The Mathematical Association of Victoria

SPECIALIST MATHEMATICS 2012

Trial Written Examination 1 - SOLUTIONS

Question 1

Solution 1: Take the **upwards** direction as positive.

Data: $u = 20$ m/s

$a = -g$ m/s^2

$s = -60$ m — Correct data **[M1]**

$t = ?$

Substitute the above data into $s = ut + \frac{1}{2}at^2$:

$-60 = 20t - \frac{1}{2}gt^2$ **[M1]**

$\Rightarrow -120 = 40t - gt^2$

$\Rightarrow gt^2 - 40t - 120 = 0$

where $b = -40$ and $c = -120$.

Total 2 marks

Solution 2: Take the **downwards** direction as positive.

Data: $u = -20$ m/s

$a = g$ m/s^2

$s = 60$ m — Correct data **[M1]**

$t = ?$

Substitute the above data into $s = ut + \frac{1}{2}at^2$:

$60 = -20t + \frac{1}{2}gt^2$ **[M1]**

$\Rightarrow 120 = -40t + gt^2$

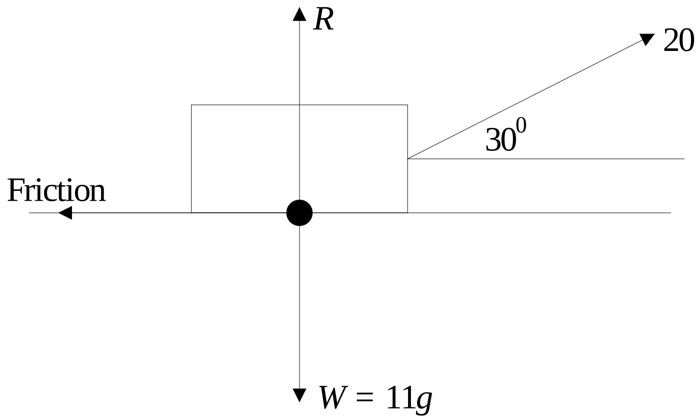
$\Rightarrow gt^2 - 40t - 120 = 0$

where $b = -40$ and $c = -120$.

Total 2 marks

Question 2

a.
Normal reaction force R.
Weight force $W = 11g$.
Friction force.
Pulling force of 20 Newton acting in a direction 30 degrees to the horizontal.



All forces labeled **[A1]**

Do NOT deduct 1 mark if the friction force is labeled as μR . This (incorrect) assumption gets penalised in **part b**.

b.
Despite knowing that $\mu = 0.2$ it is NOT known whether or not the object is on the point of sliding. Therefore the friction force CANNOT be assumed to be equal to μR .

Horizontal component of pulling force: $20\cos(30^0) = 20\left(\frac{\sqrt{3}}{2}\right) = 10\sqrt{3}$. **[M1]**

Net force in **horizontal direction:** $\left.\begin{aligned} F_{net} &= ma = 0 \\ F_{net} &= \text{Friction} - 10\sqrt{3} \end{aligned}\right\} \Rightarrow 0 = \text{Friction} - 10\sqrt{3}$

$\Rightarrow \text{Friction} = 10\sqrt{3}$. **[A1]**

Total 3 marks

NOTE:

This answer is consistent with $0 \le \text{Friction} = \mu R$ and shows that the object is not on the point of sliding:

Vertical component of pulling force: $20\sin(30^0) = 20\left(\frac{1}{2}\right) = 10$.

Net force in **vertical direction**: $\left.\begin{aligned} F_{net} &= ma = 0 \\ F_{net} &= R + 10 - 11g \end{aligned}\right\} \Rightarrow 0 = R + 10 - 11g$

$\Rightarrow R = 11g - 10$

$\Rightarrow \mu R = 0.2(11g - 10) = 2.2g - 2 > 10\sqrt{3}$.

Question 3

a.

Let $y = \tan^{-1}\left(\frac{2}{\sqrt{x+1}}\right)$.

Chain rule: Let $u = \frac{2}{\sqrt{x+1}} \Rightarrow y = \tan^{-1}(u)$.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{1+u^2} \times \left(-\frac{1}{(x+1)^{3/2}}\right)$$ **[M1]**

$$= \frac{1}{1+\frac{4}{x+1}} \times \left(-\frac{1}{(x+1)^{3/2}}\right)$$

$$= \frac{x+1}{x+5} \times \left(-\frac{1}{(x+1)^{3/2}}\right)$$

$$= \frac{1}{x+5} \times \left(-\frac{1}{(x+1)^{1/2}}\right)$$

$$= \frac{-1}{(x+5)(x+1)^{1/2}}$$ **[A1]**

where $a = -1$, $b = 5$ and $c = \frac{1}{2}$.

b.

$$\tan^{-1}\left(\frac{2}{\sqrt{x+1}}\right) = \frac{\pi}{3}$$

$$\Rightarrow \frac{2}{\sqrt{x+1}} = \sqrt{3}$$

$$\Rightarrow \frac{4}{x+1} = 3$$

$$\Rightarrow x = \frac{1}{3}.$$ **[A1]**

Total 3 marks

Question 4

a.

$w^3 + 4 - i4\sqrt{3} = 0$

$\Rightarrow w^3 = -4 + i4\sqrt{3}$.

Therefore the required numbers are the cube roots of $-4 + i4\sqrt{3}$.

Polar form: $-4 + i4\sqrt{3} = 8\text{cis}\left(\frac{2\pi}{3}\right)$ **[M1]**

$= 8\text{cis}\left(\frac{2\pi}{3} + 2n\pi\right)$ where $n \in Z$.

Let $w = r\text{cis}(\theta) \Rightarrow w^3 = r^3\text{cis}(3\theta)$.

Therefore $r^3\text{cis}(3\theta) = 8\text{cis}\left(\frac{2\pi}{3} + 2n\pi\right)$.

Equate modulus and argument:

$r^3 = 8 \Rightarrow r = 2$.

$3\theta = \frac{2\pi}{3} + 2n\pi \Rightarrow \theta = \frac{2\pi}{9} + \frac{2n\pi}{3}$.

$n = 0$: $w = 2\text{cis}\left(\frac{2\pi}{9}\right)$. **[A1]**

$n = 1$: $w = 2\text{cis}\left(\frac{8\pi}{9}\right)$.

$n = -1$: $w = 2\text{cis}\left(\frac{-4\pi}{9}\right)$. Both of the remaining values of *w* **[A1]**

b.

$u^3 = -4 - i4\sqrt{3}$

$\Rightarrow \overline{u^3} = -4 + i4\sqrt{3}$

$\Rightarrow \overline{u}^3 = -4 + i4\sqrt{3}$.

Therefore $\overline{u} = w \Rightarrow u = \overline{w}$:

$u = 2\text{cis}\left(\frac{-2\pi}{9}\right)$, $2\text{cis}\left(\frac{-8\pi}{9}\right)$, $w = 2\text{cis}\left(\frac{4\pi}{9}\right)$.

Complex conjugate of answers to **part a.** **[A1]**

Total 4 marks

Question 5

$$\frac{d\underset{\sim}{r}}{dt} = 2\sin\left(\frac{t}{2}\right)\underset{\sim}{i} + \cos(t)\underset{\sim}{j} + 2t\underset{\sim}{k}$$

$$\Rightarrow \underset{\sim}{r} = \int 2\sin\left(\frac{t}{2}\right)\underset{\sim}{i} + \cos(t)\underset{\sim}{j} + 2t\underset{\sim}{k}\ dt$$

$$= -4\cos\left(\frac{t}{2}\right)\underset{\sim}{i} + \sin(t)\underset{\sim}{j} + t^2\underset{\sim}{k} + \underset{\sim}{C}.$$ **[M1]**

Substitute $\underset{\sim}{r} = -\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ at $t = \pi$:

$$-\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} = -4\cos\left(\frac{\pi}{2}\right)\underset{\sim}{i} + \sin(\pi)\underset{\sim}{j} + \pi^2\underset{\sim}{k} + \underset{\sim}{C}$$

$$\Rightarrow -\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} = \pi^2\underset{\sim}{k} + \underset{\sim}{C}$$

$$\Rightarrow \underset{\sim}{C} = -\underset{\sim}{i} - \underset{\sim}{j} + (1-\pi^2)\underset{\sim}{k}.$$

Therefore:

$$\underset{\sim}{r} = -4\cos\left(\frac{t}{2}\right)\underset{\sim}{i} + \sin(t)\underset{\sim}{j} + t^2\underset{\sim}{k} + (-\underset{\sim}{i} - \underset{\sim}{j} + (1-\pi^2)\underset{\sim}{k}) \quad = \left(-4\cos\left(\frac{t}{2}\right) - 1\right)\underset{\sim}{i} + (\sin(t)-1)\underset{\sim}{j} + \left(t^2 + 1 - \pi^2\right)\underset{\sim}{k}.$$ **[M1]**

Substitute $t = \dfrac{3\pi}{2}$:

$$\underset{\sim}{r} = \left(-4\cos\left(\frac{3\pi}{4}\right) - 1\right)\underset{\sim}{i} + (\sin\left(\frac{3\pi}{4}\right) - 1)\underset{\sim}{j} + \left(\frac{9\pi^2}{4} + 1 - \pi^2\right)\underset{\sim}{k}$$

$$= \left(\frac{4}{\sqrt{2}} - 1\right)\underset{\sim}{i} + \left(\frac{1}{\sqrt{2}} - 1\right)\underset{\sim}{j} + \left(\frac{5\pi^2}{4} + 1\right)\underset{\sim}{k} \quad = (2\sqrt{2} - 1)\underset{\sim}{i} + \left(\frac{\sqrt{2} - 2}{2}\right)\underset{\sim}{j} + \left(\frac{5\pi^2 + 4}{4}\right)\underset{\sim}{k}.$$ **[A1]**

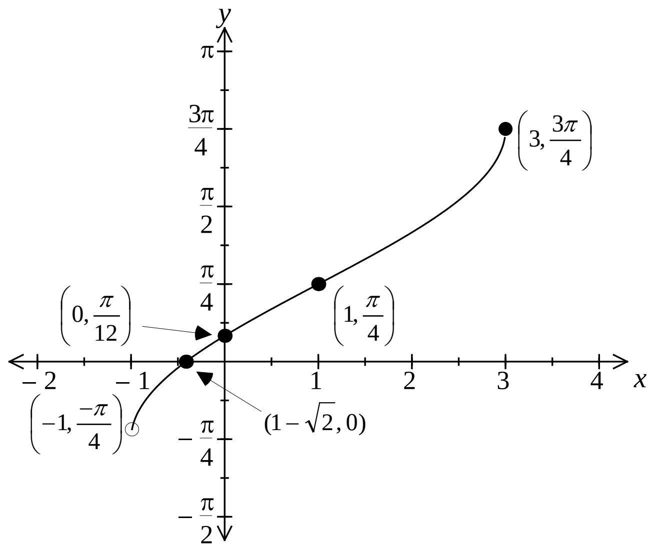
Total 3 marks

Question 6

a.

$$f(x) = \frac{3\pi}{4} - \cos^{-1}\left(\frac{x-1}{2}\right) = -\cos^{-1}\left(\frac{1}{2}(x-1)\right) + \frac{3\pi}{4}.$$

The graph of $y = \cos^{-1}(x)$ is dilated from the y-axis by a factor of 2, translated along the x-axis by 1 unit, reflected in the x-axis and translated along the y-axis by $\frac{3\pi}{4}$ units:



x-intercept: $0 = \frac{3\pi}{4} - \cos^{-1}\left(\frac{x-1}{2}\right)$

$\Rightarrow \frac{3\pi}{4} = \cos^{-1}\left(\frac{x-1}{2}\right)$

$\Rightarrow \frac{-1}{\sqrt{2}} = \frac{x-1}{2}$

$\Rightarrow x = 1 - \sqrt{2}$. **[A1]**

y-intercept: $y = \frac{3\pi}{4} - \cos^{-1}\left(\frac{-1}{2}\right) = \frac{3\pi}{4} - \frac{2\pi}{3} = \frac{\pi}{12}$. **[A $\frac{1}{2}$]**

Endpoints **[A $\frac{1}{2}$]**

Shape **[A $\frac{1}{2}$**]

Inflection point **[A $\frac{1}{2}$]**

Marks totaled and rounded down.

b. i.

$$V = \pi \int_a^b x^2 dy \,.$$

$x = 1$: $\quad y = \frac{3\pi}{4} - \cos^{-1}(0) = \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$.

$x = 1 + \sqrt{2}$: $\quad y = \frac{3\pi}{4} - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}$.

Both values of y **[M1]**

$$y = \frac{3\pi}{4} - \cos^{-1}\left(\frac{x-1}{2}\right) \Rightarrow \frac{3\pi}{4} - y = \cos^{-1}\left(\frac{x-1}{2}\right) \Rightarrow \cos\left(\frac{3\pi}{4} - y\right) = \frac{x-1}{2}$$

$$\Rightarrow x = 2\cos\left(\frac{3\pi}{4} - y\right) + 1 \,.$$

Therefore: $\quad V = \pi \int_{\pi/4}^{\pi/2} \left(2\cos\left(\frac{3\pi}{4} - y\right) + 1\right)^2 dy \,.$ **[A1]**

b. ii.
Solution 1:

Substitute $u = \frac{3\pi}{4} - y$:

$$V = -\pi \int_{\pi/2}^{\pi/4} (2\cos(u) + 1)^2 du = \pi \int_{\pi/4}^{\pi/2} (2\cos(u) + 1)^2 du \,.$$ **[M1]**

Expand:

$$V = \pi \int_{\pi/4}^{\pi/2} 4\cos(u)^2 + 4\cos(u) + 1 \, du$$

$$= \pi \int_{\pi/4}^{\pi/2} 4\cos^2(u) \, du + \pi \int_{\pi/4}^{\pi/2} 4\cos(u) + 1 \, du$$

$$\boxed{= \pi \int_{\pi/4}^{\pi/2} 2\cos(2u) + 2 \, du} + \pi \int_{\pi/4}^{\pi/2} 4\cos(u) + 1 \, du$$ **[M1]**

$$= \pi \int_{\pi/4}^{\pi/2} 2\cos(2u) + 3 + 4\cos(u)\, du$$

$$= \pi\left[\sin(2u) + 3u + 4\sin(u)\right]_{\pi/4}^{\pi/2}$$ **[M1]**

$$= \pi\left[\left(0 + \frac{3\pi}{2} + 4\right) - \left(1 + \frac{3\pi}{4} + 2\sqrt{2}\right)\right]$$

$$= \pi\left[\frac{3\pi}{4} + 3 - 2\sqrt{2}\right] = \pi\left[\frac{12 + 3\pi - 8\sqrt{2}}{4}\right] \text{ cubic units.}$$ **[A1]**

Total 9 marks

Solution 2:

Substitute $\cos\left(\frac{3\pi}{4} - y\right) = \cos\left(-\left(y - \frac{3\pi}{4}\right)\right) = \cos\left(y - \frac{3\pi}{4}\right)$:

$$V = \pi \int_{\pi/4}^{\pi/2} \left(2\cos\left(y - \frac{3\pi}{4}\right) + 1\right)^2 dy\,.$$

Substitute $u = y - \frac{3\pi}{4}$:

$$V = \pi \int_{-\pi/2}^{-\pi/4} \left(2\cos(u) + 1\right)^2 du\,.$$ **[M1]**

Expand:

$$V = \pi \int_{-\pi/2}^{-\pi/4} 4\cos(u)^2 + 4\cos(u) + 1\, du$$

$$= \pi \int_{-\pi/2}^{-\pi/4} 4\cos^2(u)\, du + \pi \int_{-\pi/2}^{-\pi/4} 4\cos(u) + 1\, du$$

$$= \pi \int_{-\pi/2}^{-\pi/4} 2\cos(2u) + 2\, du + \pi \int_{-\pi/2}^{-\pi/4} 4\cos(u) + 1\, du$$ **[M1]**

where the first integral follows from using $4\cos^2(u) = 2\cos(2u) + 2$

$$= \pi \int_{-\pi/2}^{-\pi/4} 2\cos(2u) + 3 + 4\cos(u)\, du$$

$$= \pi \left[\sin(2u) + 3u + 4\sin(u)\right]_{-\pi/2}^{-\pi/4}$$ **[M1]**

$$= \pi\left[\left(-1 - \frac{3\pi}{4} - 2\sqrt{2}\right) - \left(0 - \frac{3\pi}{2} - 4\right)\right]$$

$$= \pi\left[\frac{3\pi}{4} + 3 - 2\sqrt{2}\right] = \pi\left[\frac{12 + 3\pi - 8\sqrt{2}}{4}\right]$$ cubic units. **[A1]**

Total 9 marks

Question 7

Vertical asymptote at $x = -1$: $(-1)^2 + 2a(-1) + b = 0 \Rightarrow 2a - b = 1$. …. (1) **[M1]**

Range of $(-\infty, 0) \cup \left[\frac{1}{4}, +\infty\right) \Rightarrow$ *y*-coordinate of the turning point is $y = \frac{1}{4}$.

The *x*-coordinate of the turning point of the reciprocal quadratic function graph is the same as the *x*-coordinate of the turning point of $g(x) = x^2 + 2ax + b$:

$$x = -a.$$

Therefore the *y*-coordinate of the turning point is $y = \frac{a}{(-a)^2 + 2a(-a) + b} = \frac{a}{b - a^2}$: **[M1]**

$$\frac{1}{4} = \frac{a}{b - a^2} \Rightarrow b - a^2 = 4a.$$ …. (2) **[M1]**

Solve equations (1) and (2) simultaneously for *a* and *b*. Substitute equation (1) into equation (2):

$2a - 1 - a^2 = 4a$

$\Rightarrow a^2 + 2a + 1 = 0$

$\Rightarrow (a + 1)^2 = 0$

$\Rightarrow a = -1$.

Substitute $a = -1$ into equation (1): $b = -3$.

Therefore $a = -1$ and $b = -3$. **[A1]**

Total 4 marks

Question 8

Solution 1:

$-\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}=\alpha(2m\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k})+\beta(5m\underset{\sim}{i}-11\underset{\sim}{j}+5\underset{\sim}{k})$ where $\alpha,\beta\in R$

$=(2m\alpha+5m\beta)\underset{\sim}{i}+(-\alpha-11\beta)\underset{\sim}{j}+(3\alpha+5\beta)\underset{\sim}{k}$. **[M1]**

Equate components:

$\underset{\sim}{i}$-component: $-1=2m\alpha+5m\beta$ …. (1)

$\underset{\sim}{j}$-component: $2=-\alpha-11\beta$ …. (2)

$\underset{\sim}{k}$-component: $1=3\alpha+5\beta$ …. (3)

Solve equation (2) and equation (3) simultaneously for α and β:

$\alpha=\frac{3}{4}$ and $\beta=\frac{1}{4}$. **[M1]**

Substitute into equation (1) and solve for *m*:

$m=-4$. **[A1]**

Total 3 marks

Solution 2:

$2m\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}=\mu(-\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k})+\lambda(5m\underset{\sim}{i}-11\underset{\sim}{j}+5\underset{\sim}{k})$ where $\mu,\lambda\in R$

$=(-\mu+5m\lambda)\underset{\sim}{i}+(2\mu-11\lambda)\underset{\sim}{j}+(\mu+5\lambda)\underset{\sim}{k}$. **[M1]**

$\mu=\frac{4}{3}$ and $\lambda=\frac{1}{3}$. **[M1]**

$m=-4$. **[A1]**

Total 3 marks

Solution 3:

$5m\underset{\sim}{i}-11\underset{\sim}{j}+5\underset{\sim}{k}=\gamma(-\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k})+\delta(2m\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k})$ where $\gamma,\delta\in R$

$=(-\gamma+2m\delta)\underset{\sim}{i}+(2\gamma-\delta)\underset{\sim}{j}+(\gamma+3\delta)\underset{\sim}{k}$. **[M1]**

$\gamma=-4$ and $\delta=3$. **[M1]**

$m=-4$. **[A1]**

Total 3 marks

Question 9

a.
Solution 1:

$a = v\frac{dv}{dx} = \frac{1}{v+3}$ **[M1]**

$\Rightarrow \frac{dv}{dx} = \frac{1}{v(v+3)}$

$\Rightarrow \frac{dx}{dv} = v(v+3) = v^2 + 3v$

$\Rightarrow x = \int v^2 + 3v \, dv$

$= \frac{1}{3}v^3 + \frac{3}{2}v^2 + C$. **[M1]**

Substitute $v = 0$ and $x = 0$ when $t = 0$: $C = 0$.

Therefore $x = \frac{1}{3}v^3 + \frac{3}{2}v^2$.

Substitute $v = 1$:

$x = \frac{1}{3} + \frac{3}{2} = \frac{11}{6}$. **[A1]**

Solution 2:

From **part b.** $t = \frac{1}{2}v^2 + 3v$. Substitute $v = 1$:

$t = \frac{7}{2}$. **[M1]**

From **part b.**: $v = -3 + \sqrt{9+2t}$

$\Rightarrow \frac{dx}{dt} = -3 + \sqrt{9+2t}$

$\Rightarrow x = \int -3 + \sqrt{9+2t} \, dt$

$\Rightarrow x = -3t + \frac{1}{3}(9+2t)^{3/2} + C$. **[M1]**

Substitute $x = 0$ when $t = 0$: $0 = 9 + C \Rightarrow C = -9$.

Therefore $x = -3t + \frac{1}{3}(9+2t)^{3/2} - 9$.

Substitute $t = \frac{7}{2}$:

$$x = \frac{-21}{2} + \frac{1}{3}(16)^{3/2} - 9 = \frac{-21}{2} + \frac{64}{3} - 9$$

$= \frac{11}{6}$. **[A1]**

b.

$$a = \frac{dv}{dt} = \frac{1}{v+3}$$

$\frac{dt}{dv} = v + 3$ **[M1]**

$$\Rightarrow t = \int v + 3\, dv$$

$$= \frac{1}{2}v^2 + 3v + K.$$

Substitute $v = 0$ when $t = 0$: $K = 0$.

Therefore $t = \frac{1}{2}v^2 + 3v$. **[M1]**

Re-arrange into standard quadratic equation form:

$$v^2 + 6v - 2t = 0.$$

Solve for v:

$$v = \frac{-6 \pm \sqrt{6^2 - 4(1)(-2t)}}{2} = \frac{-6 \pm \sqrt{36 + 8t}}{2} = -3 \pm \sqrt{9 + 2t}.$$

But $v = 0$ when $t = 0$ and so the negative root solution is rejected. Expression for v and reason for rejection of negative root solution **[M1]**

Therefore $v = -3 + \sqrt{9 + 2t}$ where $b = 9$.

Total 6 marks

Question 10

Solution 1:

Let $y = f^{-1}(x)$. Then $x = ye^y$.

Substitute $x = e$: $e = ye^y$

$\Rightarrow y = 1$ (by inspection). $x = ye^y$ and $y = 1$ **[M1]**

Implicit differentiation: $x = ye^y$

$$\Rightarrow 1 = \frac{dy}{dx}e^y + ye^y\frac{dy}{dx}.$$ **[M1]**

Substitute $y = 1$: $1 = \frac{dy}{dx}e + e\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2e}.$$

Therefore $m_{\text{normal}} = -2e$. **[A1]**

Total 3 marks

Solution 2:

Let $y = f^{-1}(x)$. Then $x = ye^y$.

Substitute $x = e$: $e = ye^y$

$\Rightarrow y = 1$ (by inspection). $x = ye^y$ and $y = 1$ **[M1]**

Differentiate: $x = ye^y$

$$\Rightarrow \frac{dx}{dy} = e^y + ye^y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y + ye^y}.$$ **[M1]**

Substitute $y = 1$: $\frac{dy}{dx} = \frac{1}{2e}$.

Therefore $m_{\text{normal}} = -2e$. **[A1]**

Total 3 marks

