Neap:

Trial Examination 2011

VCE Specialist Mathematics Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

Question					
1	A	B	**C**	D	E
2	A	B	C	**D**	E
3	A	B	**C**	D	E
4	A	**B**	C	D	E
5	A	B	C	D	**E**
6	**A**	B	C	D	E
7	A	B	C	D	**E**
8	A	B	C	D	**E**
9	**A**	B	C	D	E
10	**A**	B	C	D	E
11	A	B	C	**D**	E
12	A	B	**C**	D	E
13	A	B	**C**	D	E
14	A	**B**	C	D	E
15	A	**B**	C	D	E
16	A	B	C	**D**	E
17	**A**	B	C	D	E
18	**A**	B	C	D	E
19	A	B	**C**	D	E
20	A	B	C	D	**E**
21	A	**B**	C	D	E
22	A	B	C	**D**	E

SECTION 1

Question 1 **C**

Both graphs have a vertical axis of symmetry at $x = -\frac{b}{2a}$ and so **A** is true.

$f(0) = c$ and $g(0) = \frac{1}{c}$ so **B** is true.

As $f(x) \to 0, \frac{1}{f(x)} \to \pm\infty$ and so **D** is true.

If $f(x) > 0$ then $\frac{1}{f(x)} > 0$, and if $f(x) < 0$ then $\frac{1}{f(x)} < 0$. Hence **E** is true.

Option **C** is true except where $f'(x) = 0$ and $f(x) = 0$ simultaneously. This situation occurs for $g(x) = \frac{1}{x^2}$, i.e. $f(x) = x^2$, and in this case, the reciprocal graph has a vertical asymptote at $x = 0$.

Question 2 **D**

We require $4k + 1 > 0, k + 3 > 0$ and $k + 1 > 0$, or $4k + 1 < 0, k + 3 < 0$ and $k + 1 < 0$.

From the first set of conditions we require $k > -\frac{1}{4}$, $k > -3$ and $k > -1$. Hence $k > -\frac{1}{4}$.

From the second set of conditions we require $k < -\frac{1}{4}$, $k < -3$ and $k < -1$. Hence $k < -3$.

So the curve is an ellipse for $k < -3$ or $k > -\frac{1}{4}$.

Question 3 **C**

The vertical asymptote has equation $x = 0$.

As $x \to \pm\infty, \frac{4}{x^2} \to 0$.

Hence, as $x \to \pm\infty$, the graph of the function $g(x) = 4x - \frac{4}{x^2}$ approaches the graph of $y = 4x$.

So the non-vertical asymptote has equation $y = 4x$.

Hence we can disregard options **D** and **E**.

$g'(x) = 4 + \frac{8}{x^3}$

Solving $g'(x) = 0$ for x gives $x = -\sqrt[3]{2}$.

Hence we can disregard option **A**.

$g''(-\sqrt[3]{2}) < 0$ and so a local maximum occurs at $x = -\sqrt[3]{2}$.

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Question 4 B

We require $-1 \le \frac{x-1}{2} \le 1$.

Multiplying through by 2 we obtain $-2 \le x - 1 \le 2$.

So $-2 + 1 \le x \le 2 + 1$, i.e. $-1 \le x \le 3$.

Question 5 E

$$2\cos(2x) = 2(1 - 2\sin^2(x))$$
$$= 2 - 4\sin^2(x)$$

Hence $2 - 4\sin^2(x) - \sin(x) - 1 = 0$.

Multiplying both sides by -1 , we obtain $4\sin^2(x) + \sin(x) - 1 = 0$.

Question 6 A

To form *OC* from *OA*, we rotate *OA* anticlockwise through an angle of $\frac{\pi}{2}$ and then double the length of *OA*.

Anticlockwise rotation through $\frac{\pi}{2}$ is equivalent to multiplying by *i*.

Therefore the complex number that corresponds to vertex *C* is equal to 2*iu*.

Question 7 E

Substituting $z = x + yi$ into $\text{Re}\left(\frac{z-4}{z}\right) = 0$ gives $\frac{x^2 - 4x + y^2}{x^2 + y^2} = 0$.

Given that $x^2 + y^2$ forms the denominator, $x \ne 0$ and $y \ne 0$ simultaneously, and so (0, 0) is excluded.

By completing the square, $x^2 - 4x + y^2 = 0$ can be written as $(x-2)^2 + y^2 = 4$, i.e. a circle with centre (2, 0) with radius 2.

Question 8 E

Note that $z = 0$ is a solution.

For $z \ne 0$, $|z|^{n-1} = |z|$, i.e. $|z| = 1$.

$z^{n-1} = i\bar{z}$ is equivalent to $z^n = (i\bar{z})z$.

As $z\bar{z} = 1$, the equation reduces to $z^n = i$.

For $|z| = 1$, this equation has *n* solutions and for $z = 0$, this equation has 1 solution.

Therefore the total number of solutions is $n + 1$.

Question 9 A

The required (shaded) region is the intersection of a circle and a region bounded by two rays.

$|z - i| \le 2$ describes the set of points inside or on a circle with centre $(0, 1)$ and radius 2.

$\text{Arg}(z + 1) = 0$ describes a horizontal ray starting at $z = -1$ but excluding $z = -1$.

The cartesian equation of this horizontal ray is $y = 0, x > -1$.

$\text{Arg}(z + 1) = \frac{\pi}{4}$ describes a ray also starting at $z = -1$ but also excluding $z = -1$.

The cartesian equation of this second ray is $y = x + 1, x > -1$.

Hence, the shaded region of the complex plane is $\{z : |z - i| \le 2\} \cap \left\{0 \le \text{Arg}(z + 1) \le \frac{\pi}{4}\right\}$.

Question 10 A

All gradients on the line $y = -x$ appear to be zero.

Question 11 D

Let $u = x - 2$ and so $\frac{du}{dx} = 1$.

If $u = x - 2$, then $u + 3 = x + 1$.

When $x = 6$, $u = 4$ and when $x = 2, u = 0$.

$$\int_2^6 (x + 1)\sqrt{x - 2}\,dx = \int_0^4 (u + 3)\sqrt{u}\,du$$

$$= \int_0^4 \left(u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\right) du$$

Question 12 C

If the graph of *f* touches the *x*-axis at $x = a$, i.e. $f(x) = 0$, then the graph of *F* has a stationary point of inflexion at $x = a$.

Hence **C** is not true.

Question 13 C

We are given $(x_0, y_0) = (0, 1)$ and $h = 0.25$.

$y_1 = 1 + 0.25 \times \cos^2(0) = 1.25$

$y_2 = 1.25 + 0.25 \times \cos^2(0.25) = 1.4847\ldots$

$y_3 = 1.4847\ldots + 0.25 \times \cos^2(0.5) = 1.6772\ldots$

So the value of *y*, correct to two decimal places, when $x = 0.75$, is 1.68.

Question 14 B

If $\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$, then $\frac{dy}{dx} = \int \frac{1}{(x+1)^2} dx$.

So $\frac{dy}{dx} = -\frac{1}{x+1} + c$, where c is an arbitrary constant.

When $x = 0$, $\frac{dy}{dx} = 0$ and so $c = 1$.

Hence $\frac{dy}{dx} = 1 - \frac{1}{x+1}$.

If $\frac{dy}{dx} = 1 - \frac{1}{x+1}$, then $y = \int \left(1 - \frac{1}{x+1}\right) dx$.

So $y = x - \log_e|x+1| + d$, where d in an arbitrary constant.

When $x = 0, y = 0$ and so $d = 0$.

So $y = x - \log_e|x+1|$.

When $x = 2, y = 2 - \log_e(3)$.

Question 15 B

Let $\hat{\underset{\sim}{u}} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ be a unit vector perpendicular to both vectors.

From $(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}) \cdot (2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}) = 0$, we obtain $2x - y + z = 0$. (1)

From $(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}) \cdot (\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}) = 0$, we obtain $x - y + z = 0$. (2)

From $|\hat{\underset{\sim}{u}}| = 1$, we obtain $x^2 + y^2 + z^2 = 1$. (3)

(1) – (2) gives $x = 0$ and so $y = z$.

Substituting $x = 0$, $y = z$ into (3) and solving for y gives $y = \pm\frac{1}{\sqrt{2}}$.

Hence $z = \pm\frac{1}{\sqrt{2}}$.

So $\hat{\underset{\sim}{u}} = -\frac{1}{\sqrt{2}}(\underset{\sim}{j} + \underset{\sim}{k})$.

Question 16 D

The scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is given by $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$.

$$\begin{aligned}\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} &= \frac{1}{5}(3\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}) \cdot (4\underset{\sim}{i} - 3\underset{\sim}{k}) \\ &= \frac{1}{5}(12 + 9) \\ &= \frac{21}{5}\end{aligned}$$

Question 17 A

$\underset{\sim}{r}(t) = \int ((4 - 2\cos(t))\underset{\sim}{i} - 3\sin(t)\underset{\sim}{j})dt$

$= (4t - 2\sin(t))\underset{\sim}{i} + 3\cos(t)\underset{\sim}{j} + \underset{\sim}{d}$

When $t = 0$, $\underset{\sim}{r} = 2\underset{\sim}{j}$ and so $\underset{\sim}{d} = -\underset{\sim}{j}$.

Hence $\underset{\sim}{r}(t) = (4t - 2\sin(t))\underset{\sim}{i} + (3\cos(t) - 1)\underset{\sim}{j}$.

So $\underset{\sim}{r}\left(\frac{3\pi}{2}\right) = 3(2\pi + 1)\underset{\sim}{i} - \underset{\sim}{j}$.

Question 18 A

Let the resultant force have components F_1 parallel to *AB* and F_2 perpendicular to *AB*.

Parallel to *AB*: $F_1 = 3 + 4\cos(30°) - 6\cos(60°)$, i.e. $F_1 = 4\cos(30°)$

Perpendicular to *AB*: $F_2 = 4\sin(30°) - 2 + 6\sin(60°)$, i.e. $F_2 = 6\sin(60°)$

Hence the magnitude of the resultant force is $\sqrt{16\cos^2(30°) + 36\sin^2(60°)}$.

Question 19 C

Using $R = ma$ we obtain $a = 2\pi\sin\left(\frac{\pi t}{3}\right)$.

$v = 2\pi\int \sin\left(\frac{\pi t}{3}\right)dt$

$= -6\cos\left(\frac{\pi t}{3}\right) + c$ where c is an arbitrary constant

When $t = 0$, $v = 0$ and so $c = 6$.

Hence $v = 6\left(1 - \cos\left(\frac{\pi t}{3}\right)\right)$.

After 3 seconds, the particle's velocity is 12 m/s.

Question 20 E

Specifying the equation of motion for each particle:

4 kg particle: $T - 4g = 4a$ (1)

m kg particle: $mg - T = ma$ (2)

Solving (1) and (2) simultaneously we obtain $T = \frac{8mg}{m+4}$ and $a = \frac{(m-4)g}{m+4}$.

Solving $T = \frac{8mg}{m+4}$ for m with $T = 5.5g$ gives $m = 8.8$ kg.

Alternative 2: Treating the string as a line.

We have a force mg newtons in one direction (positive direction) and $4g$ newtons in the other direction.

The equation of motion is $mg - 4g = (m+4)a$ and so $a = \frac{(m-4)g}{m+4}$.

Substituting $a = \frac{(m-4)g}{m+4}$ into $T - 4g = 4a$ and solving for m with $T = 5.5g$ gives $m = 8.8$ kg.

Question 21 B

The initial momentum (p_i) is 10×12, i.e. 120 kg m/s.

To calculate the final momentum (p_f), we need to find the particle's final velocity.

Given $u = 12$, $s = 80$ and $t = 5$ we can find v using $s = \frac{1}{2}(u+v)t$.
Solving $80 = \frac{5}{2}(12+v)$ for v we obtain $v = 20$.

The final momentum (p_f) is 10×20, i.e. 200 kg m/s.

Change in momentum $(\Delta p) = p_f - p_i = 80$ kg m/s

Question 22 D

The distance travelled by a bicycle is represented by the area under a velocity–time graph.

Let d_A be the distance travelled by bicycle *A*, and d_B be the distance travelled by bicycle *B*.

Bicycle *A*: $d_A = \frac{1}{2} \times 3 \times \frac{3V}{2} + \frac{3V}{2} \times (t-3)$.
So $d_A = \frac{9V}{4} + \frac{3V}{2}(t-3)$.

Bicycle B: $d_B = Vt$

Each bicycle will again share the same position when $d_A = d_B$.

Solving $\frac{9V}{4} + \frac{3V}{2}(t-3) = Vt$ for t $(V \neq 0)$ gives $t = 4.5$ (mins).

SECTION 2

Question 1

a. For particle A, the parametric equations are $x = 4t$ and $y = 2t$.

Substituting $t = \frac{x}{4}$ into $y = 2t$ gives $y = \frac{x}{2}$. A1

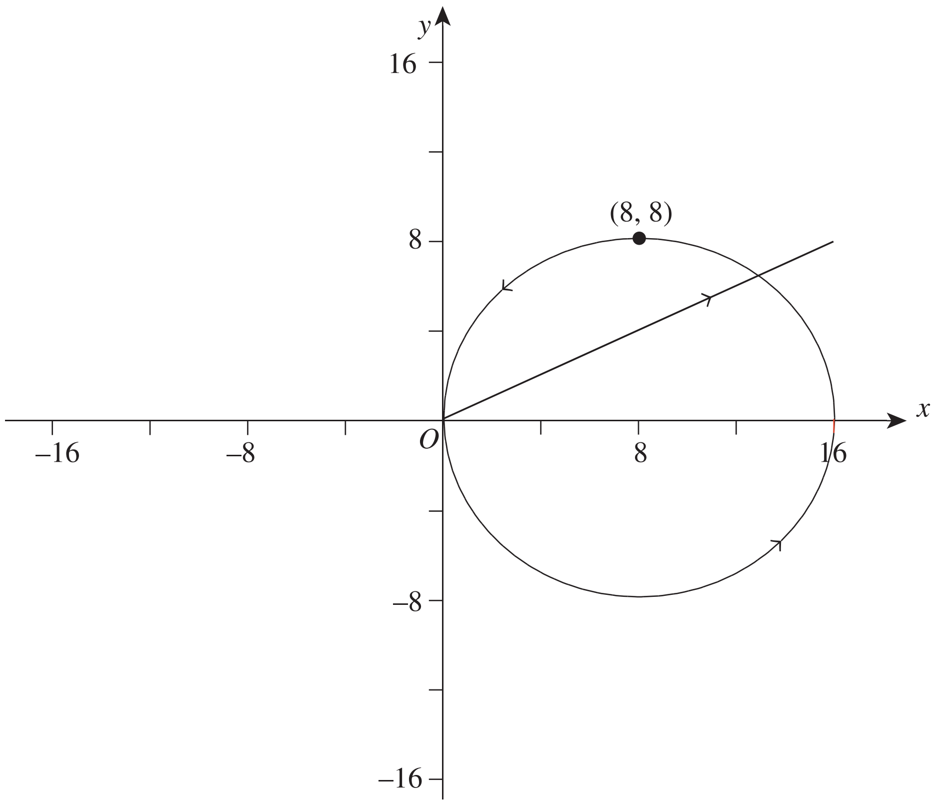
b. For particle B, the parametric equations are $x = 8 - 8\sin(nt)$ and $y = 8\cos(nt)$.

Rearranging both equations, we obtain $\sin(nt) = \frac{8 - x}{8}$ and $\cos(nt) = \frac{y}{8}$. M1

Since $\sin^2(nt) + \cos^2(nt) = 1$, we obtain $\left(\frac{8 - x}{8}\right)^2 + \frac{y^2}{64} = 1$.

Multiplying both sides by 64, we obtain $(x - 8)^2 + y^2 = 64$. A1

c.



Two correct graph shapes, i.e. $y = \frac{x}{2}$ for $x \geq 0$ and $(x - 8)^2 + y^2 = 64$ for $0 \leq x \leq 16$. A1

Particle A: as t increases, both x and y increase. A1

Particle B: at $t = 0$, particle B is at $(8, 8)$. Substituting a value of t just greater than zero, results in $x < 8$, i.e. x decreases. Hence particle B travels in an anti-clockwise direction. A1

d. Solving $y = \frac{x}{2}$ and $(x-8)^2 + y^2 = 64$ for *x* and *y*, we obtain $x = 0$ and $y = 0$ or $x = \frac{64}{5}$ and $y = \frac{32}{5}$. M1

So the paths of *A* and *B* meet at $(0, 0)$ and $\left(\frac{64}{5}, \frac{32}{5}\right)$. A1

e. Attempting to solve $8 - 8\sin(nt) = 4t$ and $8\cos(nt) = 2t$ simultaneously for *n* when $t = \frac{16}{5}$. M1

Solving $8 - 8\sin(nt) = \frac{64}{5}$ for *n* when $t = \frac{16}{5}$ gives $n = 1.1828\ldots, 1.7624\ldots, \ldots$ A1

Solving $8\cos(nt) = \frac{32}{5}$ for *n* when $t = \frac{16}{5}$ gives $n = 0.2010\ldots, 1.7624\ldots, \ldots$ A1

So the smallest value of *n* for which particle *A* and particle *B* will collide is 1.76, correct to two decimal places. A1

f. We need to show that $\underset{\sim}{a}_B(t) \cdot \underset{\sim}{v}_B(t) = 0$.

$\underset{\sim}{v}_B(t) = -8n\cos(nt)\underset{\sim}{i} - 8n\sin(nt)\underset{\sim}{j}$ and $\underset{\sim}{a}_B(t) = 8n^2\sin(nt)\underset{\sim}{i} - 8n^2\cos(nt)\underset{\sim}{j}$ A1

$$\begin{aligned}\underset{\sim}{a}_B(t) \cdot \underset{\sim}{v}_B(t) &= (8n^2\sin(nt)\underset{\sim}{i} - 8n^2\cos(nt)\underset{\sim}{j}) \cdot (-8n\cos(nt)\underset{\sim}{i} - 8n\sin(nt)\underset{\sim}{j}) \\ &= -64n^3\sin(nt)\cos(nt) + 64n^3\sin(nt)\cos(nt) \\ &= 0\end{aligned}$$ M1

As $\underset{\sim}{a}_B(t) \cdot \underset{\sim}{v}_B(t) = 0$ and $\underset{\sim}{a}_B(t), \underset{\sim}{v}_B(t) \neq 0$, the acceleration is always perpendicular to the velocity. A1

Question 2

a. $N = 8g\cos(20°)$ M1

So $N = 73.7$ (newtons) (correct to one decimal place). A1

b. Using $F = \mu N$ we obtain $F = 0.3 \times 73.7\ldots$

Hence $F = 22.1$ (newtons) (correct to one decimal place). A1

c. $T = 8g\sin(20°) - 0.3 \times 8g\cos(20°)$ M1 A1

So $T = 4.7$ (newtons) (correct to one decimal place). A1

d. $8a = 8g\sin(20°) - 0.3 \times 8g\cos(20°)$ M1 A1

So $a = 0.6$ (m/s^2) (correct to one decimal place). A1

Question 3

a. $z\text{cis}(\theta)$ is a rotation by θ anticlockwise about the origin of the point represented by *z*. A1

b. Solving $z^3 = -8i$ for *z*, gives $t = 2i$, $u = -\sqrt{3} - i$ and $v = \sqrt{3} - i$. M1 A1

c. One approach to show that *TUV* is an equilateral triangle is to use the result:

$|t - u| = |u - v| = |v - t|$

$|t - u| = |u - v| = |v - t| = 2\sqrt{3}$ A1

Hence *TUV* is an equilateral triangle.

d. $k = 2$ A1

e. $\bar{v} = \sqrt{3} + i$ A1

$|\bar{v}| = 2$ and so $\bar{v} \in S$. A1

f. Expanding $(z - a)(\bar{z} - a)$ we obtain $z\bar{z} - (z + \bar{z})a + a^2$. M1

Given that $z = x + yi$ and $\bar{z} = x - yi$, the LHS becomes $x^2 + y^2 - 2ax + a^2$. A1

Comparing $x^2 + y^2 - 2ax + a^2 = b$ to $(x - a)^2 + y^2 = 4$, we obtain $a = 0$ and $b = 4$. A1

Question 4

a. When $x = 4, y = 0$ and when $x = 8, y = 12$.

So the height of the bowl is 12 cm. A1

b. Using $V = \pi \int_0^h x^2 dy$, where $x^2 = 4y + 16$, we obtain $V = 4\pi \int_0^h (y + 4)dy$ (or equivalent). A1

$$V = 4\pi \int_0^h (y + 4)dy$$

$$= 4\pi \left[\frac{y^2}{2} + 4y\right]_0^h$$

So $V = 4\pi\left(\frac{h^2}{2} + 4h\right)$. A1

c. Solving $4\pi\left(\frac{h^2}{2} + 4h\right) = 120\pi$ for *h*, we obtain $h = 2(\sqrt{19} - 2)$ (cm), since $0 \le h \le 12$. M1 A1

So the exact depth is $2(\sqrt{19} - 2)$ cm when the bowl is exactly one-quarter full.

d. $\frac{dV}{dh} = 4\pi(h + 4)$ A1

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$= \frac{50}{4\pi(h + 4)}$$ M1

When $h = 2(\sqrt{19} - 2)$, $\frac{dh}{dt} = \frac{25\sqrt{19}}{76\pi}$ (cm per second). A1

So the height of the water is increasing at $\frac{25\sqrt{19}}{76\pi}$ cm per second when the bowl is exactly one-quarter full.

e. $\frac{dT}{dt} = -k(T - 20)$ A1

Either using CAS or a by-hand approach to solve the differential equation, we obtain

$T = Ae^{-kt} + 20$, where A is an arbitrary constant. M1

When $t = 0, T = 70$ and so $A = 50$.

When $t = 10, T = 55$ and so solving $55 = 50e^{-10k} + 20$, for k gives:

$k = -\frac{1}{10}\log_e\left(\frac{7}{10}\right)$ (or equivalent) M1 A1

Evaluating $T = 50e^{-25k} + 20$ where $k = -\frac{1}{10}\log_e\left(\frac{7}{10}\right)$, gives 40.5°C (correct to one decimal place). A1

Question 5

a. The rope always points in the direction of the speedboat and is always tangent to the curve. The gradient of the tangent, $\frac{dy}{dx}$ is given by $\frac{dy}{dx} = \frac{20t - y}{0 - x}$, i.e. $\frac{dy}{dx} = \frac{y - 20t}{x}$. A1

b. From Pythagoras' theorem, we obtain $x^2 + (20t - y)^2 = 100$. A1

Solving this equation for $20t - y$ we obtain $20t - y = \sqrt{100 - x^2}$ and substituting this into $\frac{dy}{dx} = -\frac{20t - y}{x}$, we obtain $\frac{dy}{dx} = \frac{-\sqrt{100 - x^2}}{x}$. A1

c. Let $u^2 = 100 - x^2$ and so $u = \sqrt{100 - x^2}$.

Now $2udu = -2xdx$ and so $dx = \frac{-2u}{2x}du$. M1

Hence $dx = \frac{-u}{\sqrt{100 - u^2}}du$. A1

Making the required substitutions we obtain $-\int\left(\frac{-u}{\sqrt{100 - u^2}}\right)\left(\frac{u}{\sqrt{100 - u^2}}\right)du = \int\left(\frac{u^2}{100 - u^2}\right)du$. A1

d. By division we find that $\int\left(\frac{u^2}{100 - u^2}\right)du = \int\left(-1 + \frac{100}{100 - u^2}\right)du$.

Hence $m = -1$. A1

Using partial fractions, we can express $\frac{100}{100 - u^2}$ as $\frac{A}{10 - u} + \frac{B}{10 + u}$.

So $A(10 + u) + B(10 - u) = 100$. M1

When $u = 10, A = 5$ and when $u = -10, B = 5$.

So $\int\left(\frac{u^2}{100 - u^2}\right)du = \int\left(-1 + \frac{5}{10 - u} + \frac{5}{10 + u}\right)du$, i.e. $n = 5$. A1

e. $$\int\left(-1+\frac{5}{10-u}+\frac{5}{10+u}\right)du = -u - 5\log_e(10-u) + 5\log_e(10+u) + c$$

$$= -u + 5\log_e\left(\frac{10+u}{10-u}\right) + c \quad (c \text{ is an arbitrary constant})$$ A1

When $x = 10$, $y = 0$ and so $c = 0$.

Hence $y = -\sqrt{100-x^2} + 5\log_e\left(\frac{10+\sqrt{100-x^2}}{10-\sqrt{100-x^2}}\right)$. A1

