Year 2011

VCE

Specialist Mathematics Solutions

Trial Examination 2

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SECTION 1

ANSWERS

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2	A	B	**C**	D	E
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22	A	B	**C**	D	E

SECTION 1

Question 1 **Answer C**

The equation of the hyperbola is

$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$ since its domain is R and does not cross the x-axis.

The two asymptotes intersect at the centre (h,k) of the hyperbola, so that

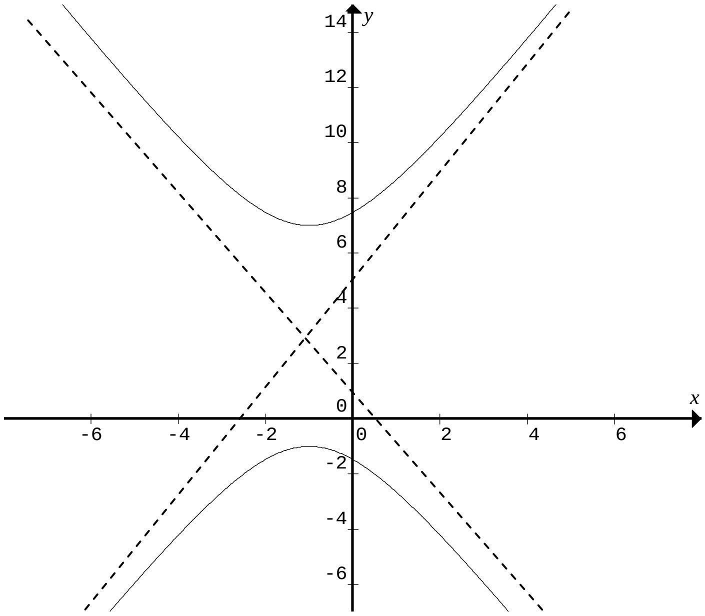
$2x+5=1-2x \quad \Rightarrow \; 4x=-4$

$\Rightarrow \; x=-1$ and $y=3 \; \Rightarrow \; h=-1$ and $k=3$

The distance from the centre to the vertices $(-1,7)$ and $(-1,-1)$ is 4 units, so that $b=4$, now the gradient of the asymptotes is $\frac{b}{a}=\pm 2$ so that $a=2$.

The equation of the hyperbola is

$\frac{(y-3)^2}{16}-\frac{(x+1)^2}{4}=1$



Question 2 **Answer C**

Let the roots be α and $\bar{\alpha}$, since the roots occur in conjugate pairs.

$\alpha=\text{cis}(\theta)=\cos(\theta)+i\sin(\theta)$ and $\bar{\alpha}=\text{cis}(-\theta)=\cos(-\theta)+i\sin(-\theta)=\cos(\theta)-i\sin(\theta)$

$\alpha+\bar{\alpha}=2\cos(\theta)$ and

$\alpha.\bar{\alpha}=\cos^2(\theta)-i^2\sin^2(\theta)=\cos^2(\theta)+\sin^2(\theta)=1$

The quadratic is $z^2-(\alpha+\bar{\alpha})z+\alpha\bar{\alpha}=0$ or $z^2-2z\cos(\theta)+1=0$

Question 3 **Answer E**

The graph is that of a reciprocal inverted parabola, with asymptotes at $x=a$ and $x=b$, note that $a<0$ while $b>0$. Its rule is

$$f(x)=\frac{-1}{(x-a)(x-b)}=\frac{-1}{x^2-(a+b)x+ab}=\frac{1}{(a+b)x-ab-x^2}$$

Question 4 **Answer D**

Let $z = x + yi$ so that $\overline{z} = x - yi$ and $c = a + bi$, $\overline{c} = a - bi$

$$(z-c)(\overline{z}-\overline{c}) = r^2$$

$$z\overline{z} - \overline{z}c - z\overline{c} + c\overline{c} = r^2$$

$$(x+yi)(x-yi) - (x-yi)(a+bi) - (x+yi)(a-bi) + (a+bi)(a-bi) = r^2$$

$$x^2 + y^2 - (xa - iya + bxi + by) - (xa + iya - bxi + yb) + a^2 + b^2 = r^2$$

$$x^2 - 2xa + a^2 + y^2 - 2by + b^2 = r^2$$

$$(x-a)^2 + (y-b)^2 = r^2$$

this is a circle with centre (a,b) radius r.

Question 5 **Answer A**

If $z = a+1+ai$ $\overline{z} = a+1-ai$, then

$$\frac{2a}{1-\overline{z}} = \frac{2a}{1-(a+1-ai)} = \frac{2a}{-a+ai} = \frac{2a}{a(-1+i)} = \frac{2}{-1+i} \times \frac{-1-i}{-1-i}$$

$$\frac{2a}{1-\overline{z}} = \frac{-2(1+i)}{1-i^2} = -1-i = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$$

Question 6 **Answer A**

$$y = \frac{x^n + a}{x} = x^{n-1} + ax^{-1} \quad \Rightarrow \quad \frac{dy}{dx} = (n-1)x^{n-2} - ax^{-2}$$ for turning points

$\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad \dfrac{(n-1)x^n}{x^2} = \dfrac{a}{x^2}$ or $x^n = \dfrac{a}{n-1}$, since there are two turning points, the only possibility listed is for $n = 2$ and $a > 0$.

Question 7 **Answer D**

$u = 4\,\text{cis}(\theta)$, $v = r\,\text{cis}\left(-\dfrac{2\pi}{3}\right)$ and $uv = 12i$

$$uv = 4r\text{cis}\left(\theta - \frac{2\pi}{3}\right) = 12i = 12\text{cis}\left(\frac{\pi}{2}\right)$$

$$4r = 12 \quad \Rightarrow \quad r = 3$$

$$\theta - \frac{2\pi}{3} = \frac{\pi}{2} \quad \Rightarrow \theta = \frac{2\pi}{3} + \frac{\pi}{2} = \frac{4\pi + 3\pi}{6} = \frac{7\pi}{6}$$

or $\theta = \dfrac{7\pi}{6} - 2\pi = -\dfrac{5\pi}{6}$

Question 8 **Answer E**

All of options **A. B. C.** and **D.** are correct, **E.** is false

$$\text{Arg}\left(\frac{1}{u}\right) = -\alpha$$

Question 9 **Answer B**

Since the area is below the *x*-axis, the area *A* is equal to

$$A = -\int_{-1}^{1}\left(x^2-1\right)\sqrt{2x+3}\,dx$$

let $u = 2x+3 \quad \frac{du}{dx} = 2 \quad \Rightarrow \; dx = \frac{1}{2}du$

terminals $x=-1 \;\; u=1$ and when $x=1 \;\; u=5$

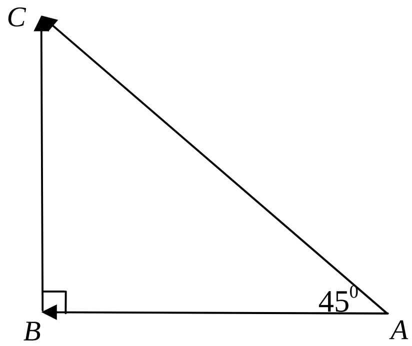
$x = \frac{1}{2}(u-3)$ so that

$$x^2-1 = \frac{1}{4}(u-3)^2 - 1 = \frac{1}{4}\left(u^2-6u+9\right)-1$$

$$= \frac{1}{4}\left(u^2-6u+5\right) = \frac{1}{4}(u-5)(u-1)$$

$$A = -\frac{1}{8}\int_{1}^{5}(u-5)(u-1)\sqrt{u}\,du = \frac{1}{8}\int_{1}^{5}(5-u)(u-1)\sqrt{u}\,du$$

Question 10 **Answer B**



Since it is an isosceles triangle, $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right|$ and

$\angle CAB = 45^0$

$$\frac{\overrightarrow{AB}.\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \cos\left(45^0\right) = \frac{1}{\sqrt{2}}$$

$$\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right| = \sqrt{2}\,\overrightarrow{AB}.\overrightarrow{AC}$$

D. States it is a right-angled triangle at *B*, but not necessarily isosceles.
E. States it is an isosceles triangle, but not necessarily right-angled.

Question 11 **Answer D**

Using parametric graphing, shows that when $n=m$ and $a=b$ the path is a circle, when $n=m$ and $a \neq b$ the path is an ellipse, and when $n=2m$ the path is part of a parabola.

Question 12 **Answer C**

Let $\underset{\sim}{a} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ $\quad |\underset{\sim}{a}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$

$$-\hat{\underset{\sim}{a}} = -\frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = -\frac{1}{3}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) = \frac{1}{3}\left(-2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$$

Question 13 **Answer E**

$y = \log_e(x-1) \quad \Rightarrow \quad x - 1 = e^y$ so that $x = 1 + e^y$, rotating about the y-axis,

$V_y = \pi\int_a^b \left(x_2^2 - x_1^2\right) dy \qquad x_2 = 1 + e^y$ and $x_1 = 1 \qquad a = 0$ and $b = 1$

$$V = \pi\int_0^1 \left(\left(e^y + 1\right)^2 - 1\right) dy = \pi\int_0^1 \left(\left(e^{2y} + 2e^y + 1\right) - 1\right) dy$$

$V = \pi\int_0^1 \left(e^{2y} + 2e^y\right) dy = \pi\int_0^1 \left(e^{2x} + 2e^x\right) dx$ by dummy variable property

Question 14 **Answer E**

$$\frac{dx}{dt} = \cos\left(\frac{1}{t}\right)$$

$x = \int_1^t \cos\left(\frac{1}{u}\right) du + C$ now to find C, $x = 3$ when $t = 1$,

$3 = \int_1^1 \cos\left(\frac{1}{u}\right) du + C \quad \Rightarrow C = 3$

$x = \int_1^t \cos\left(\frac{1}{u}\right) du + 3$ now when $t = 2 \qquad x = \int_1^2 \cos\left(\frac{1}{u}\right) du + 3$

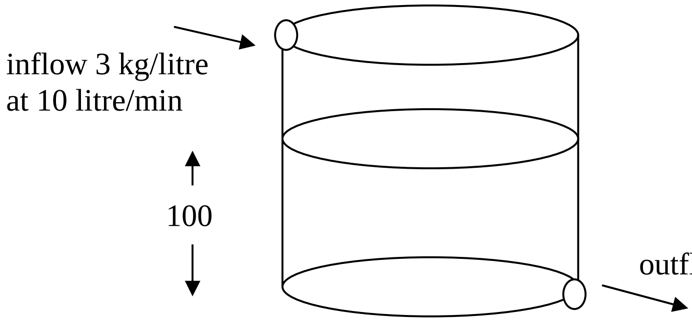
Question 15 **Answer C**

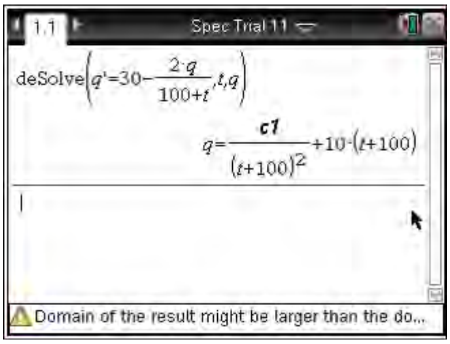
The stone takes on the initial upwards speed of the balloon, but its acceleration is just due to gravity. Taking upwards as positive and downwards as negative,

$s = -250 \quad u = 3 \quad a = -9.8 \quad t = ?$ using $s = ut + \frac{1}{2}at^2$

$-250 = 3t - 4.9t^2$ solving $\Rightarrow t = 7.46$

Question 16 **Answer B**





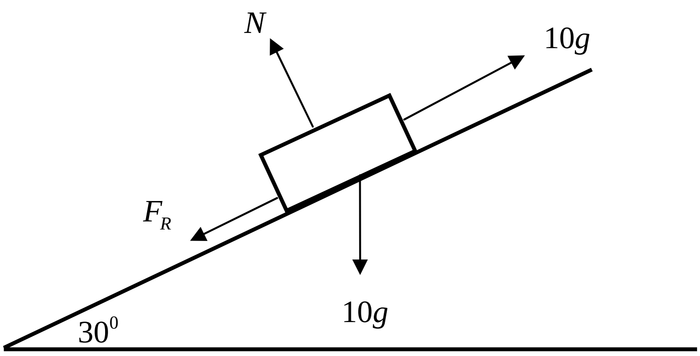
The differential equation for Q, the amount of salt in kilograms in the tank at a time t minutes, is given by

$$\frac{dQ}{dt} = \text{inflow} - \text{outflow} = 10\times 3 - \frac{2Q}{V} \quad \text{and the volume } V = V(t) = 100 + (3-2)t$$

$$\frac{dQ}{dt} = 30 - \frac{2Q}{100+t}, \quad Q(0) = 2, \text{ solving using CAS, the general solution is}$$

$$Q = \frac{C}{(100+t)^2} + 10(100+t) = C(100+t)^{-2} + 10(100+t) \qquad \text{so that } n = -2$$

Question 17 **Answer A**



Note that all forces are in newtons.
resolving perpendicular to the plane

$$N - 10g\cos(30^0) = 0 \qquad \Rightarrow N = 10g\cos(30^0) = 5\sqrt{3}g$$

resolving up and parallel to the plane

$$10g - 10g\sin(30^0) - F_R = 0 \quad \Rightarrow F_R = 10g - 10g\sin(30^0) = 5g$$

$$F_R \le \mu N$$

$$5g \le \mu 5\sqrt{3}g \qquad \Rightarrow \mu \ge \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Question 18 **Answer B**

By Newton's Law $R = ma$ using $a = \frac{dv}{dt}$ with $R = f(t)$

$$f(t) = m\frac{dv}{dt}$$

$$m\int_{v_1}^{v_2} 1.dv = m[v]_{v_1}^{v_2} = mv_2 - mv_1 = \int_{t_1}^{t_2} f(t)\,dt$$

Question 19 **Answer D**

By Newton's law, the equation of motion is given by

$ma = F - kv^3$, using $a = v\frac{dv}{dx}$

$mv\frac{dv}{dx} = F - kv^3$ integrating from $v = 0$ to $v = V$

the distance *s*, travelled from rest is given by

$$s = \int_0^V \frac{mv}{F - kv^3}\,dv$$

Question 20 **Answer A**

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{3bx - 2b^2}{x^3}$$

$$\tfrac{1}{2}v^2 = \int \frac{3bx - 2b^2}{x^3}\,dx = \int\left(3bx^{-2} - 2b^2x^{-3}\right)dx$$

$$\tfrac{1}{2}v^2 = -3bx^{-1} + b^2x^{-2} + c$$

Now when $v = 0$ $x = b$

$0 = -3 + 1 + c \quad \Rightarrow \quad c = 2$

$$\tfrac{1}{2}v^2 = \frac{-3b}{x} + \frac{b^2}{x^2} + 2 = \frac{2x^2 - 3bx + b^2}{x^2}$$

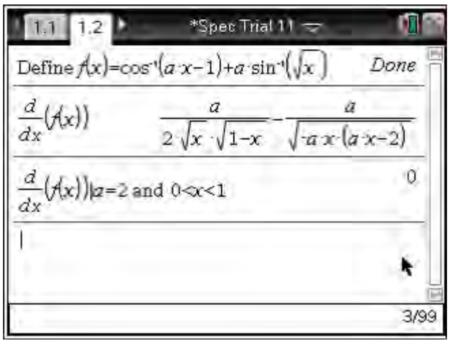
$\tfrac{1}{2}v^2 = \frac{(2x - b)(x - b)}{x^2}$ so that when $v = 0$ $x = b$ and $x = \frac{b}{2}$

Question 21 **Answer D**

$f(x) = \cos^{-1}(ax-1) + a\sin^{-1}\left(\sqrt{x}\right)$

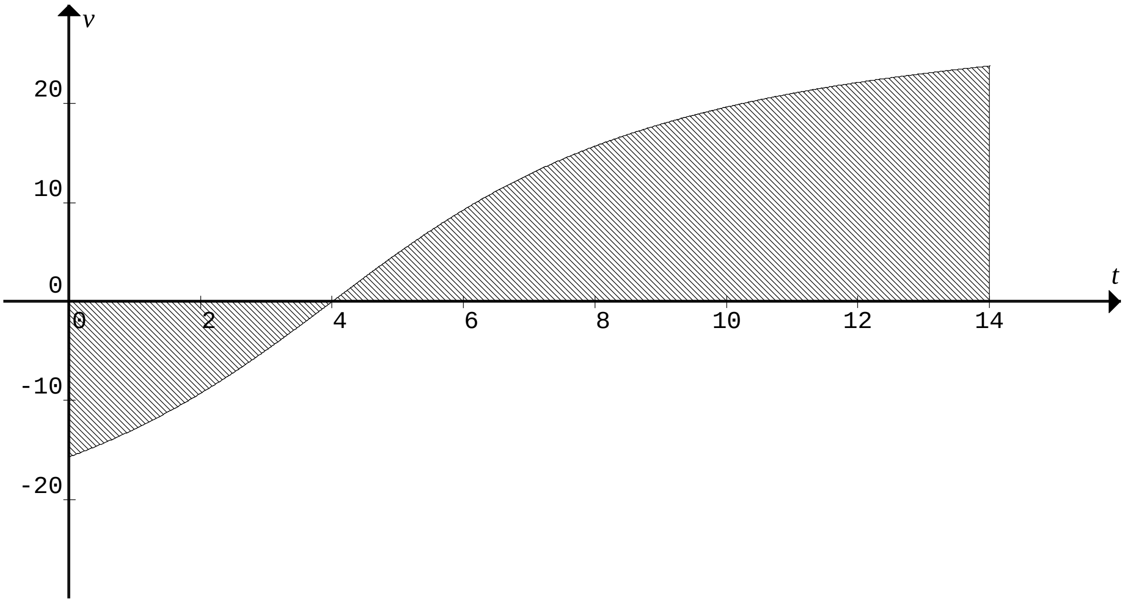
$f'(x) = \dfrac{a}{2\sqrt{x}\sqrt{1-x}} - \dfrac{a}{\sqrt{-ax(ax-2)}}$

if $a = 2$ for $x \in (0,1)$ so $b = 1$ then $f'(x) = 0$



Question 22 **Answer C**

$v(t) = 20\tan^{-1}\left(\dfrac{t}{4} - 1\right)$



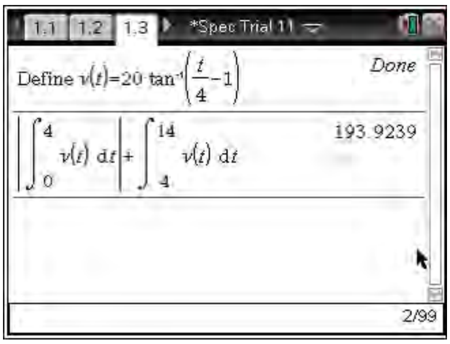
distance travelled d is given by

$d = \left|\int_0^4 v(t)\,dt\right| + \int_4^{14} v(t)\,dt$

$d = \left|\int_0^4 20\tan^{-1}\left(\dfrac{t}{4}-1\right)dt\right| + \int_4^{14} 20\tan^{-1}\left(\dfrac{t}{4}-1\right)dt$

$d = |-35.11| + 158.82 = 35.11 + 158.82 = 193.9$

$d = 194\,\text{m}$



END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a. $u = a + bi \;\Rightarrow\; |u|^2 = a^2 + b^2$ and $\bar{u} = a - bi$

$u\bar{u} = (a+bi)(a-bi) = a^2 - b^2 i^2 = a^2 + b^2$ A1

b. The cubic is $(z-1)(z-u)(z-\bar{u}) = 0$

$(z-1)\left(z^2 - (u+\bar{u})z + u\bar{u}\right) = 0$

$(z-1)\left(z^2 - 2az + a^2 + b^2\right) = 0$ M1

$z^3 - (2a+1)z^2 + \left(a^2 + b^2 + 2a\right)z - \left(a^2 + b^2\right) = 0$

$p = -2a - 1$, $q = a^2 + b^2 + 2a$ and $s = -\left(a^2 + b^2\right)$ A2

c. $iu = i(a+bi) = ai + bi^2$

$iu = -b + ai$ A1

d. $|u|^2 = a^2 + b^2$ and $|iu|^2 = (-b)^2 + a^2 = a^2 + b^2$

so that $|u|^2 + |iu|^2 = 2\left(a^2 + b^2\right)$

M1

now $|u - iu| = |(a+b) + (a-b)i|$

$|u - iu|^2 = (a+b)^2 + (a-b)^2 = a^2 + 2ab + b^2 + a^2 - 2ab + b^2$

A1

$|u - iu|^2 = 2\left(a^2 + b^2\right)$ shown

Since iu is a rotation 90^0 anticlockwise from u, and both $|u|^2 = |iu|^2 = a^2 + b^2$. In the triangle, which passes through O, u and iu it is a right angled isosceles triangle. By Pythagoras's Theorem, the hypotenuse is $\sqrt{2}$ times the two equal side lengths, and $|u - iu| = \sqrt{2\left(a^2 + b^2\right)}$ A1

and represents the distance between the complex numbers u and iu.

e. From **d.** since the angle at the origin is a right-angle, the line joining the complex numbers u and iu. is a diameter of the circle, the radius r is therefore

$r = \frac{1}{2}|u - iu| = \frac{1}{2}\sqrt{2\left(a^2 + b^2\right)} = \sqrt{\frac{a^2 + b^2}{2}}$ A1

The centre is the mid-point of the line joining the complex numbers u and iu.

$c = \frac{1}{2}(u + iu) = \frac{1}{2}(a-b) + \frac{1}{2}(a+b)i$ A1

f. The shaded area equals the area of the circle minus the area of the triangle

$\text{Area} = \pi r^2 - \frac{1}{2}|u||iu|$

$\text{Area} = \pi\left(\frac{a^2+b^2}{2}\right) - \frac{1}{2}\left(a^2+b^2\right)$ M1

$\text{Area} = \frac{1}{2}\left(a^2+b^2\right)(\pi-1)$ A1

Question 2

a. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ differentiating implicitly, gives

$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$ at $P\left(a\sec(\theta), b\tan(\theta)\right)$ A1

$\frac{dy}{dx} = \frac{b^2 a\sec(\theta)}{a^2 b\tan(\theta)} = \frac{b}{a\sin(\theta)}$, the equation of the tangent at P is

$y - b\tan(\theta) = \frac{b}{a\sin(\theta)}\left(x - a\sec(\theta)\right)$

$y = \frac{bx}{a\sin(\theta)} + \frac{b\sin(\theta)}{\cos(\theta)} - \frac{b}{\sin(\theta)\cos(\theta)}$

$y = \frac{bx}{a\sin(\theta)} - \frac{b\left(1-\sin^2(\theta)\right)}{\sin(\theta)\cos(\theta)} = \frac{bx}{a\sin(\theta)} - \frac{b\cos^2(\theta)}{\sin(\theta)\cos(\theta)}$ A1

$y = \frac{bx}{a\sin(\theta)} - \frac{b\cos(\theta)}{\sin(\theta)}$

b. This crosses the asymptote $y = \frac{bx}{a}$ at A, when

$\frac{bx}{a} = \frac{bx}{a\sin(\theta)} - \frac{b\cos(\theta)}{\sin(\theta)}$ A1

$\frac{b\cos(\theta)}{\sin(\theta)} = \frac{bx}{a\sin(\theta)} - \frac{bx}{a} = \frac{bx}{a}\left(\frac{1}{\sin(\theta)} - 1\right) = \frac{bx\left(1-\sin(\theta)\right)}{a\sin(\theta)}$

$x_A = \frac{a\cos(\theta)}{1-\sin(\theta)} \Rightarrow y_A = \frac{b\cos(\theta)}{1-\sin(\theta)}$ A1

$A\left(\frac{a\cos(\theta)}{1-\sin(\theta)}, \frac{b\cos(\theta)}{1-\sin(\theta)}\right)$

This crosses the asymptote $y = -\frac{bx}{a}$ at *B*, when

$$-\frac{bx}{a} = \frac{bx}{a\sin(\theta)} - \frac{b\cos(\theta)}{\sin(\theta)}$$ A1

$$\frac{b\cos(\theta)}{\sin(\theta)} = \frac{bx}{a\sin(\theta)} + \frac{bx}{a} = \frac{bx}{a}\left(\frac{1}{\sin(\theta)} + 1\right) = \frac{bx(1+\sin(\theta))}{a\sin(\theta)}$$

$$x_B = \frac{a\cos(\theta)}{1+\sin(\theta)} \Rightarrow y_B = \frac{-b\cos(\theta)}{1+\sin(\theta)}$$ A1

$$B\left(\frac{a\cos(\theta)}{1+\sin(\theta)}, \frac{-b\cos(\theta)}{1+\sin(\theta)}\right)$$

c. Since *O* is the origin,

$$\overrightarrow{OA} = \left(\frac{a\cos(\theta)}{1-\sin(\theta)}\right)\underset{\sim}{i} + \left(\frac{b\cos(\theta)}{1-\sin(\theta)}\right)\underset{\sim}{j}$$

$$\left|\overrightarrow{OA}\right| = \sqrt{\left(\frac{a\cos(\theta)}{1-\sin(\theta)}\right)^2 + \left(\frac{b\cos(\theta)}{1-\sin(\theta)}\right)^2}$$ A1

$$= \sqrt{\frac{a^2\cos^2(\theta) + b^2\cos^2(\theta)}{(1-\sin(\theta))^2}} = \sqrt{a^2+b^2}\left(\frac{\cos(\theta)}{1-\sin(\theta)}\right)$$

$$\overrightarrow{OB} = \left(\frac{a\cos(\theta)}{1+\sin(\theta)}\right)\underset{\sim}{i} + \left(\frac{-b\cos(\theta)}{1+\sin(\theta)}\right)\underset{\sim}{j}$$

$$\left|\overrightarrow{OB}\right| = \sqrt{\left(\frac{a\cos(\theta)}{1+\sin(\theta)}\right)^2 + \left(\frac{-b\cos(\theta)}{1+\sin(\theta)}\right)^2}$$ A1

$$= \sqrt{\frac{a^2\cos^2(\theta) + b^2\cos^2(\theta)}{(1+\sin(\theta))^2}} = \sqrt{a^2+b^2}\left(\frac{\cos(\theta)}{1+\sin(\theta)}\right)$$

$$\overrightarrow{OA}.\overrightarrow{OB}=\left(\frac{a\cos(\theta)}{1-\sin(\theta)}\right)\left(\frac{a\cos(\theta)}{1+\sin(\theta)}\right)+\left(\frac{b\cos(\theta)}{1-\sin(\theta)}\right)\left(\frac{-b\cos(\theta)}{1+\sin(\theta)}\right)$$

$$\overrightarrow{OA}.\overrightarrow{OB}=\frac{\left(a^2-b^2\right)\cos^2(\theta)}{1-\sin^2(\theta)}=a^2-b^2$$

Now $\cos(\alpha)=\dfrac{\overrightarrow{OA}.\overrightarrow{OB}}{\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|}$ M1

$$=\frac{a^2-b^2}{\sqrt{a^2+b^2}\left(\dfrac{\cos(\theta)}{1-\sin(\theta)}\right)\sqrt{a^2+b^2}\left(\dfrac{\cos(\theta)}{1+\sin(\theta)}\right)}$$

M1

$$=\frac{a^2-b^2}{\left(a^2+b^2\right)\dfrac{\cos^2(\theta)}{1-\sin^2(\theta)}}=\frac{a^2-b^2}{a^2+b^2}$$

so that $\alpha=\angle AOB=\cos^{-1}\left(\dfrac{a^2-b^2}{a^2+b^2}\right)$

d. Area of the triangle $\Delta AOB=\dfrac{1}{2}\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\sin(\alpha)$, where

$$\sin(\alpha)=\frac{\sqrt{\left(a^2+b^2\right)^2-\left(a^2-b^2\right)^2}}{\left(a^2+b^2\right)}$$

$$\sin(\alpha)=\frac{\sqrt{\left(a^4+2a^2b^2+b^4\right)-\left(a^4-2a^2b^2+b^4\right)}}{\left(a^2+b^2\right)}$$

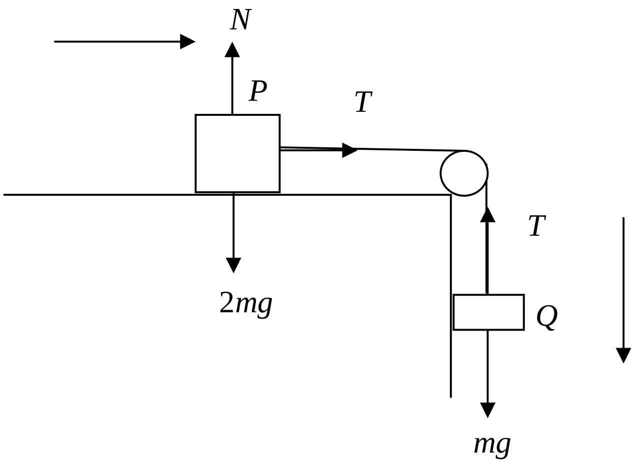
$\sin(\alpha)=\dfrac{\sqrt{4a^2b^2}}{\left(a^2+b^2\right)}=\dfrac{2ab}{\left(a^2+b^2\right)}$ since $a>b>0$ A1

$$\text{Area } \Delta AOB=\frac{1}{2}\left(\sqrt{a^2+b^2}\left(\frac{\cos(\theta)}{1-\sin(\theta)}\right)\right)\times\left(\sqrt{a^2+b^2}\left(\frac{\cos(\theta)}{1+\sin(\theta)}\right)\right)\times\frac{2ab}{\left(a^2+b^2\right)}$$

Area $\Delta AOB=ab$ A1

Question 3

a. smooth surface



resolving downwards around Q, (1) $mg - T = ma$

resolving around P, (2) $N - 2mg = 0$ A1

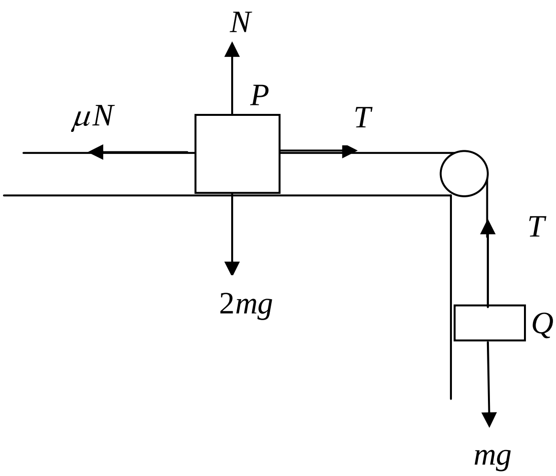
(3) $T = 2ma$

From (3) into (1) $mg - 2ma = ma$ $\Rightarrow 3ma = mg$ $\Rightarrow a = \dfrac{g}{3}$ A1

$a = \dfrac{g}{3}$ $u = 0$ $t = 0.5$ $s = ?$, using $s = ut + \dfrac{1}{2}at^2$

$s = 0 + \dfrac{1}{2} \times \dfrac{g}{3} \times 0.5^2 = 0.41$ m A1

b. rough surface



T is the tension in the string in newtons, N the normal reaction and
μN the frictional force. correct forces and labelled A1

i. resolving downwards around Q, (1) $mg - T = 0$

since the acceleration of the system is zero.

$T = mg$ newtons A1

ii. consider when the acceleration is non-zero

resolving downwards around Q, (1) $mg - T = ma$

resolving around P, (2) $N - 2mg = 0$ A1

(3) $T - \mu N = 2ma$

from (2) $N = 2mg$ substituting into (3) gives

$T - 2\mu mg = 2ma$ but from (1) $mg - T = ma$ adding eliminating T gives M1

$3ma = mg - 2\mu mg = mg(1-2\mu)$

$a = \dfrac{g}{3}(1-2\mu)$

for $a \ge 0$, it follows that the maximum value of M1

μ, is $\mu = \dfrac{1}{2}$, and since $\mu > 0$,

$0 < \mu \le \dfrac{1}{2}$ A1

Question 4

$\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$, $\overrightarrow{OB} = \alpha\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\overrightarrow{OC} = -5\underset{\sim}{i} + 8\underset{\sim}{j} + 11\underset{\sim}{k}$

a. for perpendicular $\overrightarrow{OA}.\overrightarrow{OB} = 0$

$2\alpha - 2 - 4 = 0$

$2\alpha = 6$

$\alpha = 3$ A1

b. for linearly dependant $\overrightarrow{OC} = x\overrightarrow{OA} + y\overrightarrow{OB}$

$\underset{\sim}{c} = -5\underset{\sim}{i} + 8\underset{\sim}{j} + 11\underset{\sim}{k} = x\left(2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}\right) + y\left(\alpha\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}\right)$ M1

$\underset{\sim}{i}$ (1) $-5 = 2x + y\alpha$ (2) $8 = x - 2y$

$\underset{\sim}{j}$ (2) $8 = x - 2y$ $-2\text{x}(3)$ $-22 = -8x + 2y$ M1

$\underset{\sim}{k}$ (3) $11 = 4x - y$ adding $-7x = -14$

so that $x = 2$, $y = -3$ and $\alpha = 3$ A1

c. $\left|\overrightarrow{OA}\right| = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{21}$

$\left|\overrightarrow{OB}\right| = \sqrt{\alpha^2 + (-2)^2 + (-1)^2} = \sqrt{5+\alpha^2}$ A1

$\left|\overrightarrow{OA}\right| = \left|\overrightarrow{OB}\right| \Rightarrow \quad \sqrt{21} = \sqrt{5+\alpha^2}$

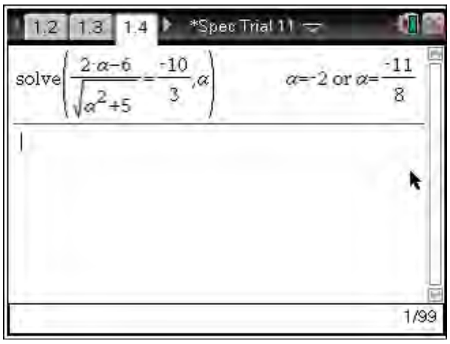
$5 + \alpha^2 = 21$

$\alpha^2 = 16$

$\alpha = \pm 4$ both answers are acceptable A1

d. the scalar resolute of $\overrightarrow{OA}$ parallel to $\overrightarrow{OB}$ is $\dfrac{\overrightarrow{OA}.\overrightarrow{OB}}{\left|\overrightarrow{OB}\right|} = -\dfrac{10}{3}$

$$\frac{\overrightarrow{OA}.\overrightarrow{OB}}{\left|\overrightarrow{OB}\right|} = \frac{2\alpha - 6}{\sqrt{\alpha^2 + 5}} = -\frac{10}{3}$$



A1

solving on a CAS gives

$\alpha = -2$ or $\alpha = -\dfrac{11}{8}$ both answers are acceptable. A1

e. Let θ be the angle the vector $\overrightarrow{OB}$ makes with the *x*-axis, then

$$\theta = \cos^{-1}\left(-\frac{2}{3}\right) \quad \Rightarrow \cos(\theta) = -\frac{2}{3} = \frac{\overrightarrow{OB}.\underset{\sim}{i}}{\left|\overrightarrow{OB}\right|}$$

$-\dfrac{2}{3} = \dfrac{\alpha}{\sqrt{\alpha^2+5}}$, and $\alpha < 0$

M1

$-2\sqrt{\alpha^2 + 5} = 3\alpha$

$4(\alpha^2 + 5) = 9\alpha^2$

$20 = 5\alpha^2$

$\alpha^2 = 4$

solving gives $\alpha = -2$ as the only solution A1

Question 5

a. $\dot{x} = \dfrac{dx}{dt} = V\cos(\alpha) - kx$

$\dfrac{dt}{dx} = \dfrac{1}{V\cos(\alpha) - kx}$

$t = \displaystyle\int \dfrac{1}{V\cos(\alpha) - kx}\,dx$ M1

$t = -\dfrac{1}{k}\log_e\left(V\cos(\alpha) - kx\right) + c$

Now when $t = 0$ $x = 0$

$0 = -\dfrac{1}{k}\log_e\left(V\cos(\alpha)\right) + c \quad \Rightarrow c = \dfrac{1}{k}\log_e\left(V\cos(\alpha)\right)$

$t = \dfrac{1}{k}\log_e\left(V\cos(\alpha)\right) - \dfrac{1}{k}\log_e\left(V\cos(\alpha) - kx\right)$ M1

$t = \dfrac{1}{k}\log_e\left(\dfrac{V\cos(\alpha)}{V\cos(\alpha) - kx}\right)$

$e^{kt} = \dfrac{V\cos(\alpha)}{V\cos(\alpha) - kx}$

$\dfrac{V\cos(\alpha) - kx}{V\cos(\alpha)} = e^{-kt}$ M1

$V\cos(\alpha) - kx = V\cos(\alpha)e^{-kt}$

$kx = V\cos(\alpha) - V\cos(\alpha)e^{-kt} = V\cos(\alpha)\left(1 - e^{-kt}\right)$

$x = x(t) = \dfrac{V\cos(\alpha)}{k}\left(1 - e^{-kt}\right)$ A1

b. $\dot{y} = V\sin(\alpha) - gt$ since $y(0) = 0$

$y = Vt\sin(\alpha) - \dfrac{1}{2}gt^2 = t\left(V\sin(\alpha) - \dfrac{gt}{2}\right)$ M1

when $y = 0$ $t = T = \dfrac{2V\sin(\alpha)}{g}$

c. $x = x(t) = \dfrac{V\cos(\alpha)}{k} - \dfrac{V\cos(\alpha)}{k}e^{-kt}$ $\qquad y(t) = Vt\sin(\alpha) - \dfrac{1}{2}gt^2$

$\dot{x}(t) = -k \times \dfrac{-V\cos(\alpha)}{k}e^{-kt} = V\cos(\alpha)e^{-kt}$ $\qquad \dot{y}(t) = V\sin(\alpha) - gt$

when it hits the ground $T = \dfrac{2V\sin(\alpha)}{g}$ M1

$\dot{x}(T) = V\cos(\alpha)e^{-\frac{2Vk\sin(\alpha)}{g}}$

$\dot{y}(T) = V\sin(\alpha) - g \times \dfrac{2V\sin(\alpha)}{g} = -V\sin(\alpha)$ A1

the angle at which it hits the ground is

$$\tan^{-1}\left(\left|\frac{\dot{y}(T)}{\dot{x}(T)}\right|\right) = \tan^{-1}\left(\frac{V\sin(\alpha)}{V\cos(\alpha)e^{-\frac{2Vk\sin(\alpha)}{g}}}\right)$$

$$= \tan^{-1}\left(\tan(\alpha)e^{\frac{2Vk\sin(\alpha)}{g}}\right)$$ A1

d. since $\underset{\sim}{\ddot{r}}(t) = -k\dot{x}(t)\underset{\sim}{i} - g\underset{\sim}{j}$ the golf ball is subjected to gravity, in the vertical y direction, but an air resistance force, proportional to the velocity in the horizontal x direction of motion. A1

e. $V = 49 \quad \alpha = 25^0 \quad k = 0.2$

$T = \dfrac{2V\sin(\alpha)}{g} = \dfrac{2 \times 49 \times \sin(25^0)}{9.8}$

$T = 4.23$ sec

$R = x(T) = \dfrac{V\cos(\alpha)}{k}\left(1 - e^{-kT}\right) = \dfrac{49 \times \cos(25^0)}{0.2}\left(1 - e^{-0.2\times4.23}\right)$

$R = 126.69$ m A1

f. at maximum height

$\dot{y} = 0 \ \Rightarrow t = \dfrac{1}{2}T = 2.11$ sec A1

$x\left(\dfrac{1}{2}T\right) = x(2.11) = x(t) = \dfrac{49\cos 25^0}{0.2}\left(1 - e^{-0.2\times2.11}\right) = 76.53$ m A1

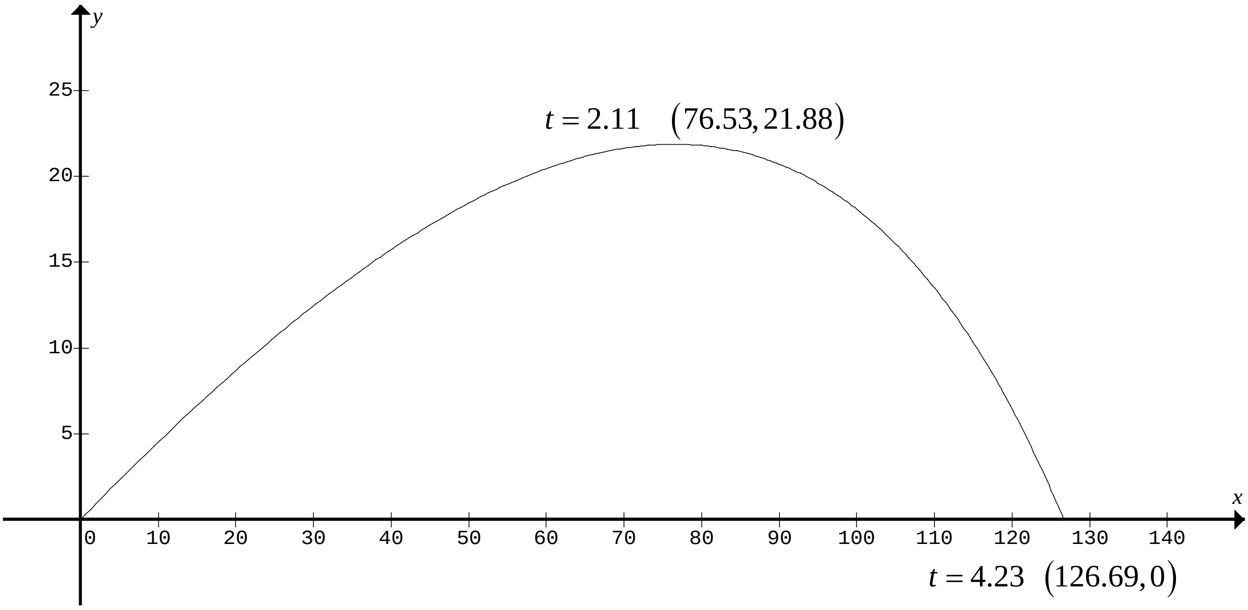
$y_{\max} = H = \dfrac{V^2\sin^2(\alpha)}{2g} = \dfrac{49^2\sin^2(25^0)}{2\times9.8} = 21.88$ m A1

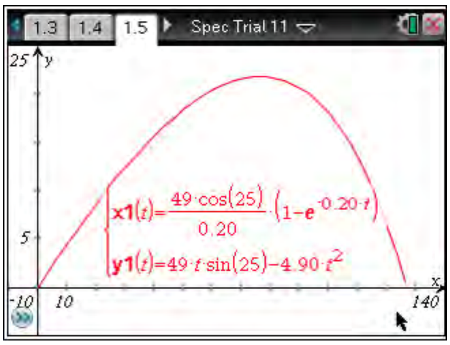
g. correct graph, shape, critical points,

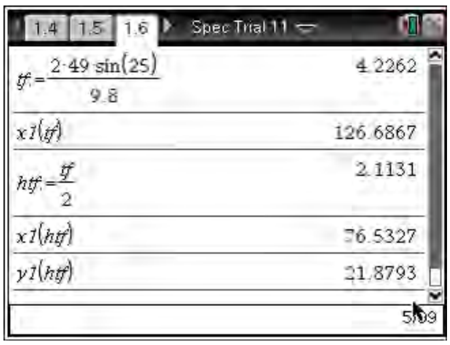
the maximum height at maximum $(76.53, 21.88)$ when $t = 2.11$

and the range, hits the ground at $(126.69, 0)$ when $t = 4.23$ A1

correct shape of the graph (not parabolic) A1







END OF SECTION 2 SUGGESTED ANSWERS