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Unit 3 and 4 Specialist Mathematics: Exam 2

Practice Exam Question and Answer Booklet

Duration: 15 minutes reading time, 2 hours writing time

Structure of book:

Section	Number of questions	Number of questions to be answered	Number of marks
A	22	22	22
B	6	6	58
		Total	80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory does not need to be cleared) and, if desired, one scientific calculator.
- Students are not permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied:

- This question and answer booklet of 23 pages, including a sheet of miscellaneous formulas.

Instructions:

- You must complete all questions of the examination.
- Write all your answers in the spaces provided in this booklet.

Section A – Multiple-choice questions

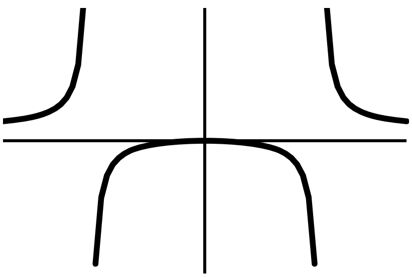
Instructions

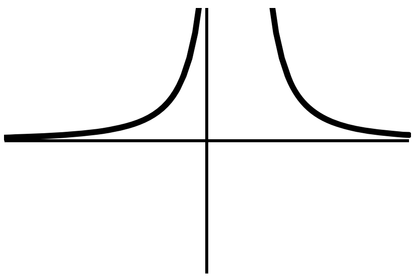
Answer all questions by circling your choice.
Choose the response that is correct or that best answers the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Take the acceleration due to gravity to have magnitude g m/s^2, where $g = 9.8$.

Questions

Question 1

Which one of the following graphs is not that of $f(x) = \frac{1}{ax^2+bx+c}$, where a≠0?

A.


B.


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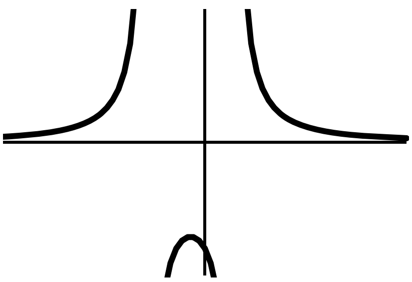

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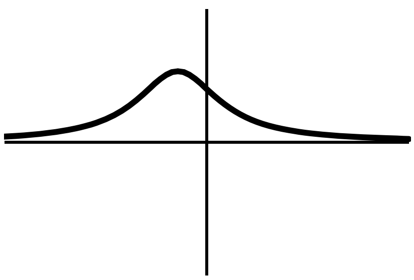

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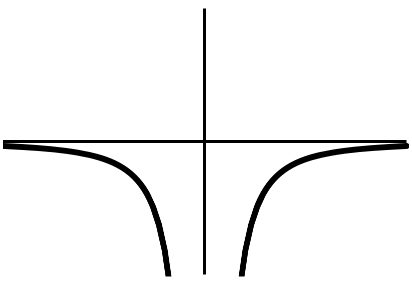


Question 2

$\cos(x - y + z)$ can be expressed as:

A. $\cos(x)\cos(y+z) + \sin(x)\sin(y+z)$
B. $\cos(x+z)\cos(y) + \sin(x+z)\sin(y)$
C. $\cos(x)\cos(y+z) - \sin(x)\sin(y+z)$
D. $\cos(x-y)\cos(z) + \sin(x-y)\sin(z)$
E. $\cos(x)\cos(y-z) + \sin(x)\sin(y-z)$

Question 3

An anti-derivative of $\frac{1}{2}\sin(2x)\sqrt{1 - cosx}$ is:

A. $\frac{2}{3}(1 - cosx)^{1.5} + \frac{2}{5}(1 - cosx)^{2.5}$
B. $\frac{2}{3}(1 - cosx)^{1.5} - \frac{2}{5}(1 - cosx)^{2.5}$
C. $\frac{2}{5}(1 - cosx)^{2.5} - \frac{2}{3}(1 - cosx)^{1.5}$
D. $\frac{4}{5}(1 - cosx)^{2.5} - \frac{4}{3}(1 - cosx)^{1.5}$
E. $\frac{4}{3}(1 - cosx)^{1.5} + \frac{4}{5}(1 - cosx)^{2.5}$

Question 4

Given $f(x) = \tan^{-1}(x)$, $f'(x) = f''(x)$ when $x =$

A. -1
B. $-\frac{1}{2}$
C. 0
D. $\frac{1}{2}$
E. 1

Question 5

Let $\frac{dy}{dx} = \tan^{-1}(x^2)$ and $y = c$ when $x = -1$. When $x = -2$, the value of y is closest to:

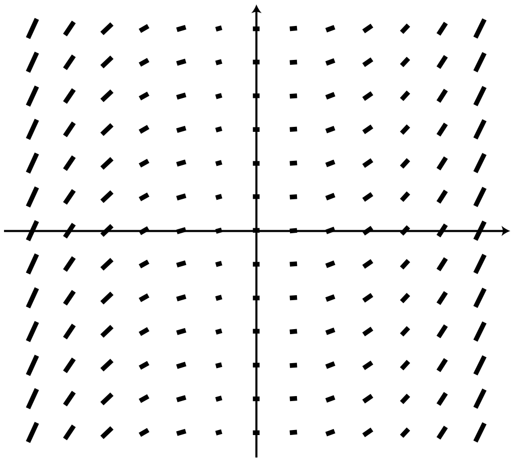
A. $c + 0.79$
B. $0.79 - c$
C. $c - 0.79$
D. $0.79c$
E. $\frac{c}{0.79}$

Question 6

For the vectors $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}, \boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = 4\boldsymbol{i} - 9\boldsymbol{j}$, which one of the following statements is false?

A. $\boldsymbol{a}$ and $\boldsymbol{b}$ are independent.
B. $\boldsymbol{b}$ and $\boldsymbol{c}$ are independent.
C. $\boldsymbol{c}$ and $\boldsymbol{a}$ are independent.
D. $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are independent.
E. $\frac{1}{2}\boldsymbol{c} + \boldsymbol{b}$ and $\boldsymbol{a}$ are dependent.

Question 7



A slope field for a certain differential equation is shown above. The differential equation could be:

A. $\frac{dy}{dx} = \frac{k}{x}$, where k is a real constant.
B. $\frac{dy}{dx} = k\log_e x$, where k is a real constant.
C. $\frac{dy}{dx} = e^{kx}$, where k is a real constant.
D. $\frac{dy}{dx} = kx^3$, where k is a real constant.
E. $\frac{dy}{dx} = kx^2$, where k is a real constant.

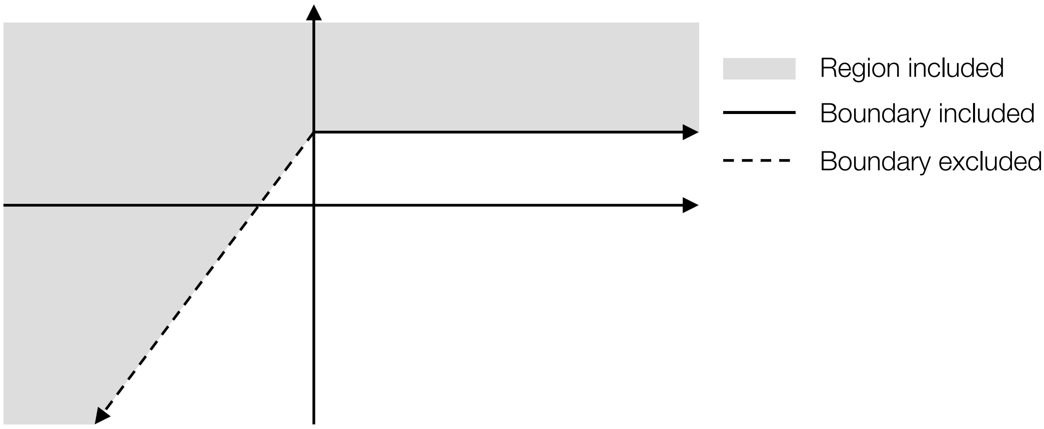
Question 8

The position of a moving particle is given by $\boldsymbol{r}(t) = \tan(t)\,\boldsymbol{i} + \frac{1}{2}\sec^2(t)\boldsymbol{j} + \boldsymbol{k}$.

At $t = \frac{3\pi}{4}$, the direction of the moving particle is given by:

A. $2(\boldsymbol{i} - 2\boldsymbol{j})$
B. $-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$
C. $\frac{\sqrt{3}}{3}(-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$
D. $\frac{\sqrt{2}}{2}(\boldsymbol{i} + \boldsymbol{j})$
E. $\frac{\sqrt{3}}{3}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

Question 9



The shaded region in the complex plane $\mathbb{C}$ could represent:

A. $\mathbb{C} \setminus \{z: -\frac{2\pi}{3} \leq \text{Arg}(z - i\sqrt{3}) < 0)\}$
B. $\mathbb{C} \setminus \{z: -\frac{2\pi}{3} < \text{Arg}(z - i\sqrt{3}) < 0)\}$
C. $\{z: 0 \leq \text{Arg}\left(z - i\sqrt{3}\right) < \pi)\} \cup \{z: -\pi \leq \text{Arg}(z - i\sqrt{3}) < -\frac{2\pi}{3}\}$
D. $\{z: 0 \leq \text{Arg}(z - i\sqrt{3}) < \pi)\} \cup \{z: -\pi \leq \text{Arg}(z - i\sqrt{3}) < -\frac{2\pi}{3}\}$
E. $\{z: 0 \leq \text{Arg}(z + i\sqrt{3}) < \frac{4\pi}{3}\}$

Question 10

The area bounded by the y-axis, the x-axis and $y = 2\cos^{-1}(2x)$ is:

A. 4
B. 3
C. 2
D. 1
E. 0

Question 11

An anti-derivative of $\frac{e^{2x}-1}{e^{2x}+1} - \frac{e^{2x}+1}{e^{2x}-1}$ is:

A. $\log_e\left(\frac{e^{2x}+1}{e^{2x}-1}\right)$
B. $\log_e\left(\frac{e^{2x}-1}{e^{2x}+1}\right)$
C. $\log_e\left(\frac{e^{2x}}{e^{2x}-1}\right)$
D. $\log_e\left(\frac{e^{2x}}{e^{2x}+1}\right)$
E. $\log_e\left(\frac{e^{4x}}{e^{4x}-1}\right)$

Question 12

$P(x, y)$ is a point on the ellipse $\frac{x^2}{2} + y^2 = 1$, and $\frac{x^2}{2} + y = c$. The maximum value of c is:

A. 1
B. $\frac{6}{5}$
C. $\frac{5}{4}$
D. $\frac{4}{3}$
E. $\frac{3}{2}$

Question 13

The volume of the solid revolving the curve $y = \log_e(-x)$ where $-2 \leq x \leq -1$ about the x-axis is given by:

A. $2\pi \int_{-2}^{-1} \log_e(-x)\, dx$
B. $2\pi \int_{-1}^{-2} \log_e(-x)\, dx$
C. $\pi \int_{1}^{2} \log_e(x)^2\, dx$
D. $\pi \int_{1}^{2} e^{2x} dx$
E. $\pi \int_{-2}^{-1} e^{-2x} dx$

Question 14

$\sqrt{-1}$ equals:

A. i only
B. $-i$ only
C. $-i$ or i
D. i^5 only
E. i or i^5

Question 15

Given $a \in R^-, b \in R^+$ and $z = a - bi, |z|$ equals:

A. $|a + b|$
B. $|a - b|$
C. $\sqrt{a^2 - b^2}$
D. $\sqrt{(a - b)^2}$
E. $\sqrt{a^2 + b^2}$

Question 16

The angle between vector $\boldsymbol{p} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ and the xz-plane is:

A. $\tan^{-1}\left(\frac{b}{\sqrt{c^2+a^2}}\right)$
B. $\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2+c^2}}\right)$
C. $\tan^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)$
D. $\tan^{-1}\left(\frac{b}{\sqrt{b^2+c^2}}\right)$
E. $\sin^{-1}\left(\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$

Question 17

$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$ are non-zero vectors independent of each other. If $\boldsymbol{a}+\boldsymbol{b}$ is perpendicular to $\boldsymbol{c}+\boldsymbol{d}$, and $\boldsymbol{b}+\boldsymbol{c}$ is perpendicular to $\boldsymbol{a}+\boldsymbol{d}$, then:

A. $\boldsymbol{b}+\boldsymbol{d}$ is perpendicular to $\boldsymbol{c}-\boldsymbol{a}$
B. $\boldsymbol{b}-\boldsymbol{d}$ is perpendicular to $\boldsymbol{c}+\boldsymbol{a}$
C. $\boldsymbol{b}+\boldsymbol{d}$ is perpendicular to $\boldsymbol{c}+\boldsymbol{a}$
D. $\boldsymbol{b}-\boldsymbol{d}$ is perpendicular to $\boldsymbol{c}-\boldsymbol{a}$
E. $\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}$ is perpendicular to $\boldsymbol{b}-\boldsymbol{c}+\boldsymbol{d}$

Question 18

The acceleration of a particle is given by $a=-2(x-3)^3$ at displacement x from the origin O. At $x=3+\sqrt{2}$, its velocity is 0. Its minimum displacement from O and its maximum velocity are, respectively:

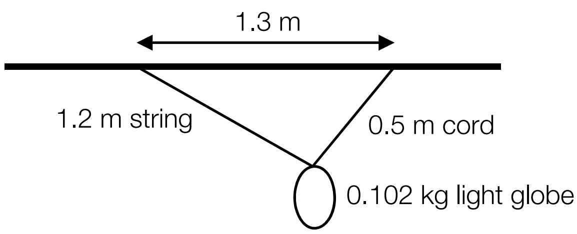
A. $3, 2$
B. $3+\sqrt{2}, 2$
C. $-\sqrt{2}, 4$
D. $\sqrt{2}, 2$
E. $\sqrt[3]{3}, 4$

Question 19

The momentum of a 0.2 kg particle changes from $(3\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})\ \mathrm{kg\ m\ s^{-1}}$ to $(\boldsymbol{i}-\boldsymbol{k})\ \mathrm{kg\ m\ s^{-1}}$. The change of speed in $\mathrm{m\ s^{-1}}$ of the particle is closest to:

A. 10
B. -10
C. 15
D. -15
E. 3

Question 20



The diagram above shows a 0.102 kg light globe suspended from the ceiling by a 1.2 m string and a 0.5 m cord. The tension in the string is T_s newtons and the tension in the cord is T_c newtons. The value of the ratio $\frac{Ts}{Tc}$ is closest to:

A. 2.4
B. 3.8
C. 1.08
D. 0.92
E. 0.42

Question 21
Given that $z = a + bi, a, b \in \mathbb{R}$, $z + \bar{z}$ is equal to:

A. 0
B. $2a$
C. $2b$
D. $2bi$
E. $2a + 2bi$

Question 22
Geometrically, iz represents:

A. A reflection of z in the x-axis
B. A reflection of z in the y-axis
C. A reflection of z in the line $y = x$
D. A rotation of z by 90° clockwise around the origin
E. A rotation of z by 90° anticlockwise around the origin

Section B – Short-answer questions

Instructions

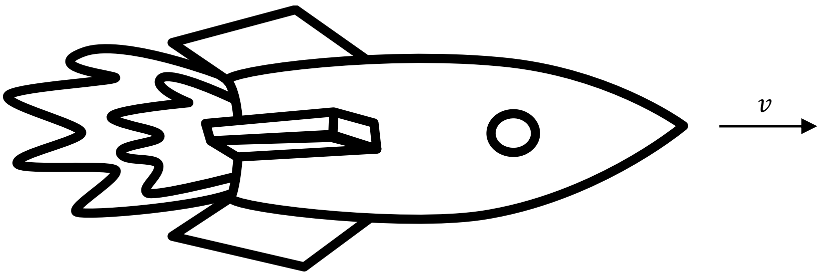
Answer all questions in the spaces provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise indicated, the diagrams in this book are not drawn to scale.
Take the acceleration due to gravity to have magnitude g m/s^2, where $g = 9.8$.

Questions

Question 1

A 1000 kg rocket has a constant forward thrust of 2500 newtons. It experiences air resistance at k^2v^2 newtons when travelling at v m s^{-1}, where k is a constant.

a. Label the forces on the rocket below.



1 mark

b.

i. Write down the acceleration of the rocket in terms of k and v.

1 mark

ii. Hence show that if at $t = 0, v = 0$, $v = \frac{50(\alpha-1)}{k(\alpha+1)}$, where $\alpha = e^{\frac{kt}{10}}$.

5 marks

iii. Show that $k = \frac{e-1}{e+1}$ given $v = 50$ at $t = \frac{10}{k}$.

1 mark

c. Find t to the nearest second when $v = 50$.

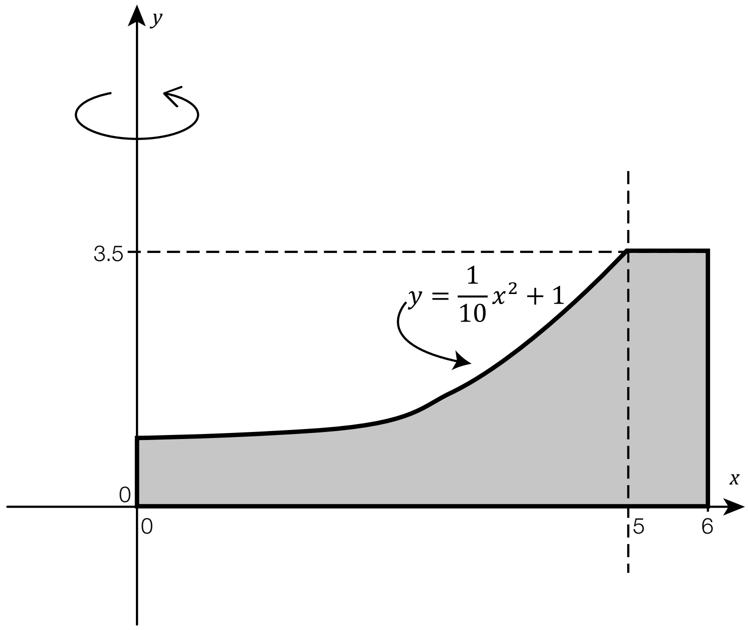
1 mark

d. Find the speed v_m, the maximum speed of the rocket.

1 mark

Question 2

The following diagram shows a cross section of a small structure used for storing water. The structure is modelled by a solid of revolution around the y-axis. All lengths are in metres.



a. Find the exact area of the shaded region.

2 marks

b. Find the maximum amount of water the structure can hold.

3 marks

Water is now filling the structure at a constant rate of $\frac{\pi}{4}$ m^3 per minute.

c. Find the average rate in mm per second at which the water level is rising.

2 marks

d.

i. When the structure is filled to a depth of h m, express the volume of water $V(h)$ m^3 as a definite integral.

1 mark

ii. Hence find an expression for $\frac{dh}{dt}$ in terms of h.

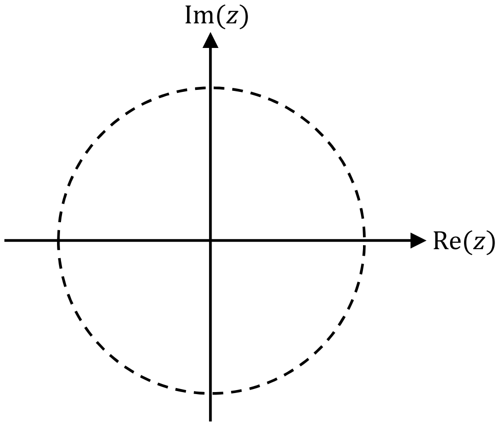
4 marks

Question 3

a. $\frac{z^7+1}{z+1}$ can be expressed as a polynomial of $z, P(z) = Az^6 + Bz^5 + Cz^4 + Dz^3 + Ez^2 + Fz + G$. Find the values A,B,C,D,E,F and G.

2 marks

b. Plot the roots $P(z) = 0$ on the complex plane below.



2 marks

c. Express the roots of $P(z) = 0$ in polar form.

3 marks

Question 4

The coefficient of friction between a 2.5 kg mass and a plank is 0.7. Express your answers to 2 decimal places in the following questions.

a. What is the force of friction in newtons if the object is at rest on a horizontal plank?

1 mark

b. What is the force of friction if the plank is tilted, making a 30° angle with the horizontal?

2 marks

The angle is now increased to 40°.

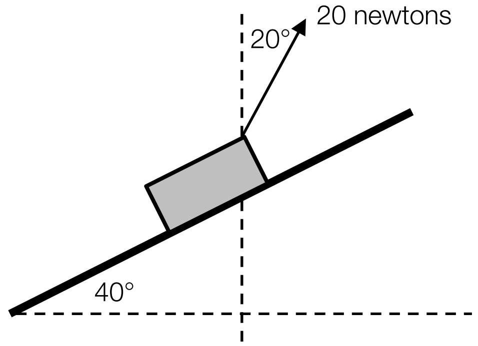
c. Calculate the acceleration of the object down the plank.

2 marks

d. If the object starts from rest 3.0 m from the lower end of the plank, calculate the momentum when the object reaches the lower end of the plank.

2 marks

Keeping the box on the 40° plank, a rope is attached to the object at an angle of 20° to the vertical. A student pulls the rope with a force of 20 newtons, as shown in the diagram below:



e. Calculate the magnitude of the resultant force on the object.

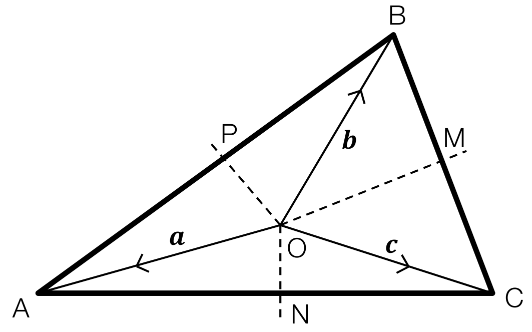
4 marks

f. Determine the direction of motion of the object (to the nearest degree, with 0° being up). Briefly describe the motion of the box.

1 mark

Question 5

Consider a scalene triangle ABC. There exists a point O where OM and ON are the perpendicular bisectors of BC and AC respectively, and OP bisects AB.



Let OA = $\boldsymbol{a}$, OB = $\boldsymbol{b}$ and OC = $\boldsymbol{c}$.

a. Express AC, BC and BA in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

2 marks

b. Express OM, ON and OP in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

2 marks

c. Hence show that:

i. $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}|$

3 marks

ii. OP is perpendicular to BA

1 mark

d. Show that $|AC|^2 + |BC|^2 + |BA|^2 = 2d^2[3 - (\cos\alpha + \cos\beta + \cos\gamma)]$, where $d = |\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}|$, and α, β and γ are angles of your choice.

4 marks

Question 6

a. Use Euler's method with a step size of 0.1 to find $y(1.2)$ if $\frac{dy}{dx} = \frac{2x+1}{x+1}$, given that $y(1) = 1$.

2 marks

b. Solve the differential equation given in part a to find the exact value of $y(1.2)$. Give your answer in the form $y = a - \log_e(b)$, where $a, b \in \mathbb{R}$.

3 marks

Formula sheet

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder	$2\pi rh$
volume of a cylinder	$\pi r^2 h$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3}Ah$
volume of a sphere	$\frac{4}{3}\pi r^3$
area of a triangle	$\frac{1}{2}bc\sin A$
sine rule	$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$
cosine rule	$c^2=a^2+b^2-2ab\cos C$

Coordinate geometry

ellipse $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Circular (trigonometric) functions

$\cos^2(x)+\sin^2(x)=1$

$1+\tan^2(x)=\sec^2(x)$ $\cot^2(x)+1=\operatorname{cosec}^2(x)$

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$ $\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$

$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$ $\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$

$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$ $\tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x)$

$\sin(2x)=2\sin(x)\cos(x)$ $\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$

function	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1,1]$	$[-1,1]$	$\mathbb{R}$
range	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0,\pi]$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$	$z^n = r^n \operatorname{cis}(n\theta)$ (de Moivre's theorem)
$\lvert z\rvert = \sqrt{x^2 + y^2} = r$	$-\pi < \operatorname{Arg} z \le \pi$
$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$	$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$

Calculus

$\dfrac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \dfrac{1}{n+1}x^{n+1} + c, n \ne -1$
$\dfrac{d}{dx}(e^{ax}) = ae^{ax}$	$\int e^{ax} dx = \dfrac{1}{a}e^{ax} + c$
$\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$	$\int \dfrac{1}{x} dx = \log_e\lvert x\rvert + c$
$\dfrac{d}{dx}(\sin(ax)) = a\cos(ax)$	$\int \sin(ax)\,dx = -\dfrac{1}{a}\cos(ax) + c$
$\dfrac{d}{dx}(\cos(ax)) = -a\sin(ax)$	$\int \cos(ax)\,dx = \dfrac{1}{a}\sin(ax) + c$
$\dfrac{d}{dx}(\tan(ax)) = \dfrac{a}{\cos^2(ax)} = a\sec^2(ax)$	$\int \sec^2(ax)\,dx = \dfrac{1}{a}\tan(ax) + c$
$\dfrac{d}{dx}(\sin^{-1}(x)) = \dfrac{1}{\sqrt{1-x^2}}$	$\int \dfrac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\dfrac{x}{a}\right) + c, a > 0$
$\dfrac{d}{dx}(\cos^{-1}(x)) = -\dfrac{1}{\sqrt{1-x^2}}$	$\int \dfrac{-1}{\sqrt{a^2-x^2}}dx = \cos^{-1}\left(\dfrac{x}{a}\right) + c, a > 0$
$\dfrac{d}{dx}(\tan^{-1}(x)) = \dfrac{1}{1+x^2}$	$\int \dfrac{a}{a^2+x^2}dx = \tan^{-1}\left(\dfrac{x}{a}\right) + c$

product rule	$\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
quotient rule	$\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{\left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)}{v^2}$
chain rule	$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
Euler's method	If $\frac{dy}{dx} = f(x), x_0 = a$ and $y_0 = a$, then $y_{n+1} = y_n + hf(x_n)$
acceleration	$a = \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$
constant (uniform) acceleration	$v = u + at$, $s = ut + \frac{1}{2}at^2$, $v^2 = u^2 + 2as$, $s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

$\dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}$

$|\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2} = r$

$\boldsymbol{r}_1 . \boldsymbol{r}_2 = r_1 r_2 \cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2$

Mechanics

momentum	$\boldsymbol{p} = m\boldsymbol{v}$
equation of motion	$\boldsymbol{R} = m\boldsymbol{a}$
friction	$F \leq \mu N$