Neap:

Trial Examination 2012

VCE Specialist Mathematics Units 3 & 4

Written Examination 1

Suggested Solutions

Question 1

Using the chain rule $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ where $\frac{dy}{dt} = 3\cos(3t)$ and $\frac{dx}{dt} = e^t$ we get $\frac{dy}{dt} = 3\cos(3t) \cdot e^{-t}$ M1

At $(1, 0)$, $1 = e^t \Rightarrow t = 0$

Thus the gradient of the tangent at $(1, 0)$ is $\frac{dy}{dx} = 3\cos(0) \cdot e^0 = 3$

$\therefore$ the equation of the tangent is $y - 0 = 3(x - 1) \Rightarrow y = 3x - 3$ A1

Question 2

Let $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}}$

Changing the terminals: $x = 1 \Rightarrow u = 1$, $x = 4 \Rightarrow u = 2$ M1

So $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$ is equivalent to $\int_1^2 2e^u du = [2e^u]_1^2 = 2(e^2 - e) = 2e(e - 1)$ A1

Question 3

a. $\frac{dh}{dt} = -\frac{\sqrt{h}}{10} \Rightarrow \frac{dt}{dh} = -\frac{10}{\sqrt{h}}$

Thus $t = \int -10h^{-\frac{1}{2}} dh \Rightarrow t = -20h^{\frac{1}{2}} + c$ or $t = \left[-20h^{\frac{1}{2}}\right]_4^h$ M1

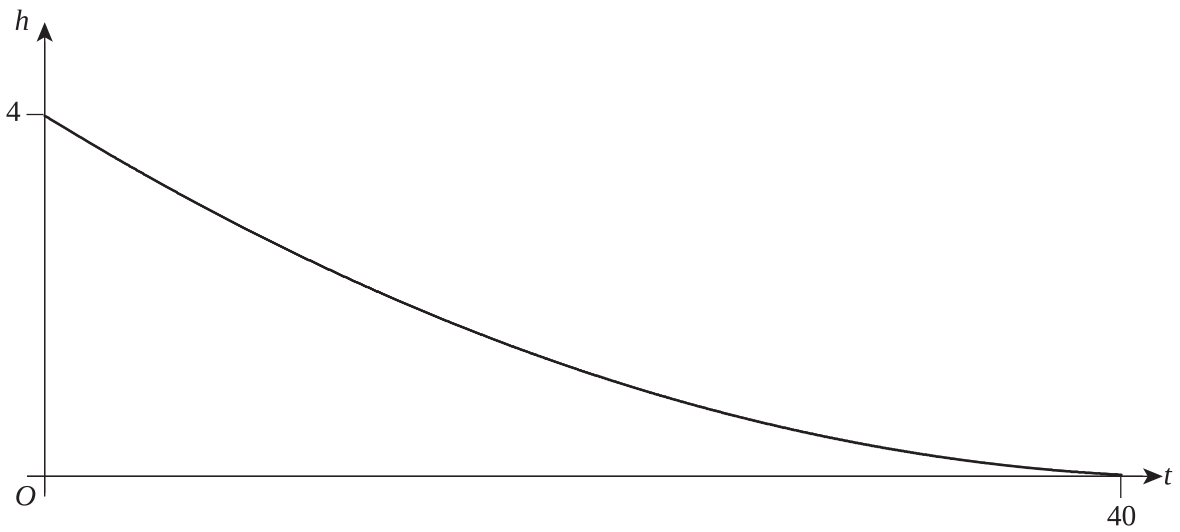
As $t = 0$, $h = 4$, $0 = -20\sqrt{4} + c \Rightarrow c = 40$ $= -20\sqrt{h} + 40$

Thus $t = 40 - 20\sqrt{h}$

So we have $20\sqrt{h} = 40 - t \Rightarrow \sqrt{h} = 2 - \frac{t}{20}$

Thus $h = \left(2 - \frac{t}{20}\right)^2$ A1

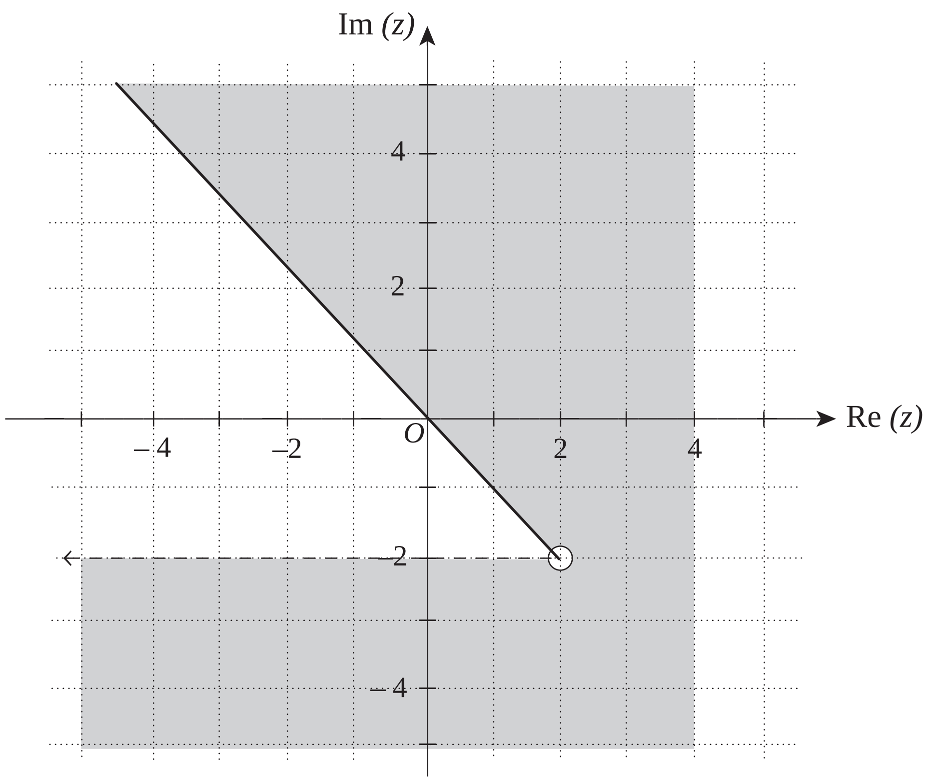
b.



parabola over domain $[0, 40]$ A1

Question 4

a.

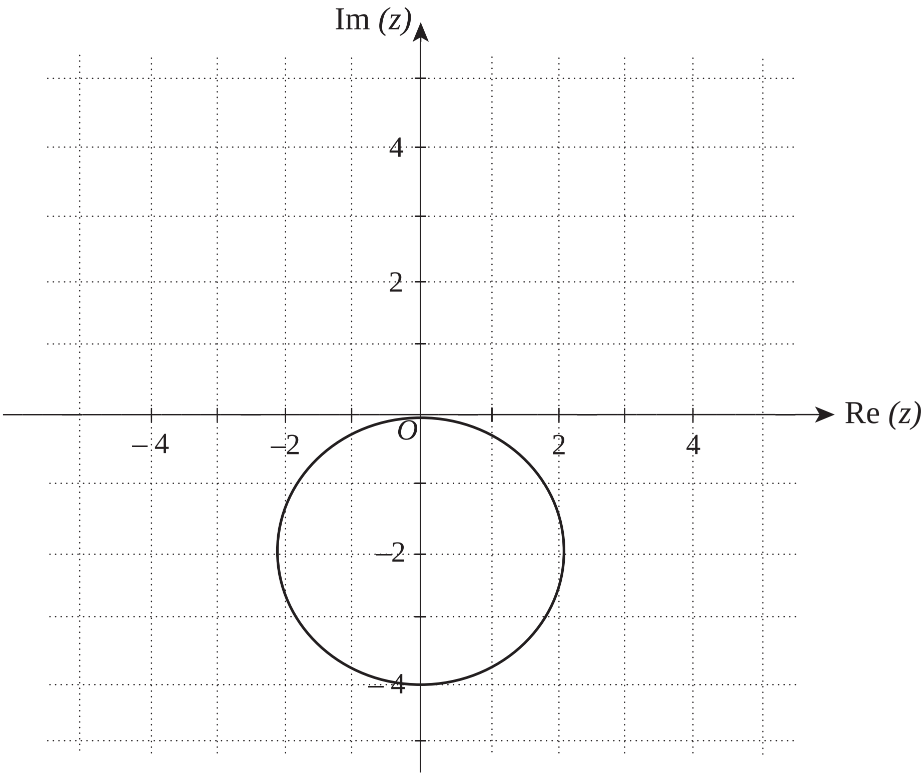


$\text{Arg}\left(z - 2 + 2i\right) \le \frac{3\pi}{4}$

Ray starting at $2 - 2i$*, at angle of* $\frac{3\pi}{4}$ A1

Shaded region, with dotted ray starting at $(2, -2)$ *extending horizontally* A1

b.



Standard form of circle centre z_1, radius r^2 is $(z - z_1)(\bar{z} - \bar{z}_1) = r^2$

so $(z + 2i)(\bar{z} - 2i) = 4$ is a circle with centre (0, –2) and radius 2.

OR

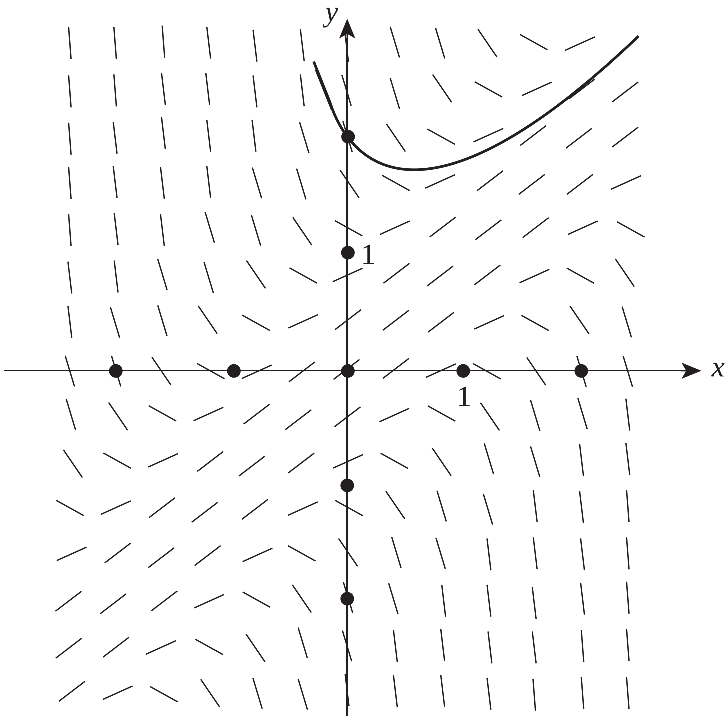
$$(x + yi + 2i)(x - yi - 2i) = 4$$
$$x^2 - xyi - 2xi + xyi + y^2 + 2y + 2xi + 2y + 4 = 4$$
$$x^2 + y^2 + 4y + 4 = 4$$
$$x^2 + (y + 2)^2 = 4$$

circle centre $(-2, 0)$ A1

radius = 2 A1

Question 5

a.



correct shape and passing through $(0, 2)$ A1

b. **i.** Equation (A) is not correct because it represents a differential equation, which does not depend on *x*. The slopes would need to be constant in the horizontal direction, for each value of *y*. A1

ii. Equation (B) is correct. It predicts that if $y = x$, $\frac{dy}{dx} = 1$, which is observed for all the slopes along the line $y = x$. Equation (C) does not give this result. A1

Question 6

Let $y = x\arctan(x) - \frac{1}{2}\log_e(x^2 + 1)$

$$\frac{dy}{dx} = x \cdot \frac{1}{x^2 + 1} + \arctan(x) - \frac{1}{2}\left(\frac{2x}{x^2 + 1}\right) = \frac{x}{x^2 + 1} + \arctan(x) - \frac{x}{x^2 + 1} = \arctan(x)$$ M1 A1

Thus $$\int_{-\sqrt{3}}^{1} \arctan(x)dx = \left[x\arctan(x) - \frac{1}{2}\log_e(x^2 + 1)\right]_{-\sqrt{3}}^{1}$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\log_e(2)\right) - \left(-\sqrt{3}\left(-\frac{\pi}{3}\right) - \frac{1}{2}\log_e(4)\right)$$ M1

$$= \frac{\pi}{4} - \frac{\sqrt{3}}{3}\pi + \frac{1}{2}\log_e(2)$$ A1

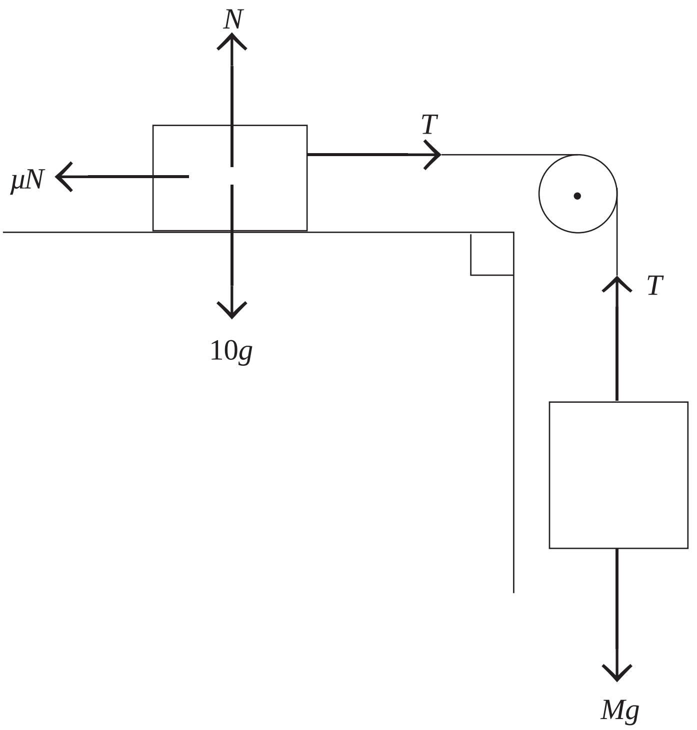
OR

$$= \pi\left(\frac{1}{4} - \frac{\sqrt{3}}{3}\right) + \frac{1}{2}\log_e(2)$$

or

A1

Question 7



a. $N = 10g$, $Fr = \mu N$

$= 0.2 \times 10g$

$= 2g$ M1

$T = 2g$

$Mg = 2g$

$M = 2$ A1

b. If $M = 4$, the resultant force $= 4g - 2g$

$= 2g$ M1

Using $R_F = ma$

$a = \frac{R_F}{M}$

$= \frac{2g}{14}$

$= \frac{g}{7}$ A1

OR

At 10 kg, $T - 2g = 10a$

At 4 kg, $4g - T = 4a$

$2g = 14a$

$a = \frac{g}{7}$ A1

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Question 8

$\cos(2x) = \cos^2(x)$

$2\cos^2(x) - 1 = \cos^2(x)$ M1

$\cos^2(x) = 1$

$\cos(x) = \pm 1$ A1

$x = 0, \pi, 2\pi$

$\therefore$ the points of intersection are $(0, 1)$, $(\pi, 1)$ and $(2\pi, 1)$. A1

Question 9

a. $\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$

$= (3\underset{\sim}{i} - 4\underset{\sim}{j} + m\underset{\sim}{k}) - (\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k})$

$= 2\underset{\sim}{i} - 2\underset{\sim}{j} + (m - 3)\underset{\sim}{k}$ A1

b. $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$

$= (-\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}) - (\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k})$

$= -2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ M1

We require $\overrightarrow{QP} \cdot \overrightarrow{QR} = 0$

$(-2\underset{\sim}{i} - \underset{\sim}{j} + 2k) \cdot (2\underset{\sim}{i} - 2\underset{\sim}{j} + (m - 3)\underset{\sim}{k}) = 0$ M1

$-4 + 2 + 2(m - 3) = 0$

$m = 4$ A1

c. $\left|\overrightarrow{QP}\right| = \left|-2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}\right| = \sqrt{(-2)^2 + (-1)^2 + (2)^2} = 3$

$\left|\overrightarrow{QR}\right| = \left|2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right| = \sqrt{(2)^2 + (-2)^2 + (1)^2} = 3$

$\therefore$ PQR is an isosceles triangle. A1

Question 10

a. The area A of the region is given by $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{e^{\tan(x)}}{1-\sin^2 x}dx$

Notice that $\frac{e^{\tan(x)}}{1-\sin^2 x} = \frac{e^{\tan(x)}}{\cos^2(x)} = e^{\tan(x)}\sec^2(x)$ M1

Thus $A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{\tan(x)}\sec^2(x)dx = [e^{\tan(x)}]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$ A1

$A = \left[e^{\tan\left(\frac{\pi}{4}\right)} - e^{\tan\left(-\frac{\pi}{4}\right)}\right] = e - e^{-1} = e - \frac{1}{e}$ A1

OR

$u = \tan(x)$

$\frac{dy}{dx} = \frac{1}{\cos^2(x)}$

$A = \int_{-1}^{1} e^u du$ M1

$= [e^u]_{-1}^{1}$ A1

$= e - \frac{1}{2}$ A1

b. $V\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{e^{\tan(x)}}{1-\sin^2(x)}\right)^2 dx = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{e^{\tan(x)}}{\cos^2(x)}\right)^2 dx = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (e^{\tan(x)}\sec^2(x))^2 dx$ M1

Thus $V = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{2\tan(x)}\sec^4(x)dx$, giving $k = 2, m = 4$. A1

Question 11

a. Substituting $x = \frac{\pi}{2}$ into $\cos(x) + e^{xy} = 2$ gives $\cos\left(\frac{\pi}{2}\right) + e^{\frac{\pi y}{2}} = 2 \Rightarrow \frac{\pi y}{2} = \log_e(2)$

Thus $y = \frac{2\log_e(2)}{\pi}$ A1

b. Using implicit differentiation,

$$-\sin(x) + \frac{d}{dx}(e^{xy}) = 0$$

$$-\sin(x) + e^{xy}\left(x\frac{dy}{dx} + y\right) = 0$$

$$x\frac{dy}{dx} + y = \frac{\sin(x)}{e^{xy}} \Rightarrow \frac{dy}{dx} = \frac{1}{x}\left(\frac{\sin(x)}{e^{xy}} - y\right)$$ M1 A1

c. At $x = \frac{\pi}{2}$, $y = \frac{2\log_e(2)}{\pi}$,

$$\frac{dy}{dx} = \frac{1}{\frac{\pi}{2}}\left(\frac{\sin\left(\frac{\pi}{2}\right)}{e^{\frac{\pi}{2} \times \frac{2\log_e(2)}{\pi}}} - \frac{2\log_e(2)}{\pi}\right)$$

$$= \frac{2}{\pi}\left(\frac{1}{e^{\log_e(2)}} - \frac{2\log_e(2)}{\pi}\right)$$

$$= \frac{2}{\pi}\left(\frac{1}{2} - \frac{2\log_e(2)}{\pi}\right)$$

$$= \frac{1}{\pi} - \frac{4\log_e(2)}{\pi^2}$$ M1

Thus the gradient of the tangent is $\frac{\pi - 4\log_e(2)}{\pi^2}$, giving the value for the gradient of the normal as

$\frac{\pi^2}{4\log_e(2) - \pi}$ as required. A1