclined plane is 0.25. Assume that the pulley is frictionless, the rope, the chain and the pulley have negligible mass.



a. Determine the force of friction (nearest newton) when the crate moves up the inclined plane at a constant speed of 0.2 m s^{-1}.

2 marks

b. Determine the force (nearest newton) applied by the person to pull the crate up the inclined plane at a constant speed of 0.2 m s^{-1}.

1 mark

c i. Determine $\theta°$, the angle between the chain and the ceiling. 1 mark

c ii. Determine the tension (nearest newton) in the chain. 2 marks

The rope breaks before the crate reaches the horizontal platform.

d. Determine the magnitude of the acceleration (in m s^{-2}, 1 decimal place) of the crate when it slides down the inclined plane.

2 marks

e i. Determine the speed (m s^{-1}, 1 decimal place) of the crate 0.25 s after the rope breaks, assuming it is still on its way down the inclined plane.

2 marks

e ii. Find the momentum of the crate 0.25 s after the rope breaks, assuming it is still on its way down the inclined plane.

1 mark

End of Exam 2