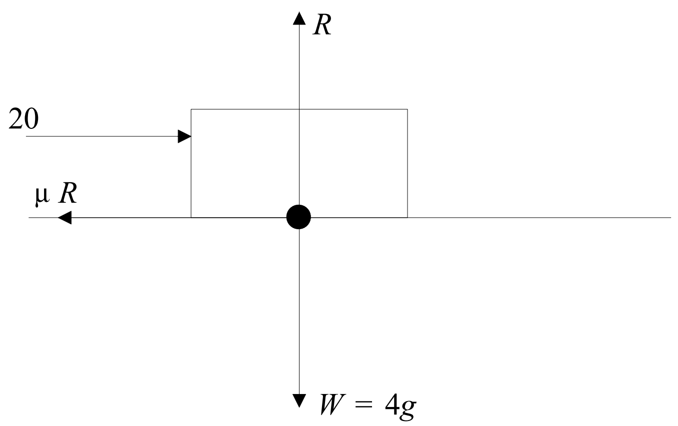
The Mathematical Association of Victoria
Trial Examination 2011
Specialist Maths Examination 1 - SOLUTIONS

Question 1

a.
Normal reaction force R.
Weight force $W = 4g$.
Friction force μR (since object is moving).
Pushing force of 20 Newton.



All forces labeled **[A1]**

b.

Net force in **vertical direction**: $\left.\begin{array}{l} F_{net} = 0 \\ F_{net} = R - 4g \end{array}\right\} \Rightarrow R = 4g$. .... (1)

Net force in **horizontal direction**: $\left.\begin{array}{l} F_{net} = ma = (4)(2) = 8 \\ F_{net} = 20 - \mu R \end{array}\right\} \Rightarrow 8 = 20 - \mu R$ **[M1]**

$\Rightarrow 12 = \mu R$. .... (2)

Substitute equation (1) into equation (2):

$12 = \mu 4g$

$\Rightarrow \mu = \dfrac{3}{g}$. **[A1]**

Total 3 marks

Question 2

Let $\underset{\sim}{a} = -\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - 6\underset{\sim}{k}$.

Parallel vector resolute:

$\underset{\sim}{u} = \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$.

$\left|\underset{\sim}{b}\right| = \sqrt{3^2 + (-2)^2 + (-6)^2} = \sqrt{49} = 7$.

Therefore $\hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{b}}{|\underset{\sim}{b}|} = \dfrac{3\underset{\sim}{i} - 2\underset{\sim}{j} - 6\underset{\sim}{k}}{7}$. **[A1]**

Therefore $\underset{\sim}{u} = \left(\dfrac{\left(-\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right) \cdot \left(3\underset{\sim}{i} - 2\underset{\sim}{j} - 6\underset{\sim}{k}\right)}{7}\right)\left(\dfrac{3\underset{\sim}{i} - 2\underset{\sim}{j} - 6\underset{\sim}{k}}{7}\right) = \left(\dfrac{-3 - 2 + 12}{7}\right)\left(\dfrac{3\underset{\sim}{i} - 2\underset{\sim}{j} - 6\underset{\sim}{k}}{7}\right)$

$= \dfrac{1}{7}\left(3\underset{\sim}{i} - 2\underset{\sim}{j} - 6\underset{\sim}{k}\right)$. **[M1]**

Perpendicular vector resolute:

$\underset{\sim}{a} = \underset{\sim}{u} + \underset{\sim}{v} \Rightarrow \underset{\sim}{v} = \underset{\sim}{a} - \underset{\sim}{u}$

$\Rightarrow \underset{\sim}{v} = -\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k} - \left(\dfrac{3}{7}\underset{\sim}{i} - \dfrac{2}{7}\underset{\sim}{j} - \dfrac{6}{7}\underset{\sim}{k}\right)$

$= -\dfrac{10}{7}\underset{\sim}{i} + \dfrac{9}{7}\underset{\sim}{j} - \dfrac{8}{7}\underset{\sim}{k}$. **[A1]**

Total 3 marks

Question 3

Substitute $y=\sqrt{3}-2$ into $(x+1)^2+\dfrac{(y+2)^2}{4}=1$:

$$(x+1)^2+\frac{3}{4}=1$$

$$\Rightarrow x+1=\pm\frac{1}{2}$$

$$\Rightarrow x=-\frac{1}{2},\ -\frac{3}{2}.$$

But $x>-1$.

Therefore $x=-\dfrac{1}{2}$. **[A1]**

Implicit differentiation: $2(x+1)+\dfrac{2(y+2)}{4}\times\dfrac{dy}{dx}=0$. **[M1]**

Substitute $x=-\dfrac{1}{2}$ and $y=\sqrt{3}-2$ into $2(x+1)+\dfrac{2(y+2)}{4}\times\dfrac{dy}{dx}=0$:

$$1+\frac{2\sqrt{3}}{4}\times\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{4}{2\sqrt{3}}=-\frac{2}{\sqrt{3}}.$$

$m_{normal}=\dfrac{-1}{\frac{dy}{dx}}=\dfrac{\sqrt{3}}{2}$. **[A1]**

Total 3 marks

Question 4

$f(-i)=0 \Rightarrow z+i$ is a factor of $f(z)=z^3+2iz^2-2z-i$.
Note: $z-i$ is NOT a second factor because the conjugate root theorem is NOT valid (the coefficients of the polynomial are not all real).

Polynomial long division:

$$
\begin{array}{r l}
 & z^2+iz-1 \\
z+i \,\big) & z^3+2iz^2-2z-i \\
 & \underline{z^3+iz^2} \\
 & \quad iz^2-2z-i \\
 & \quad \underline{iz^2-z} \\
 & \qquad -z-i \\
 & \qquad \underline{-z-i} \\
 & \qquad\quad 0
\end{array}
$$

[M2]

Therefore the remaining roots of $f(z)=z^3+2iz^2-2z-i$ are solutions to $z^2+iz-1=0$.

Quadratic formula:

$$z=\frac{-i \pm \sqrt{(i)^2-4(1)(-1)}}{2}$$

$$=\frac{-i \pm \sqrt{3}}{2}.$$

Roots: $z=\dfrac{-i \pm \sqrt{3}}{2}, \ -i$. **[A1]**

Total 3 marks

Question 5

a.
Option 1:

$(z-\sqrt{2})(\bar{z}-\sqrt{2})=2$

$\Rightarrow (z-\sqrt{2})\overline{(z-\sqrt{2})}=2$

$\Rightarrow |z-\sqrt{2}|^2=2$

$\Rightarrow |z-\sqrt{2}|=\sqrt{2}$. **[M1]**

Option 2:

Substitute $z=x+iy$:

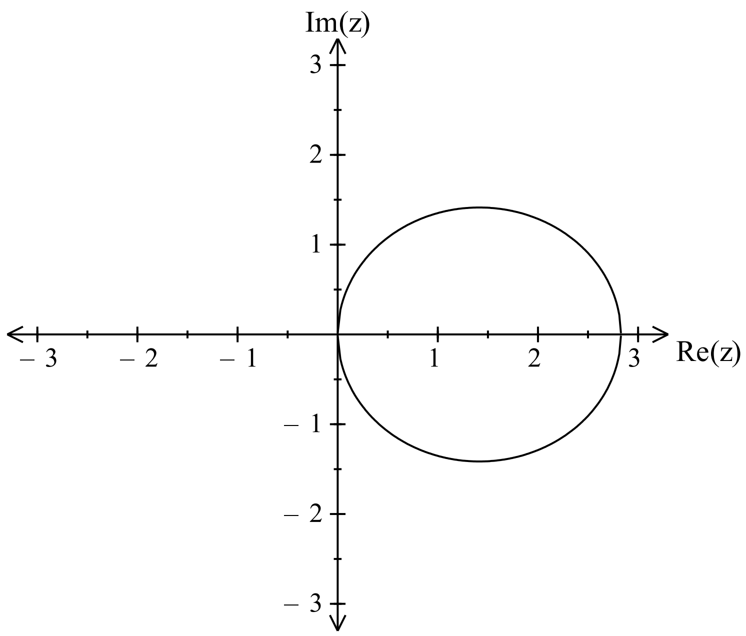
$(z-\sqrt{2})(\bar{z}-\sqrt{2})=2$

$\Rightarrow (x-\sqrt{2}+iy)(x-\sqrt{2}-iy)=2$

$\Rightarrow x^2-2\sqrt{2}x+y^2+2=2$

$\Rightarrow (x-\sqrt{2})^2+y^2=2$. **[M1]**

Circle with radius $r=\sqrt{2}$ and centre at $(\sqrt{2},0)$.



Note: Shape must be consistent with the scale on imaginary and real axes.

Shape, centre and radius **[A1]**

b.

Option 1:

$|z|=\left|z-3\sqrt{2}\right|$ defines the perpendicular bisector of the line segment joining $z=0$ and $z=3\sqrt{2}$:

$x=\dfrac{3\sqrt{2}}{2}$. **[A1]**

Option 2:

Substitute $z=x+iy$:

$$|z|=\left|z-3\sqrt{2}\right| \Rightarrow |x+iy|=\left|x-3\sqrt{2}+iy\right|$$

$$\Rightarrow \sqrt{x^2+y^2}=\sqrt{(x-3\sqrt{2})^2+y^2}$$

$$\Rightarrow x^2+y^2=x^2-6\sqrt{2}x+18+y^2$$

$$\Rightarrow x=\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{2}$$ **[A1]**

Substitute $x=\dfrac{3\sqrt{2}}{2}$ into $(x-\sqrt{2})^2+y^2=2$:

$$\frac{1}{2}+y^2=2$$

$$\Rightarrow y=\pm\frac{\sqrt{3}}{\sqrt{2}}.$$

[A1]

Therefore $z=\dfrac{3\sqrt{2}}{2}\pm i\dfrac{\sqrt{3}}{\sqrt{2}}$.

Polar form:

$$r=\sqrt{\left(\frac{3\sqrt{2}}{2}\right)^2+\frac{3}{2}}=\sqrt{\frac{18}{4}+\frac{3}{2}}=\sqrt{6}.$$

$$\tan(\theta)=\frac{\pm\frac{\sqrt{3}}{\sqrt{2}}}{\frac{3\sqrt{2}}{2}}=\pm\frac{1}{\sqrt{3}}.$$

$$z=\sqrt{6}\,\text{cis}\left(\frac{\pi}{6}\right),\ \sqrt{6}\,\text{cis}\left(-\frac{\pi}{6}\right).$$ **[A1]**

Total 5 marks

Question 6

a.
Option 1:

$$a = v\frac{dv}{dx} = v\left(-\frac{1}{x^2}\right)$$

$$= \left(\frac{1}{x} + \frac{1}{2}\right)\left(-\frac{1}{x^2}\right)$$ **[A1]**

$$= -\left(\frac{2+x}{2x^3}\right).$$

Option 2:

$$a = \frac{1}{2}\frac{d}{dx}\left(v^2\right) = \frac{1}{2}\frac{d}{dx}\left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(-\frac{2}{x^3} - \frac{1}{x^2}\right)$$ **[A1]**

$$= -\left(\frac{2+x}{2x^3}\right).$$

b.

$$\frac{dx}{dt} = \frac{2+x}{2x}$$

$$\Rightarrow \frac{dt}{dx} = \frac{2x}{2+x}$$ **[M1]**

$$= 2\left(1 - \frac{2}{x+2}\right)$$

$$\Rightarrow t = 2\int\left(1 - \frac{2}{x+2}\right)dx$$

$$= 2x - 4\log_e|x+2| + C.$$ **[A1]**

Substitute $t = 0$ and $x = 2$: $C = 4\ln(4) - 4$.

Therefore:

$$t = 2x - 4 + 4\log_e(4) - 4\log_e|x+2|$$

$$= 2x - 4 + 4\log_e\left|\frac{4}{x+2}\right|.$$ **[A1]**

Total 4 marks

Question 7

Let $y = \arccos\left(\frac{1}{2\sqrt{x}}\right)$.

Chain rule: Let $u = \frac{1}{2\sqrt{x}} \Rightarrow y = \arccos(u)$.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{-1}{\sqrt{1-u^2}} \times \left(-\frac{1}{4x^{3/2}}\right)$$ **[M1]**

$$= \frac{-1}{\sqrt{1-\frac{1}{4x}}} \times \left(-\frac{1}{4x\sqrt{x}}\right)$$

$$= \frac{1}{\sqrt{\frac{4x-1}{4x}}} \times \left(\frac{1}{4x\sqrt{x}}\right)$$ **[M1]**

$$= \frac{2\sqrt{x}}{\sqrt{4x-1}} \times \left(\frac{1}{4x\sqrt{x}}\right)$$

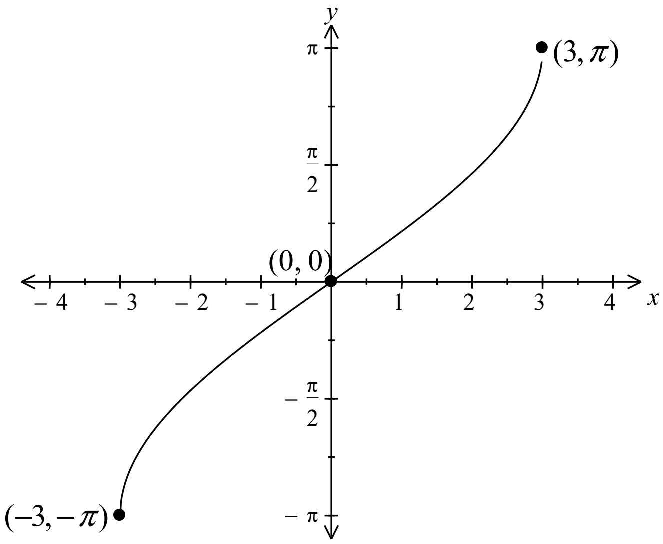
$$= \frac{1}{2x\sqrt{4x-1}}.$$ **[A1]**

Total 3 marks

Question 8

a.

The graph of $y = \arcsin(x)$ is dilated from the x-axis by a factor of 2 and dilated from the y-axis by a factor of 3:



Endpoints **[A1]**
Shape and inflection point **[A1]**

b.

$x = \frac{3}{2} \Rightarrow y = 2\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

$x = 3 \Rightarrow y = 2\arcsin(1) = \pi$.

$$V = \pi \int_{\pi/3}^{\pi} x^2 \, dy.$$ **[M1]**

$y = 2\arcsin\left(\frac{x}{3}\right) \Rightarrow 3\sin\left(\frac{y}{2}\right) = x$.

Therefore:

$$V = 9\pi \int_{\pi/3}^{\pi} \sin^2\left(\frac{y}{2}\right) dy$$ **[M1]**

$$= \frac{9\pi}{2} \int_{\pi/3}^{\pi} (1 - \cos(y)) \, dy$$ **[M1]**

$$= \frac{9\pi}{2} \left[y - \sin(y)\right]_{\pi/3}^{\pi} = \frac{9\pi}{2}\left[\pi - \frac{\pi}{3} + \frac{\sqrt{3}}{2}\right] = \frac{9\pi}{2}\left[\frac{4\pi + 3\sqrt{3}}{6}\right]$$

$$= \frac{(27\sqrt{3} + 36\pi)\pi}{12} = \frac{(9\sqrt{3} + 12\pi)\pi}{4}.$$ **[A1]**

Total 6 marks

Question 9

a.

$$\frac{x^2}{(x+1)^2} = \frac{(x^2+2x+1)-(2x+1)}{x^2+2x+1}$$

$$= 1 - \frac{2x+1}{x^2+2x+1}$$ **[M1]**

$$\frac{2x+1}{x^2+2x+1} = \frac{2x+1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2}.$$

$$2x+1 \equiv A(x+1)+B = Ax+(A+B)$$

$$\Rightarrow A = 2, \quad B = -1.$$

Therefore $1 - \frac{2x+1}{x^2+2x+1} = 1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}$. **[A1]**

b.

$$\frac{dx}{dt} = \left(\frac{x+1}{x}\right)^2 = \frac{(x+1)^2}{x^2}$$

$$\Rightarrow \frac{dt}{dx} = \frac{x^2}{(x+1)^2} = 1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}$$ **[M1]**

$$\Rightarrow t = \int \left(1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}\right) dx$$

$$= x - 2\log_e |x+1| - \frac{1}{x+1} + C.$$ **[M1]**

Substitute $x = 0$ and $t = 0$: $C = 1$.

Therefore:

$$t = x - 2\log_e |x+1| - \frac{1}{x+1} + 1.$$ **[A1]**

Total 5 marks

Question 10

a.

$x = \sin(t)$ …. (1)

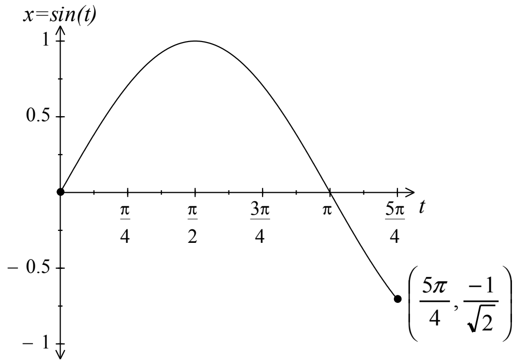
$y = -\cos(2t)$

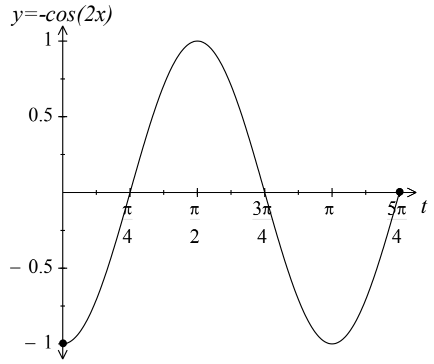
$= 2\sin^2(t) - 1$ …. (2) **[M1]**

Substitute equation (1) into equation (2):

$y = 2x^2 - 1.$ **[A1]**

Domain: Range:





Domain: $-\frac{1}{\sqrt{2}} \le x \le 1.$

Range: $-1 \le y \le 1.$ Domain and range **[A1]**

b.

$\dot{\underset{\sim}{r}} = \cos(t)\,\underset{\sim}{i} + 2\sin(2t)\,\underset{\sim}{j}.$ **[M1]**

At $t = \frac{\pi}{4}$: $\dot{\underset{\sim}{r}} = \frac{1}{\sqrt{2}}\,\underset{\sim}{i} + 2\,\underset{\sim}{j}.$ **[A1]**

Total 5 marks

END OF SOLUTIONS

