2012 Specialist Mathematics Trial Exam 1 Solutions

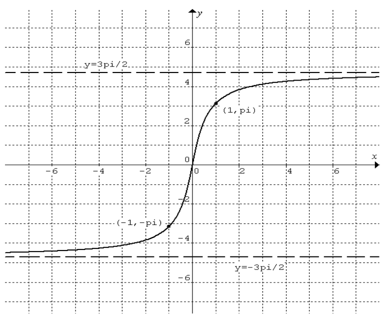
Q1a $\tan^{-1}(1)=\frac{\pi}{4}$, $0<\tan^{-1}(2)<\frac{\pi}{2}$, $0<\tan^{-1}(3)<\frac{\pi}{2}$

$\tan\left(\tan^{-1}(2)+\tan^{-1}(3)\right)=\frac{\tan\left(\tan^{-1}(2)\right)+\tan\left(\tan^{-1}(3)\right)}{1-\tan\left(\tan^{-1}(2)\right)\times\tan\left(\tan^{-1}(3)\right)}$

$=\frac{2+3}{1-2\times 3}=-1$, .: $\tan^{-1}(2)+\tan^{-1}(3)=\frac{3\pi}{4}$

.: $\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)=\pi$

Q1b Sketch $y=\tan^{-1}(x)$, $y=\tan^{-1}(2x)$ and $y=\tan^{-1}(3x)$ on the same set of axes. By addition of ordinates:



Q2a $\frac{(x+2)^2}{4}+\frac{\left(y-\sqrt{2}\right)^2}{2}=1$, an ellipse

Domain: $[-4,0]$, range: $\left[0,2\sqrt{2}\right]$

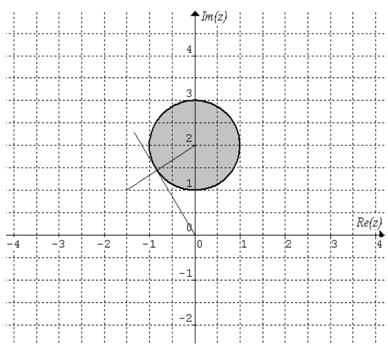
Q2b $\frac{(x+2)^2}{4}+\frac{\left(y-\sqrt{2}\right)^2}{2}=1\rightarrow\frac{(x+2)^2}{4}+\frac{\left(\frac{y}{\sqrt{2}}-\sqrt{2}\right)^2}{2}=1$

Simplify $\frac{(x+2)^2}{4}+\frac{\left(\frac{y}{\sqrt{2}}-\sqrt{2}\right)^2}{2}=1$ to $(x+2)^2+(y-2)^2=4$

Q2c $(x+2)^2+(y-2)^2=4$ is a circle of radius 2.

$Area=\pi r^2=4\pi$

Q3a



Q3b $Max\ Arg(z)=\frac{\pi}{2}+\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{2}+\frac{\pi}{6}=\frac{2\pi}{3}$

Q4 By inspection, $z=0$ is a solution.

For $z\neq 0$, $z^5=\bar{z}$, $zz^5=z\bar{z}$, $z^6=|z|^2$

Let $z=rcis\theta$, $r^6cis(6\theta)=r^2$, $r^4cis(6\theta)=1$

.: $r=1$ and $6\theta=2k\pi$ where $k=0, 1, 2, 3, 4, 5$

.: $z=cis0$, $cis\frac{\pi}{3}$, $cis\frac{2\pi}{3}$, $cis\pi$, $cis\frac{4\pi}{3}$ and $cis\frac{5\pi}{3}$

.: $z=1$, $\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, -1, $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ and $\frac{1}{2}-\frac{\sqrt{3}}{2}i$

Q5a Let $\tilde{p}=a\tilde{i}+b\tilde{j}+c\tilde{k}$ be a vector perpendicular to $\tilde{r}=2\tilde{i}-\tilde{j}+3\tilde{k}$ where *a*, *b* and *c* are non-zero scalars.

$\tilde{p}.\tilde{r}=0$, .: $2a-b+3c=0$

Let $a=-1$ and $b=1$, .: $c=1$, .: $\tilde{p}=-\tilde{i}+\tilde{j}+\tilde{k}$

Note: Infinitely many such vectors can be obtained by choosing values for *a* and *b*, and then find *c*.

Q5b $|\tilde{p}|=\sqrt{(-1)^2+1^2+1^2}=\sqrt{3}$, $\hat{p}=\frac{1}{\sqrt{3}}\left(-\tilde{i}+\tilde{j}+\tilde{k}\right)$

Q5c A possible point is $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.

Q6a The position of the centre of motion is given by $\tilde{r}_c=3\tilde{k}$.

Q6b The position of the particle from the centre of motion is

$\tilde{R}=\tilde{r}-\tilde{r}_c=(\cos t)\tilde{i}-(2\sin t)\tilde{j}$

$\dot{\tilde{R}}=-(\sin t)\tilde{i}-(2\cos t)\tilde{j}$, $\ddot{\tilde{R}}=-(\cos t)\tilde{i}+(2\sin t)\tilde{j}$, .: $\ddot{\tilde{R}}=-\tilde{R}$

i.e. the particle's acceleration always towards the centre of motion.

Q7a $f(x)=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$, $-1<x<1$

$$f'(x)=\frac{\sqrt{1-x^2}\times\frac{1}{\sqrt{1-x^2}}-\left(\sin^{-1}x\right)\frac{-x}{\sqrt{1-x^2}}}{\left(\sqrt{1-x^2}\right)^2}$$

$$=\frac{\sqrt{1-x^2}+x\left(\sin^{-1}x\right)}{\left(\sqrt{1-x^2}\right)^3}$$

$$f'\left(\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{1-\frac{1}{2}}+\frac{\sqrt{2}}{2}\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)}{\left(\sqrt{1-\frac{1}{2}}\right)^3}=\frac{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(\frac{\pi}{4}\right)}{\left(\frac{\sqrt{2}}{2}\right)^3}$$

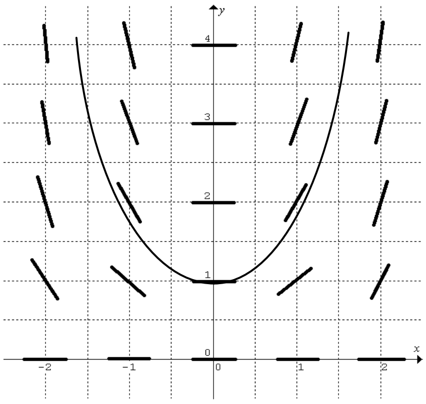
$$=\frac{1+\frac{\pi}{4}}{\frac{1}{2}}=2+\frac{\pi}{2}$$

Q7b Note that $f(-a) = -f(a)$ for $-1 < a < 1$.

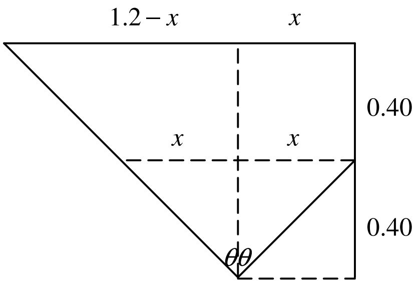
$Area = 2\times\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx$, let $u = \sin^{-1} x$, $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\therefore\ area = 2\times\int_0^{\frac{\pi}{2}} u\,du = 2\times\left[\frac{u^2}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{4}$

Q8a, b



Q9



$\frac{0.40}{x} = \frac{0.80}{1.2-x}$, $\therefore\ x = 0.40$, $\therefore\ \theta = 45°$

Let *T* be the tension in the thread.

$2T\cos 45° - 0.010\times 9.8 = 0$, $\sqrt{2}T = 0.098$, $1.414T = 0.098$,
$T \approx 0.069$ N

Q10a $\Delta v = -4-8 = -12$, $|p| = m|\Delta v| = 0.50\times 12 = 6$ kg m s^{-1}

Q10b Gradient at $t = 5$ is -0.8, i.e. $a = -0.8$ m s^{-2}
$|F| = m|a| = 0.50\times 0.8 = 0.4$ N

Q10c $Distance = \int_0^5 \frac{8}{1+0.04t^2}dt + 2\left(\frac{1}{2}\times 4\times 5\right)$

$= \int_0^5 \frac{8}{1+(0.2t)^2}dt + 20 = \left[\frac{8\tan^{-1}(0.2t)}{0.2}\right]_0^5 + 20 = 40\tan^{-1}1 + 20$

$= 10\pi + 20$ metres

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