The Mathematical Association of Victoria

Trial Exam 2012

SPECIALIST MATHEMATICS

Written Examination 1

STUDENT NAME ____

Reading time: 15 minutes
Writing time: 1 hour

QUESTION AND ANSWER BOOK

Structure of Book

Number of questions	*Number of questions to be answered*	*Number of marks*
10	10	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers,
- Students are NOT permitted to bring into the examination room: notes of any kind, a calculator of any type, blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 13 pages with a detachable sheet of miscellaneous formulas at the back.

Instructions
- Detach the formula sheet from the back of this book during reading time.
- Write your name in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$

Question 1

A helicopter is rising with a constant velocity of 20 m/s. When the helicopter is 60 m above the ground a parcel is dropped out of it. Find an equation in the form

$gt^2 + bt + c = 0$, where g is the magnitude of the acceleration due to gravity and b and c are integers,

which, when solved will give the exact time it takes for the parcel to hit the ground.

2 marks

Question 2

An object of mass 11 kg **at rest** on a rough horizontal surface is being pulled by a force of magnitude 20 newtons acting in a direction 30 degrees to the horizontal. The coefficient of friction between the object and the surface is 0.2.

a. In the space below draw a diagram that shows all the forces acting on the object. You must clearly label these forces.

1 mark

b. Find the exact size of the friction force acting on the object.

2 marks

Total 3 marks

Question 3

Consider the function $f:(-1,+\infty)\to R$, $f(x)=\tan^{-1}\left(\dfrac{2}{\sqrt{x+1}}\right)$.

a. Find $f'(x)$ in the form $\dfrac{a}{(x+b)(x+1)^c}$ where $a,b,c\in R$.

2 marks

b. Find the exact value of x when $f(x)=\dfrac{\pi}{3}$.

1 mark

Total 3 marks

Question 4

a. Find in polar form all numbers $w \in C$ such that $w^3 + 4 - i4\sqrt{3} = 0$.

3 marks

b. Hence state in polar form all numbers $u \in C$ such that $u^3 + 4 + i4\sqrt{3} = 0$.

1 mark

Total 4 marks

Question 5

Relative to an origin O, a toy train is at the point with coordinates $(-1, -1, 1)$ at time $t = \pi$ and has velocity $\underset{\sim}{v} = 2\sin\left(\frac{t}{2}\right)\underset{\sim}{i} + \cos(t)\underset{\sim}{j} + (2t)\underset{\sim}{k}$ at time t.

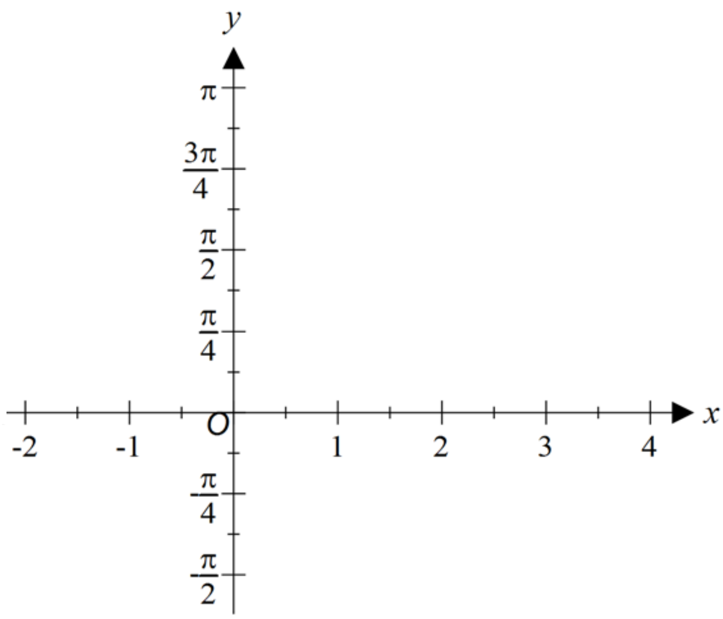
Find the position vector of the toy train when $t = \frac{3\pi}{2}$.

3 marks

Question 6

Consider the function *f* with rule $f(x) = \frac{3\pi}{4} - \cos^{-1}\left(\frac{x-1}{2}\right)$.

a. Sketch the graph of $y = f(x)$ on the axes below.
Label the endpoints and the point of inflection with their **exact coordinates**, and label all axes intercepts with their exact values.



3 marks

The part of the graph drawn above over the interval $1 \le x \le 1+\sqrt{2}$ is rotated about the *y*-axis to form a solid of revolution.

b. **i.** Write down a **definite integra**l in terms of *y*, which when evaluated will give the volume of this solid.

ii. Find the exact value of the integral in **part i.**

2 + 4 = 6 marks

Total 9 marks

Question 7

The graph of $y = \dfrac{a}{x^2 + 2ax + b}$, where $a, b \in R$, has a range of $(-\infty, 0) \cup \left[\dfrac{1}{4}, +\infty\right)$ and one of its vertical asymptotes has the equation $x = -1$.

Find the exact values of *a* and *b*.

4 marks

Question 8

Consider the three vectors

$$-\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k},\ \ 2m\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}\ \text{ and }\ 5m\underset{\sim}{i}-11\underset{\sim}{j}+5\underset{\sim}{k},\ \text{where } m\in R.$$

Find the value(s) of *m* so that the three vectors are linearly **dependent**.

3 marks

Question 9

A mass has acceleration a m/s^2 given by $a = \frac{1}{v+3}$ where v m/s is the velocity of the mass after t seconds. The displacement of the mass from the origin after t seconds is x m. Given that $v = 0$ and $x = 0$ when $t = 0$:

a. Find the exact position of the mass when $v = 1$.

3 marks

b. Show that $v = -3 + \sqrt{2t + b}$ where b is an integer.

3 marks
Total 6 marks

Question 10

Consider the function $f:[-1,+\infty)\to R$, $f(x)=xe^{x}$.

Find, in simplest form, the gradient of the normal to the **inverse function** f^{-1} at the point where $x=e$.

3 marks

END OF QUESTION AND ANSWER BOOK

SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.

Specialist Mathematics Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
cosine rule:	$c^2 = a^2 + b^2 - 2ab\cos C$

Coordinate geometry

ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

hyperbola: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Circular (trigonometric) functions

$\cos^2(x) + \sin^2(x) = 1$

$1 + \tan^2(x) = \sec^2(x)$

$\cot^2(x) + 1 = \text{cosec}^2(x)$

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\sin(2x) = 2\sin(x)\cos(x)$

$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

function	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1, 1]$	$[-1, 1]$	R
range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$

$|z| = \sqrt{x^2 + y^2} = r$ $\qquad -\pi < \text{Arg}\, z \le \pi$

$z_1 z_2 = r_1 r_2\,\text{cis}(\theta_1 + \theta_2)$ $\qquad \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$

$z^n = r^n\,\text{cis}(n\theta)$ (de Moivre's theorem)

Calculus

$\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$ $\qquad \int x^n dx = \dfrac{1}{n+1}x^{n+1} + c, n \ne -1$

$\dfrac{d}{dx}\left(e^{ax}\right) = ae^{ax}$ $\qquad \int e^{ax} dx = \dfrac{1}{a}e^{ax} + c$

$\dfrac{d}{dx}\left(\log_e(x)\right) = \dfrac{1}{x}$ $\qquad \int \dfrac{1}{x} dx = \log_e|x| + c$

$\dfrac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$ $\qquad \int \sin(ax)\,dx = -\dfrac{1}{a}\cos(ax) + c$

$\dfrac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$ $\qquad \int \cos(ax)\,dx = \dfrac{1}{a}\sin(ax) + c$

$\dfrac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$ $\qquad \int \sec^2(ax)\,dx = \dfrac{1}{a}\tan(ax) + c$

$\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad \int \dfrac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\dfrac{x}{a}\right) + c, a > 0$

$\dfrac{d}{dx}\left(\cos^{-1}(x)\right) = \dfrac{-1}{\sqrt{1-x^2}}$ $\qquad \int \dfrac{-1}{\sqrt{a^2 - x^2}}dx = \cos^{-1}\left(\dfrac{x}{a}\right) + c, a > 0$

$\dfrac{d}{dx}\left(\tan^{-1}(x)\right) = \dfrac{1}{1+x^2}$ $\qquad \int \dfrac{a}{a^2 + x^2}dx = \tan^{-1}\left(\dfrac{x}{a}\right) + c$

product rule: $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

quotient rule: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

Euler's method: If $\dfrac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$

acceleration: $a = \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at \qquad s = ut + \dfrac{1}{2}at^2 \qquad v^2 = u^2 + 2as \qquad s = \dfrac{1}{2}(u + v)t$

TURN OVER

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$|\underset{\sim}{r}| = \sqrt{x^2 + y^2 + z^2} = r$

$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2$

$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$

Mechanics

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

friction: $F \le \mu N$

END OF FORMULA SHEET