SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2

2011 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer : *A*

Explanation :

$centre = (-1,1) \quad , \quad a = (-1) - (-3) = 2$

$$\frac{b}{a} = 2 \quad \therefore \quad b = 4 \quad \therefore \quad \frac{(x+1)^2}{4} - \frac{(y-1)^2}{16} = 1$$

Question 2

Answer : *B*

Explanation :

$y = \sin(2\cos^{-1}(2x))$

$$-\frac{\pi}{2} \le 2\cos^{-1}(2x) \le \frac{\pi}{2} \quad \therefore \quad -\frac{\pi}{4} \le \cos^{-1}(2x) \le \frac{\pi}{4}$$

$$But \quad 0 \le \cos^{-1}(2x) \le \pi \quad \therefore \quad 0 \le \cos^{-1}(2x) \le \frac{\pi}{4} \quad \therefore \quad \frac{\sqrt{2}}{2} \le 2x \le 1$$

$$\therefore \quad \frac{\sqrt{2}}{4} \le x \le \frac{1}{2} \quad \therefore \quad dom = \left[\frac{\sqrt{2}}{4}, \frac{1}{2}\right]$$

Question 3

$Answer: B$

$Explanation:$

$y = \sec\left(a(b-x)\right) = \sec\left(a(x-b)\right)$

dilation of factor 2 from the y axis, hence $a = \frac{1}{2}$

translation $\frac{\pi}{2}$ to the right, hence $\mathrm{b} = \frac{\pi}{2}$

Question 4

$Answer: B$

$Explanation:$

The rays $Arg(z-i) = \frac{\pi}{4}$ and $Arg(z-2) = \frac{\pi}{2}$ intersect at $z = 2+3i$

$Arg(z+1) = Arg(3+3i) = \frac{\pi}{4}$

Question 5

$Answer: D$

$Explanation:$

$1+i \quad , \quad 1-i \quad and \quad 2$

$Q(z) = P(i\,\overline{z})$

$i\,\overline{z} = 1+i \quad , \quad 1-i \quad and \quad 2$

$\overline{z} = \frac{1+i}{i} \quad , \quad \frac{1-i}{i} \quad and \quad \frac{2}{i}$

$\overline{z} = 1-i \quad , \quad -1-i \quad and \quad -2i$

$z = 1+i \quad , \quad -1+i \quad and \quad 2i$

Question 6

Explanation:

$w=\frac{2\overline{z}^2(1+i)}{z^2}$, $Arg(\overline{z})=\frac{\pi}{16}$ $\therefore$ $Arg(z)=-\frac{\pi}{16}$

$$Arg(w)=2\times\frac{\pi}{16}+\frac{\pi}{4}-2\times\left(-\frac{\pi}{16}\right)=\frac{\pi}{8}+\frac{\pi}{4}+\frac{\pi}{8}=\frac{\pi}{2}$$

Question 7

Answer: *D*
Explanation:

$$T=\{z:|z+1|\geq|z-i|,\ z\in C\}\cap\{z:|z-2|\leq|z-i|,\ z\in C\}\cap\{z:|z|\leq 1,\ z\in C\}$$

$|z+1|\geq|z-i|$ is the half plane above the left line

$|z-2|\leq|z-i|$ is the half plane below the right line

$|z|\leq 1$ is the interior of the circle

Question 8

Answer: *A*
Explanation:
When $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent, one of them, for example $\underset{\sim}{c}$, is a linear combination of the other two.
Therefore $\underset{\sim}{c}$ is in the plane determined by $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Question 9

Answer : *C*

Explanation :

$\underset{\sim}{b}\cdot\underset{\sim}{c}=|\underset{\sim}{b}|^2+\underset{\sim}{a}\cdot\underset{\sim}{b} \quad \therefore \quad \underset{\sim}{b}\cdot\underset{\sim}{c}-\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{b}|^2 \quad \therefore \quad \underset{\sim}{b}\cdot(\underset{\sim}{c}-\underset{\sim}{a})=|\underset{\sim}{b}|^2$ true

$\frac{\underset{\sim}{a}\cdot\underset{\sim}{c}}{|\underset{\sim}{c}|^2}\underset{\sim}{c}+\frac{\underset{\sim}{b}\cdot\underset{\sim}{c}}{|\underset{\sim}{c}|^2}\underset{\sim}{c}=\underset{\sim}{c} \quad \therefore \quad \frac{\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{c}}{|\underset{\sim}{c}|^2}\underset{\sim}{c}=\underset{\sim}{c}$ true

$\underset{\sim}{a}-\frac{\underset{\sim}{a}\cdot\underset{\sim}{c}}{|\underset{\sim}{c}|^2}\underset{\sim}{c}=\left(\underset{\sim}{b}-\frac{\underset{\sim}{b}\cdot\underset{\sim}{c}}{|\underset{\sim}{c}|^2}\underset{\sim}{c}\right)$ false

the vector resolutes of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in the direction $\perp$ to $\underset{\sim}{c}$ are opposite not equal.

$\frac{\underset{\sim}{c}\cdot\underset{\sim}{a}}{|\underset{\sim}{c}|^2}\underset{\sim}{c}+\frac{\underset{\sim}{c}\cdot\underset{\sim}{b}}{|\underset{\sim}{c}|^2}\underset{\sim}{c}=\underset{\sim}{c} \quad \therefore \quad \frac{\underset{\sim}{c}\cdot(\underset{\sim}{a}+\underset{\sim}{b})}{|\underset{\sim}{c}|^2}\underset{\sim}{c}=\underset{\sim}{c} \quad \therefore \quad \frac{\underset{\sim}{c}\cdot\underset{\sim}{c}}{|\underset{\sim}{c}|^2}\underset{\sim}{c}=\underset{\sim}{c}$ true

$\underset{\sim}{c}\cdot\underset{\sim}{a}=|\underset{\sim}{c}|^2-\underset{\sim}{b}\cdot\underset{\sim}{c} \quad \therefore \quad \underset{\sim}{c}\cdot(\underset{\sim}{a}+\underset{\sim}{b})=|\underset{\sim}{c}|^2$ true

Question 10

Answer : *B*

Explanation :

$\underset{\sim}{a}=\underset{\sim}{i}+\sqrt{3}\underset{\sim}{j} \quad , \quad \underset{\sim}{b}=3\underset{\sim}{k}$

$\hat{\underset{\sim}{a}}=\frac{1}{2}\left(\underset{\sim}{i}+\sqrt{3}\underset{\sim}{j}\right) \quad , \quad \hat{\underset{\sim}{b}}=\underset{\sim}{k}$

$\hat{\underset{\sim}{a}}+\hat{\underset{\sim}{b}}=\frac{1}{2}\left(\underset{\sim}{i}+\sqrt{3}\underset{\sim}{j}\right)+\underset{\sim}{k}$

$\left|\hat{\underset{\sim}{a}}+\hat{\underset{\sim}{b}}\right|=\sqrt{2}$

Hence the required vector is $\frac{1}{2\sqrt{2}}\left(\underset{\sim}{i}+\sqrt{3}\underset{\sim}{j}\right)+\frac{1}{\sqrt{2}}\underset{\sim}{k}$

Question 11

Answer : *B*
Explanation :

$$\frac{1-\sin(2\theta)}{\cos(2\theta)}=\tan\left(\frac{\pi}{4}-\theta\right) \quad \text{Change } \theta \text{ to } \theta-\frac{\pi}{4}$$

$$\frac{1-\sin\left(2\theta-\frac{\pi}{2}\right)}{\cos\left(2\theta-\frac{\pi}{2}\right)}=\tan\left(\frac{\pi}{4}-\theta+\frac{\pi}{4}\right) \quad \therefore \quad \frac{1+\cos(2\theta)}{\sin(2\theta)}=\cot(\theta)$$

Question 12

Answer : *E*
Explanation :
$\underset{\sim}{r}(t)=\tan(t)\underset{\sim}{i}+\sec(t)\underset{\sim}{j}$
$x=\tan(t) \quad y=\sec(t)$
$\sec^2(t)-\tan^2(t)=1 \quad \therefore \quad y^2-x^2=1 \qquad \therefore \quad hyperbola$

Question 13

Answer : *D*
Explanation :
$v^2=-4x$
$a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\frac{d}{dx}\left(\frac{1}{2}\times(-4x)\right)=-2 \quad \therefore \quad a=const$
Initially $v=8$. Since $a<0 \quad v$ decreases

Question 14

Answer : *D*
Explanation :

$\int_0^1 x^3 e^{x^2}\,dx \qquad u = x^2 \quad \therefore \quad du = 2x\,dx$

$\int_0^1 x^3 e^{x^2}\,dx = \int_0^1 x^2 x\, e^{x^2}\,dx = \int_0^1 u\, e^u \frac{du}{2} = \frac{1}{2}\int_0^1 u\, e^u\, du$

Question 15

Answer : *A*
Explanation :

$\underset{\sim}{r}(t) = -\sin(2\pi t)\underset{\sim}{i} + \cos(\pi t)\underset{\sim}{j} \quad \therefore \quad \underset{\sim}{v}(t) = -2\pi\cos(2\pi t)\underset{\sim}{i} - \pi\sin(\pi t)\underset{\sim}{j}$

$\underset{\sim}{v}\left(\frac{3}{2}\right) = -2\pi\cos\left(2\pi \times \frac{3}{2}\right)\underset{\sim}{i} - \pi\sin\left(\pi \times \frac{3}{2}\right)\underset{\sim}{j}$

$\underset{\sim}{v}\left(\frac{3}{2}\right) = 2\pi\underset{\sim}{i} + \pi\underset{\sim}{j} \quad \therefore \quad \left|\underset{\sim}{v}\left(\frac{3}{2}\right)\right| = \pi\sqrt{5}$

Question 16

Answer : *D*
Explanation :
For $y = const$ the gradient follows a sin or cos shape
when $x = y$, gradient is zero, hence sin shape
when $x = 1$ and $y = 0$, gradient is positive,
hence $\sin(x - y)$ applies

Question 17

Answer : *B*

Explanation :

$d = \frac{1}{2}at^2 \quad , \quad t = 1 \quad \therefore \quad d = \frac{1}{2}a$

$d_n = \frac{1}{2}an^2 - \frac{1}{2}a(n-1)^2 = \frac{1}{2}a\left(n^2 - (n-1)^2\right) \quad \therefore \quad d_n = d(2n-1)$

Question 18

Answer : *C*

Explanation :

$mg - mkv = ma \quad \therefore \quad a = g - kv$

$\frac{g}{2} = g - kv \quad \therefore \quad v = \frac{g}{2k}$

$\frac{dv}{dt} = g - kv \quad \therefore \quad t = \int_0^{\frac{g}{2k}} \frac{dv}{g - kv} = \frac{1}{k}\log_e 2$

Question 19

Answer : *E*

Explanation :

$\left.\begin{aligned} 2mg - T &= 2ma \\ T - mg &= ma \end{aligned}\right\}$

$\Rightarrow mg = 3ma \quad \therefore \quad a = \frac{g}{3}$

$T - mg = m\frac{g}{3} \quad \therefore \quad T = \frac{4}{3}mg$

Question 20

Answer : *C*
Explanation :

$$\left.\begin{array}{l} F-\mu(m+M)g-\mu mg=Ma \\ \mu mg=ma \end{array}\right\}\Rightarrow F-\mu(m+M)g-\mu mg=\mu mg \quad \therefore \quad F=2\mu(M+m)g$$

Question 21

Answer : *A*
Explanation :

$$Q^2=R^2+P^2$$

$$if \quad \theta=45^0 \quad then \quad R=P=\frac{Q}{\sqrt{2}}$$

Question 22

Answer : *A*
Explanation :

$$\underset{\sim}{p}=m\underset{\sim}{v} \quad , \quad \underset{\sim}{F}=\frac{d\underset{\sim}{p}}{dt}=\frac{d}{dt}\left(m\underset{\sim}{v}\right) \quad \therefore \quad \underset{\sim}{F}=m\frac{d\underset{\sim}{v}}{dt}+\underset{\sim}{v}\frac{dm}{dt}$$

$$\underset{\sim}{a}=\frac{d\underset{\sim}{v}}{dt} \quad \therefore \quad \underset{\sim}{F}=m\underset{\sim}{a}+\underset{\sim}{v}\frac{dm}{dt}$$

SECTION 2

Question 1

a.

$$\left.\begin{array}{l} z+\dfrac{1}{z}=-2\sin\theta \quad , \quad -\dfrac{\pi}{4}\le\theta\le 0 \quad \therefore \quad z^2+2z\sin\theta+1=0 \\ z=-\sin\theta\pm\sqrt{\sin^2\theta-1} \quad \therefore \quad z=-\sin\theta\pm i\cos\theta \end{array}\right\} \quad ....[M1]$$

$$\left.\begin{array}{l} z=-\sin\theta+i\cos\theta \quad \text{or} \quad z=-\sin\theta-i\cos\theta \\ \text{But } \cos\left(\dfrac{\pi}{2}+\theta\right)=-\sin\theta \text{ and } \sin\left(\dfrac{\pi}{2}+\theta\right)=\cos\theta \end{array}\right\} \quad ....[M1]$$

$$\therefore \quad u=cis\left(\frac{\pi}{2}+\theta\right) \quad \text{as} \quad \frac{\pi}{2}+\theta>0 \quad \therefore \quad w=\overline{u} \qquad ....[M1]$$

$$\therefore \quad u=cis\left(\frac{\pi}{2}+\theta\right) \quad \text{and} \quad w=cis\left(-\frac{\pi}{2}-\theta\right) \quad \text{as required} \quad ....[A1]$$

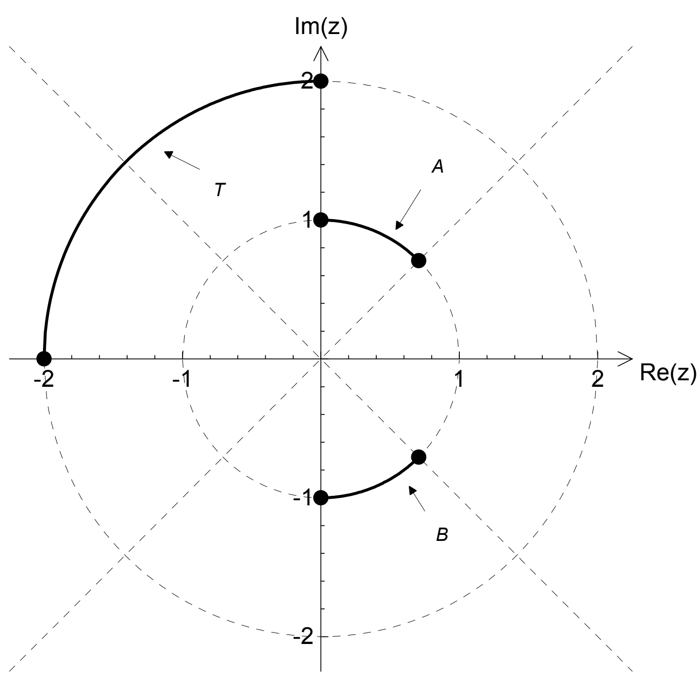
b.

See the diagram to the left

region A ....[A1]

region B ....[A1]

region T ....[A1]



c.

$|z_1 - z_2|_{\min} = |2i - i| = 1$ ....[A1]

$|z_1 - z_2|_{\max} = \left|cis\left(\frac{\pi}{4}\right) + 2\right| = \sqrt{\left(\cos\left(\frac{\pi}{4}\right) + 2\right)^2 + \sin^2\left(\frac{\pi}{4}\right)} = \sqrt{2\sqrt{2} + 5}$ ....[A1]

d.

i.

$u^n - w^n = u^n - (\overline{u})^n = 2i\,\mathrm{Im}(u^n)$ ....[M1]

$u^n - w^n = 2i\sin\left(n\left(\frac{\pi}{2} + \theta\right)\right) = 2i\sin\left(\frac{n\pi}{2} + n\theta\right)$ as required ....[A1]

ii.

$u^4 - w^4 = 2i\sin\left(\frac{4\pi}{2} + 4 \times \frac{\pi}{4}\right) = 2i\sin(3\pi) = 0$........[A1]

Question 2

a.

To show that $MNPQ$ is a parallelogram, it is enough to show that $\overrightarrow{MN} = \overrightarrow{QP}$

$\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$ But $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DC} = -\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ .....[M1]

$\overrightarrow{MN} = \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}(-\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) \quad \therefore \quad \overrightarrow{MN} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c})$

$\overrightarrow{QP} = \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c}) \quad \therefore \quad \overrightarrow{MN} = \overrightarrow{QP}$ ......[A1]

b.

i.

$\overrightarrow{\mathrm{AC}} = \overrightarrow{\mathrm{AD}} + \overrightarrow{\mathrm{DC}} \quad \therefore \quad \overrightarrow{\mathrm{AC}} = 5\underset{\sim}{i} + 3\underset{\sim}{j}$

$\overrightarrow{DB} = \overrightarrow{\mathrm{DA}} + \overrightarrow{\mathrm{AB}} \quad \therefore \quad \overrightarrow{DB} = -(2\underset{\sim}{i} + 3\underset{\sim}{j}) + 8\underset{\sim}{i} \quad \therefore \quad \overrightarrow{DB} = 6\underset{\sim}{i} - 3\underset{\sim}{j}$ ......[A1]

ii.

$$\left.\begin{array}{l}\overrightarrow{AC} = 5\underset{\sim}{i} + 3\underset{\sim}{j} \quad \overrightarrow{DB} = 6\underset{\sim}{i} - 3\underset{\sim}{j} \\ \overrightarrow{AC} \cdot \overrightarrow{DB} = \left|\overrightarrow{AC}\right| \times \left|\overrightarrow{DB}\right| \times \cos\theta \\ 5\times 6 - 3\times 3 = \sqrt{5^2+3^2} \times \sqrt{6^2+3^2} \times \cos\theta\end{array}\right\} \ldots\ldots[\text{M1}]$$

$$\cos\theta = \frac{21}{\sqrt{34}\times\sqrt{45}} \quad \therefore \quad \theta = 57.53^0 \approx 58^0 \quad \ldots\ldots[\text{A1}]$$

c.

i.

$$\left.\begin{array}{l}\overrightarrow{AC} = 5\underset{\sim}{i} + 3\underset{\sim}{j} \quad \overrightarrow{DB} = 6\underset{\sim}{i} - 3\underset{\sim}{j} \quad , \ \overrightarrow{AE} = m\overrightarrow{AC} \text{ and } \overrightarrow{DE} = n\overrightarrow{DB} \\ \overrightarrow{AE} + \overrightarrow{EB} = \overrightarrow{AB} \quad \therefore \quad m\overrightarrow{AC} + (1-n)\overrightarrow{DB} = 8\underset{\sim}{i}\end{array}\right\} \ldots\ldots[\text{M1}]$$

$$m(5\underset{\sim}{i} + 3\underset{\sim}{j}) + (1-n)(6\underset{\sim}{i} - 3\underset{\sim}{j}) = 8\underset{\sim}{i} \quad \therefore \quad (5m - 6n + 6)\underset{\sim}{i} + (3m + 3n - 3)\underset{\sim}{j} = 8\underset{\sim}{i} \quad \ldots\ldots[\text{M1}]$$

$$\left.\begin{array}{l}5m - 6n + 6 = 8 \\ 3m + 3n - 3 = 0\end{array}\right\} \Rightarrow \left.\begin{array}{l}5m - 6n + 6 = 8 \\ 6m + 6n - 6 = 0\end{array}\right\} \quad \therefore \quad 11m = 8 \quad \therefore \quad m = \frac{8}{11} \quad \ldots\ldots[\text{A1}]$$

$$n = 1 - m \quad \therefore \quad n = \frac{3}{11} \quad \ldots\ldots[\text{A1}]$$

ii.

$$\left.\begin{array}{l}\overrightarrow{AC} = 5\underset{\sim}{i} + 3\underset{\sim}{j} \\ \overrightarrow{AE} = m\overrightarrow{AC} \quad \therefore \quad \overrightarrow{AE} = \frac{8}{11}\left(5\underset{\sim}{i} + 3\underset{\sim}{j}\right) = \frac{40}{11}\underset{\sim}{i} + \frac{24}{11}\underset{\sim}{j}\end{array}\right\} \ldots\ldots[\text{M1}]$$

$$A_{\Delta ABE} = \frac{1}{2} \times 8 \times \frac{24}{11} = \frac{96}{11}\ sq.\,units \quad \ldots\ldots[\text{A1}]$$

Question 3

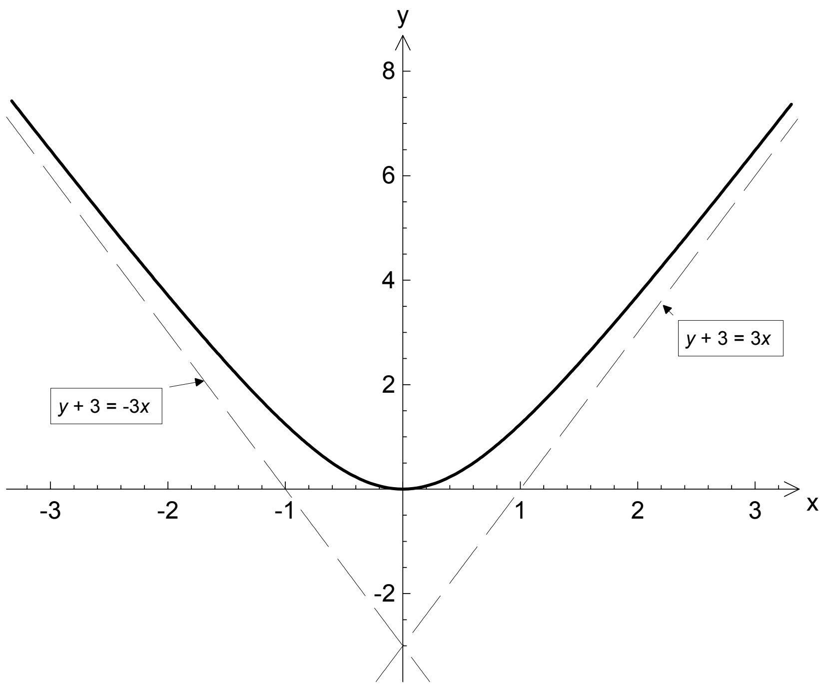
a.

See the diagram below

shape (one arm only) ......[A1]

$\frac{(y+3)^2}{9} - x^2 = 1 \quad a = 1$ and $b = 3$ ......[A1]

asymptotes $y + 3 = \pm 3x$ ......[A2]



b.

i.

$y + 3 = \pm 3\sqrt{x^2+1} \quad \therefore \quad y = -3 + 3\sqrt{x^2+1}$ as $y \geq 0$ ......[M1]

$f(x) = \frac{dy}{dx} = \frac{d}{dx}\left(-3 + 3\sqrt{x^2+1}\right) = \frac{3x}{\sqrt{x^2+1}}$ as required ......[A1]

ii.

It can be seen from the graph that the gradient to the hyperbola changes from -3 to 3 , hence $a = -3$ and $b = 3$ ......[A1]

c.

i.

$$V = \pi \int_0^h x^2 dy \quad , \quad (y+3)^2 - 9x^2 = 9 \quad \therefore \quad x^2 = \frac{(y+3)^2}{9} - 1 \quad ......[\text{M1}]$$

$$V = \pi \int_0^h \left(\frac{(y+3)^2}{9} - 1 \right) dy \quad ......[\text{A1}]$$

ii

$$V = \pi \int_0^h \left(\frac{(y+3)^2}{9} - 1 \right) dy = \pi \left[\frac{(y+3)^3}{27} - y \right]_0^h = \pi \left(\frac{(h+3)^3}{27} - h - 1 \right)$$

$$V(h) = \pi \left(\frac{h^3}{27} + \frac{h^2}{3} \right) \text{ as required} \quad ......[\text{A1}]$$

d.

Consider dV, a very small change in volume, that produces dh, a very small change in depth.

$$dV = A \times dh \quad \therefore \quad A = \frac{dV}{dh} \quad ......[\text{A1}]$$

Question 4

a.

$$\left. \begin{aligned} &m\underset{\sim}{a} = \underset{\sim}{W} + \underset{\sim}{F} \\ &m(a_x \underset{\sim}{i} + a_y \underset{\sim}{j}) = mg \underset{\sim}{j} - mk(v_x \underset{\sim}{i} + v_y \underset{\sim}{j}) \end{aligned} \right\} \quad ......[\text{M1}]$$

$$\left. \begin{aligned} &a_x \underset{\sim}{i} + a_y \underset{\sim}{j} = -k v_x \underset{\sim}{i} + (g - k v_y) \underset{\sim}{j} \\ &a_x = -k v_x \quad \text{and} \quad a_y = g - k v_y \\ &\frac{dv_x}{dt} = -k v_x \quad \text{and} \quad \frac{dv_y}{dt} = g - k v_y \quad \text{as required} \end{aligned} \right\} \quad ......[\text{A1}]$$

b.

$$\left.\begin{array}{l}\dfrac{dv_x}{dt} = -k\,v_x \quad \therefore \quad -kt = \displaystyle\int \frac{dv_x}{v_x} \quad \therefore \quad -kt = \log_e v_x + c \\ \text{At } t = 0, \quad v_x = u \quad \therefore \quad 0 = \log_e u + c \quad \therefore \quad c = -\log_e u\end{array}\right\} \quad ......[\text{M1}]$$

$$-kt = \log_e v_x - \log_e u \quad \therefore \quad \log_e\left(\frac{v_x}{u}\right) = -kt \quad \therefore \quad v_x(t) = ue^{-kt} \quad ......[\text{A1}]$$

$$\left.\begin{array}{l}\dfrac{dv_y}{dt} = g - k\,v_y \quad \therefore \quad t = \displaystyle\int \frac{dv_y}{g - k\,v_y} \quad \therefore \quad t = -\frac{1}{k}\log_e\left(g - k\,v_y\right) + c \\ \text{At } t = 0, \quad v_y = 0 \quad \therefore \quad 0 = -\dfrac{1}{k}\log_e\left(g\right) + c\end{array}\right\} \quad ......[\text{M1}]$$

$$\left.\begin{array}{l} t = -\dfrac{1}{k}\log_e\left(g - k\,v_y\right) + \dfrac{1}{k}\log_e\left(g\right) \quad \therefore \quad t = \dfrac{1}{k}\log_e\left(\dfrac{g}{g - k\,v_y}\right) \\ \dfrac{g}{g - k\,v_y} = e^{kt} \quad \therefore \quad g - k\,v_y = ge^{-kt} \quad \therefore \quad v_y(t) = \dfrac{g}{k}\left(1 - e^{-kt}\right)\end{array}\right\} \quad ......[\text{A1}]$$

c.

$$v_y(t) = \frac{g}{k}\left(1 - e^{-kt}\right) \quad \therefore \quad y = \int \frac{g}{k}\left(1 - e^{-kt}\right)dt \quad \therefore \quad y = \frac{g}{k}\left(t + \frac{1}{k}e^{-kt}\right) + c \quad ......[\text{M1}]$$

$$\left.\begin{array}{l}\text{At } t = 0, \quad y = 0 \quad \therefore \quad 0 = \dfrac{g}{k} \times \dfrac{1}{k} + c \quad \therefore \quad c = -\dfrac{g}{k} \times \dfrac{1}{k} \\ y = \dfrac{g}{k}\left(t + \dfrac{1}{k}e^{-kt}\right) - \dfrac{g}{k^2} \quad \therefore \quad y(t) = \dfrac{g}{k}t + \dfrac{g}{k^2}\left(e^{-kt} - 1\right)\end{array}\right\} \quad ......[\text{A1}]$$

d.

i.

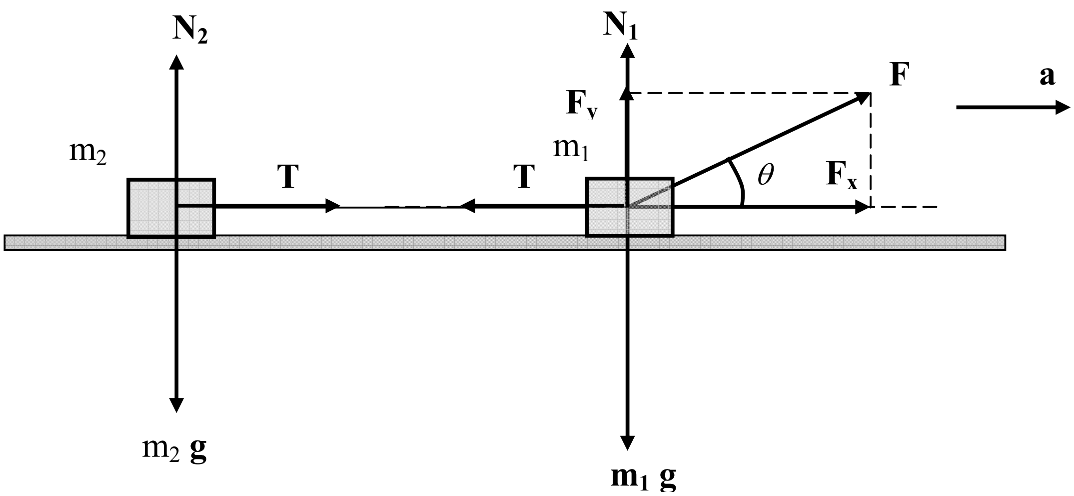
$$h = \frac{gt^2}{2} \quad \therefore \quad t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 490}{9.8}} = 10\,s \quad ......[\text{A1}]$$

ii.

$$490 = 100t + \frac{10000}{9.8}\left(e^{-0.01\times 9.8t} - 1\right) \quad \therefore \quad t = 11.9\,s \quad \ldots\ldots[\text{A1}]$$

Question 5

a.



See the diagram above

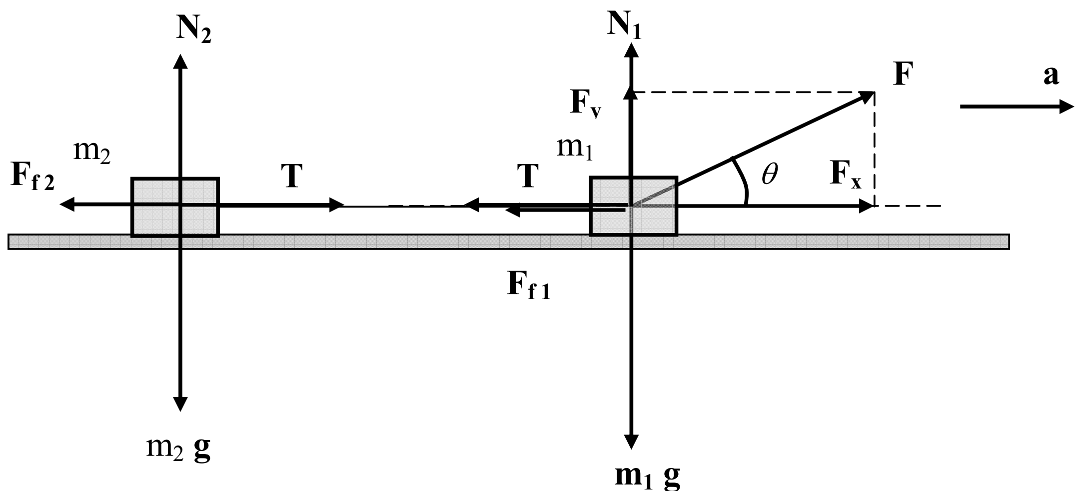
Apply Newton's laws on both axes for both objects

$$\left.\begin{array}{l} \begin{cases} N_1 + F\sin\theta - m_1 g = 0 \\ F\cos\theta - T = m_1 a \end{cases} \\ \begin{cases} N_2 - m_2 g = 0 \\ T = m_2 a \end{cases} \end{array}\right\} \quad \ldots\ldots[\text{M1}]$$

$$F\cos\theta = (m_1 + m_2)a \quad \therefore \quad a = \frac{F\cos\theta}{m_1 + m_2} \quad \ldots\ldots[\text{M1}]$$

$$\therefore \quad a = \frac{300\cos 45^0}{60} \quad \therefore \quad a = 3.5ms^{-2} \quad \ldots\ldots[\text{A1}]$$

b.



Apply Newton's laws on both axes for both objects

$$\left.\begin{matrix}\begin{cases} N_1 + F\sin\theta - m_1 g = 0 \\ F\cos\theta - T - F_{f_1} = m_1 a \\ F_{f_1} = \mu N_1 \end{cases} \\ \begin{cases} N_2 - m_2 g = 0 \\ T - F_{f_2} = m_2 a \\ F_{f_2} = \mu N_2 \end{cases}\end{matrix}\right\} \quad \ldots..[\text{M1}]$$

$$\Rightarrow \left.\begin{matrix}\begin{cases} N_1 = m_1 g - F\sin\theta \\ F\cos\theta - T - \mu\left(m_1 g - F\sin\theta\right) = m_1 a \end{cases} \\ \begin{cases} N_2 = m_2 g \\ T - \mu m_2 g = m_2 a \end{cases}\end{matrix}\right\} \quad \ldots..[\text{M1}]$$

$$\left.\begin{matrix} F\cos\theta - \mu\left(m_1 g - F\sin\theta\right) - \mu m_2 g = (m_1 + m_2)a \\ a = \dfrac{F\cos\theta - \mu\left(m_1 g - F\sin\theta\right) - \mu m_2 g}{m_1 + m_2} \\ a = \dfrac{F(\cos\theta + \mu\sin\theta) - \mu m_1 g - \mu m_2 g}{m_1 + m_2} \\ a = \dfrac{F(\cos\theta + \mu\sin\theta)}{m_1 + m_2} - \mu g \end{matrix}\right\} \quad \ldots..[\text{M1}]$$

$$\left.\begin{matrix} \mu = \tan\varphi \quad \therefore \quad a = \dfrac{F(\cos\theta + \tan\varphi\sin\theta)}{m_1 + m_2} - g\tan\varphi \\ a = \dfrac{F\left(\cos\theta + \dfrac{\sin\varphi}{\cos\varphi}\sin\theta\right)}{m_1 + m_2} - g\tan\varphi = \dfrac{F\left(\cos\theta\cos\varphi + \sin\varphi\sin\theta\right)}{(m_1 + m_2)\cos\varphi} - g\tan\varphi \end{matrix}\right\} \quad \ldots..[\text{M1}]$$

$$a = \frac{F\cos(\theta - \varphi)}{(m_1 + m_2)\cos\varphi} - g\tan\varphi \text{ as required}\ldots..[\text{A1}]$$

c.

i.

$$a = \frac{F\cos\left(\theta - 10^0\right)}{(m_1 + m_2)\cos 10^0} - g\tan 10^0$$

$a = \max$ when $\cos\left(\theta - 10^0\right) = 1$ $\therefore$ $\theta = 10^0$ .....[A1]

ii.

$$a_{\max} = \frac{F}{(m_1 + m_2)\cos 10^0} - g\tan 10^0 = \frac{300}{60\cos 10^0} - 9.8\tan 10^0 \quad \therefore \quad a_{\max} = 3.3ms^{-2}$$ .....[A1]

iii.

$$\left.\begin{array}{l} a_1 = \dfrac{F\cos\left(\theta_1 - 10^0\right)}{(m_1 + m_2)\cos 10^0} - g\tan 10^0 \quad \text{and} \quad a_2 = \dfrac{F\cos\left(\theta_2 - 10^0\right)}{(m_1 + m_2)\cos 10^0} - g\tan 10^0 \\ a_1 = a_2 \quad \therefore \quad \cos\left(\theta_1 - 10^0\right) = \cos\left(\theta_2 - 10^0\right) \end{array}\right\}$$ ....[M1]

$\therefore$ $\theta_1 - 10^0 = 10^0 - \theta_2$ $\therefore$ $\theta_2 = 20^0 - \theta_1$ $\therefore$ $\theta_2 = 5^0$ ....[A1]

d.

The object is on verge of losing contact with the surface
when the normal reaction N_1 becomes zero.

$N_1 = m_1 g - F\sin\theta$, $N_1 = 0$ $\therefore$ $0 = m_1 g - F\sin\theta$ .....[M1]

$\therefore$ $F = \dfrac{m_1 g}{\sin\theta} = \dfrac{40 \times 9.8}{\sin 45^0} = 554.4N$ .....[A1]