INSIGHT

YEAR 12 ***Trial Exam Paper***

2013

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

Question 1a.

Worked solution

$$z^4 + 2z^2 = 8$$
$$z^4 + 2z^2 - 8 = 0$$
$$(z^2 + 4)(z^2 - 2) = 0$$
$$(z^2 - 4i^2)(z^2 - 2) = 0$$
$$(z + 2i)(z - 2i)(z + \sqrt{2})(z - \sqrt{2}) = 0$$
$$\therefore \; z = \pm 2i \;\text{ or }\; z = \pm\sqrt{2}$$

Mark allocation: 2 marks

- 1 mark for factorising correctly.
- 1 mark for the correct answer.

Tip

- *The quadratic formula in* z^2 *could also have been used to solve the equation.*

Question 1b.

Worked solution

$$z_1 = 2\operatorname{cis}\left(\frac{\pi}{4}\right)$$

$$z_2 = \operatorname{cis}\left(\frac{-3\pi}{4}\right)$$

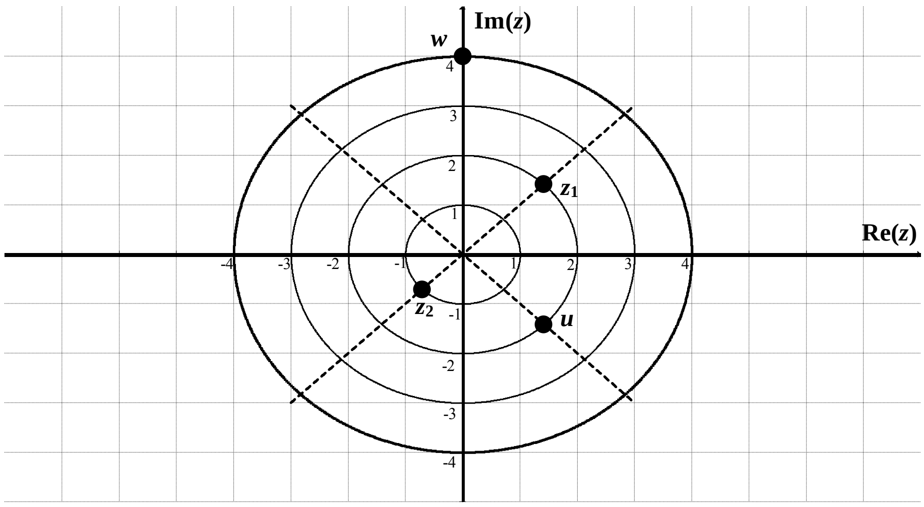
$$u = \frac{z_1}{{z_2}^2} = \frac{2\operatorname{cis}\left(\frac{\pi}{4}\right)}{\left(\operatorname{cis}\left(\frac{-3\pi}{4}\right)\right)^2}$$

$$\Rightarrow u = 2\operatorname{cis}\left(\frac{7\pi}{4}\right) = 2\operatorname{cis}\left(\frac{-\pi}{4}\right)$$

$$w = i^2\left(\overline{z}_1\right)^2 = i^2 \times \left(2\operatorname{cis}\left(\frac{-\pi}{4}\right)\right)^2$$

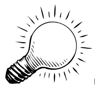
$$= \operatorname{cis}(\pi) \times 4\operatorname{cis}\left(\frac{-\pi}{2}\right)$$

$$\therefore\ w = 4\operatorname{cis}\left(\frac{\pi}{2}\right)$$



Mark allocation: 2 marks

- 1 mark for the correct position of *u*.
- 1 mark for the correct position of *w*.



Tip

- *The point* w *can also be obtained by rotating the point* $\left(\overline{z}_1\right)^2 = 4\operatorname{cis}\left(\frac{-\pi}{2}\right)$, *180° anticlockwise.*

Question 2a.

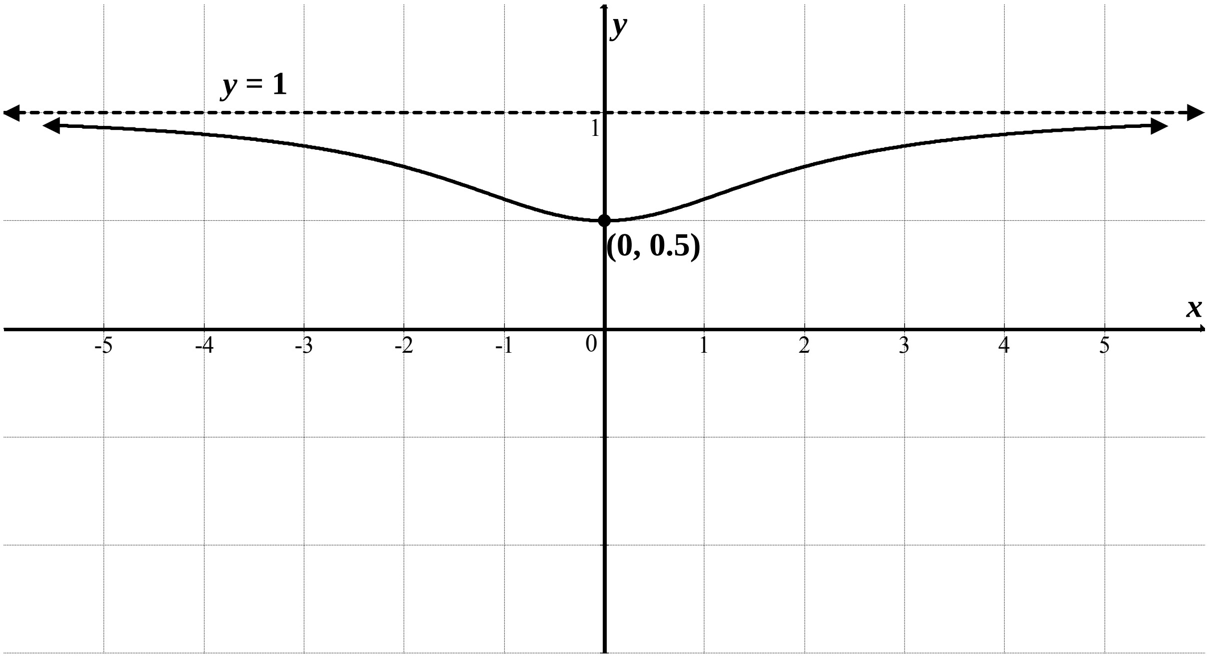
Worked solution

$f(x) = 1 - \dfrac{2}{x^2+4}$ has a horizontal asymptote $y = 1$.

$f(0) = 1 - 0.5 = 0.5$

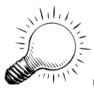
The point (0, 0.5) is also a local minimum.

As $x \to \infty,\ y \to 1$.
As $x \to -\infty,\ y \to 1$.



Mark allocation: 3 marks

- 1 mark for the correct asymptote.
- 1 mark for the correct y-intercept.
- 1 mark for the correct curve.



Tip

- *The graph of* $y = f(x)$ *can also be obtained by first sketching* $y = x^2 + 4$, *then sketching its reciprocal* $y = \dfrac{1}{x^2+4}$, *then reflecting through the* x-*axis for* $y = \dfrac{-1}{x^2+4}$, *then dilating in the vertical by factor 2 for* $y = \dfrac{-2}{x^2+4}$ *and, finally, translating 1 unit up for* $y = f(x)$.

Question 2b.

Worked solution

$$\text{Area} = \int_0^2 \left(1 - \frac{2}{x^2+4}\right) dx$$

$$= \left[x - \tan^{-1}\left(\frac{x}{2}\right) \right]_0^2$$

$$= \left(2 - \tan^{-1}(1)\right) - \left(0 - \tan^{-1}(0)\right)$$

$$= 2 - \frac{\pi}{4}$$

$$= \frac{8-\pi}{4}$$

$\therefore$ The area enclosed is $\left(\frac{8-\pi}{4}\right)$ square units.

Mark allocation: 3 marks

- 1 mark for correctly setting up the integral representing the required area.
- 1 mark for anti-differentiating correctly.
- 1 mark for the correct answer.

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Question 3

Worked solution

$$\int_0^{e-1} \frac{\sin(\pi \log_e(x+1))}{x+1}\,dx$$

Let $u = \log_e(x + 1)$

When $x = e-1 \Rightarrow u = \log_e e = 1$

When $x = 0 \Rightarrow u = \log_e 1 = 0$

$$\frac{du}{dx} = \frac{1}{x+1}$$

$$\int_0^{e-1} \frac{\sin(\pi \log_e(x+1))}{x+1}\,dx$$

$$= \int_0^1 \sin(\pi u)\frac{du}{dx}\,dx$$

$$= \int_0^1 \sin(\pi u)\,du$$

$$= \frac{-1}{\pi}\left[\cos(\pi x)\right]_0^1$$

$$= \frac{-1}{\pi}\left[\cos(\pi) - \cos(0)\right]$$

$$= \frac{-1}{\pi}(-2)$$

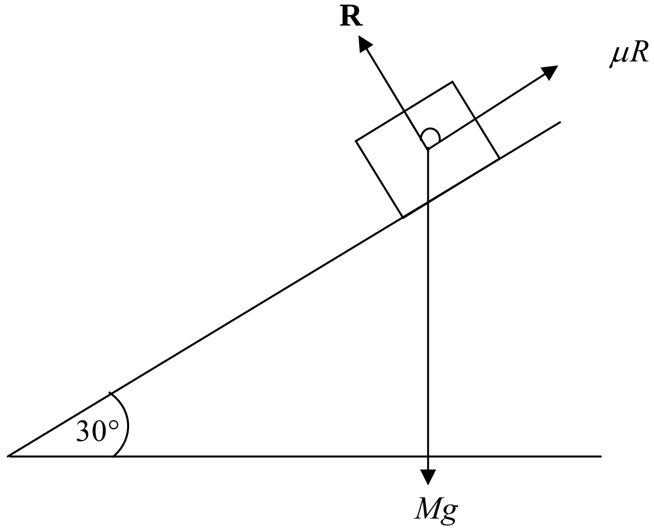
$$= \frac{2}{\pi}$$

Mark allocation: 3 marks

- 1 mark for using the correct substitution.
- 1 mark for correctly expressing the integral in terms of u.
- 1 mark for the correct answer.

Question 4a.

Worked solution

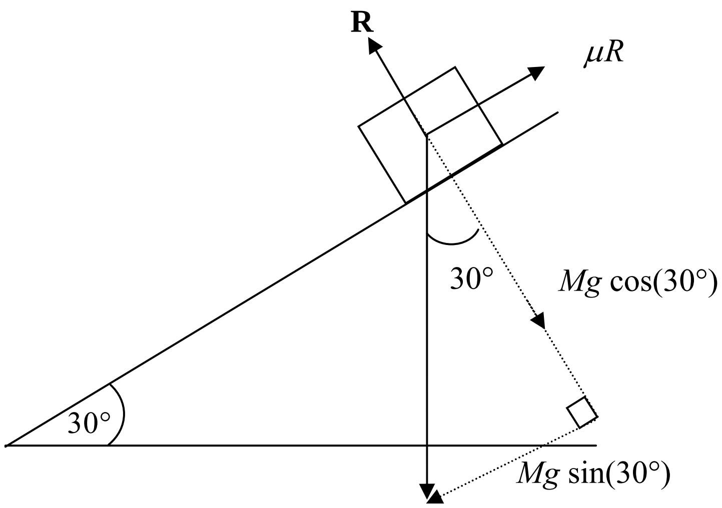


Mark allocation

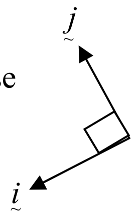
- 1 mark for showing all the forces correctly

Question 4b.

Worked solution



Use



Resolving in $\underset{\sim}{i}$ direction:

$$Mg\sin(30°) - \mu R = Ma$$

Resolving in $\underset{\sim}{j}$ direction:

$$R = Mg\cos(30°)$$

$$\therefore \frac{Mg}{2} - \mu Mg\frac{\sqrt{3}}{2} = Ma$$

$$\frac{g}{2} - \mu g\frac{\sqrt{3}}{2} = a$$

$$\text{So} \quad a = \frac{g}{2}\left(1 - \mu\sqrt{3}\right)$$

Mark allocation: 2 marks

- 1 mark for resolving in both $\underset{\sim}{i}$ and $\underset{\sim}{j}$ direction
- 1 mark for correct answer

Question 4c.

Worked solution

$$v^2 = u^2 + 2as$$

$$9 = 0 + 2 \times \frac{g}{2}\left(1 - \mu\sqrt{3}\right) \times 3$$

$$\text{So } 9 = 3g\left(1 - \mu\sqrt{3}\right)$$

$$3 = g - \mu g\sqrt{3}$$

$$\mu g\sqrt{3} = g - 3$$

$$\therefore \mu = \frac{g-3}{\sqrt{3}g}$$

Mark allocation: 2 marks

- 1 mark for determining $9 = 0 + 2 \times \frac{g}{2}\left(1 - \mu\sqrt{3}\right) \times 3$
- 1 mark for correct answer

Question 5a.

Worked solution

$$a = \frac{d}{dx}\left(\frac{v^2}{2}\right) = x + 1$$

$$\frac{v^2}{2} = \int (x+1)\,dx$$

$$= \frac{x^2}{2} + x + c$$

$$x = 0, \ v = 1$$

$$\therefore \ c = \frac{1^2}{2} = \frac{1}{2}$$

$$\frac{v^2}{2} = \frac{x^2}{2} + x + \frac{1}{2}$$

$$v^2 = x^2 + 2x + 1$$

$$v^2 = (x+1)^2$$

$\Rightarrow v = |x+1|$, but as $v = 1$ when $x = 0$

$$\therefore \ v = x + 1$$

Mark allocation: 3 marks

- 1 mark for using the appropriate expression for acceleration.
- 1 mark for correctly integrating and substituting to obtain the expression for v^2.
- 1 mark for justifying that $v = |x+1| = x + 1$.

Question 5b.

Worked solution

$$v = \frac{dx}{dt} = x + 1$$

$$\Rightarrow \frac{dt}{dx} = \frac{1}{x+1}$$

$$t = \int \frac{1}{x+1}\,dx$$

$$t = \log_e |x+1| + c$$

$$t = 0,\ \ x = 0$$

$$\Rightarrow c = 0$$

$$\Rightarrow t = \log_e |x+1|$$

$$|x+1| = e^t$$

$$\Rightarrow x + 1 = \pm e^t$$

$$x = \pm e^t - 1$$

$\therefore\ \ x = e^t - 1$ when $t = 0,\ x = 0.$

Mark allocation: 2 marks

- 1 mark for correctly expressing t as a function of x.
- 1 mark for the correct answer.

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Question 6

Worked solution

$$y^2 = 4 - 4(x-1)^2$$

$$\frac{y^2}{4} = 1 - (x-1)^2$$

$$\Rightarrow \frac{(x-1)^2}{1} + \frac{y^2}{4} = 1$$

This relation represents an ellipse with centre (1, 0), $a = 1$ and $b = 2$.

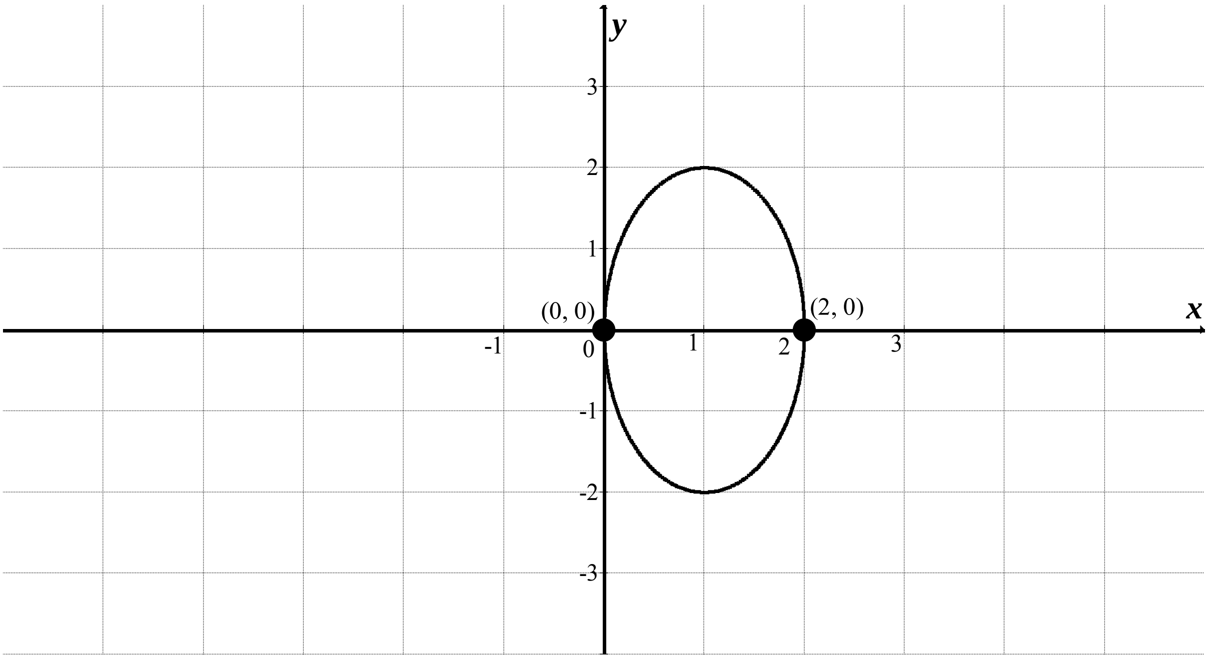
$x = 0,\ y^2 = 4 - 4 = 0$

$\Rightarrow y = 0$ only $(0,0)$

$y = 0,\ (x-1)^2 = 1$

$\Rightarrow x - 1 = \pm 1$

$\Rightarrow x = 0$ or $x = 2$ $(0,0)$ and $(2,0)$



Mark allocation: 2 marks

- 1 mark for correctly expressing the relation in standard ellipse form.
- 1 mark for the correct graph.

Question 7a.

Worked solution

$$\sec(t) = 2(x - y)$$
$$\Rightarrow \sec^2(t) = 4(x - y)^2$$
$$\tan^2(t) = y - 2$$
$$\Rightarrow 1 + \tan^2(t) = y - 1$$
$$1 + \tan^2(t) = \sec^2(t)$$
$$\Rightarrow y - 1 = 4(x - y)^2$$
$$y - 1 = 4x^2 - 8xy + 4y^2$$
$$\therefore\ y = 4x^2 - 8xy + 4y^2 + 1$$

Mark allocation: 2 marks

- 1 mark for using the identity $1 + \tan^2(t) = \sec^2(t)$
- 1 mark for simplifying the cartesian equation.

Tip

- *Use the identity* $1 + \tan^2(t) = \sec^2(t)$.

Question 7b.

Worked solution

$t = \frac{\pi}{3}, \quad y = 2 + \tan^2\left(\frac{\pi}{3}\right)$

$y = 2 + (\sqrt{3})^2 = 5$

$x - y = 0.5\sec(t)$

$\Rightarrow x - 5 = 0.5 \div \cos\left(\frac{\pi}{3}\right) = 0.5 \div 0.5 = 1$

$x = 6$

$\therefore$ The coordinate of the particle is (6, 5).

Mark allocation: 2 marks

- 1 mark for evaluating x correctly.
- 1 mark for evaluating y correctly.

Question 7c.

Worked solution

$y = 4x^2 - 8xy + 4y^2 + 1$

Differentiate both sides with respect to x.

$$\frac{dy}{dx} = 8x - 8y - 8x\frac{dy}{dx} + 8y\frac{dy}{dx}$$

$\frac{dy}{dx}(1 + 8x - 8y) = 8x - 8y$

$$\frac{dy}{dx} = \frac{8x - 8y}{1 + 8x - 8y}$$

$\Rightarrow$ At $(6,5)$, $\frac{dy}{dx} = \frac{48 - 40}{1 + 48 - 40} = \frac{8}{9}$

Mark allocation: 2 marks

- 1 mark for differentiating correctly.
- 1 mark for the correct answer.

Tip

- *Although the relation can be expressed explicitly as a function of* y, *it is easier to obtain* $\frac{dy}{dx}$ *using implicit differentiation.*

Question 8a.

Worked solution

$\overrightarrow{OA} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{OB} = 3\underset{\sim}{i} + 4\underset{\sim}{k}$

$\overrightarrow{OC} = 5\underset{\sim}{i} - 4\underset{\sim}{j} + 10\underset{\sim}{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\Rightarrow \overrightarrow{AB} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\Rightarrow \overrightarrow{BC} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + 6\underset{\sim}{k}$

$\Rightarrow \overrightarrow{BC} = 2\overrightarrow{AB}$

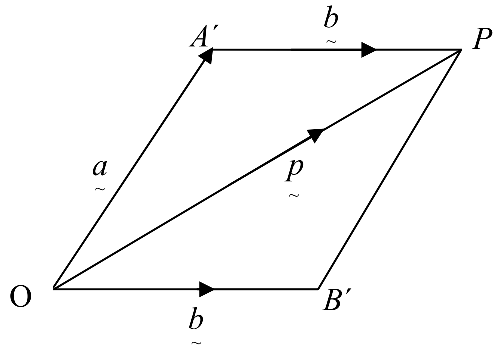
$\therefore$ A, B and C are collinear, as B is common to $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

Mark allocation: 2 marks

- 1 mark for correctly obtaining the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$
- 1 mark for recognising that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel.

Question 8b.

Worked solution



$OA'PB'$ is a rhombus as $a = b$ because $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are both unit vectors.

$\therefore\ \underset{\sim}{p}$ bisects the angle $A'OB'$.

$$\underset{\sim}{p} = \underset{\sim}{a} + \underset{\sim}{b}$$

$$\underset{\sim}{p} = \frac{1}{15}(10\underset{\sim}{i} + 10\underset{\sim}{j} + 5\underset{\sim}{k}) + \frac{1}{15}(9\underset{\sim}{i} + 12\underset{\sim}{k})$$

$\therefore\ \underset{\sim}{p} = \dfrac{1}{15}(19\underset{\sim}{i} + 10\underset{\sim}{j} + 17\underset{\sim}{k})$ is a vector that bisects the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

or $\underset{\sim}{p} = m(19\underset{\sim}{i} + 10\underset{\sim}{j} + 17\underset{\sim}{k})$, where $m \in R \setminus \{0\}$.

Mark allocation: 2 marks

- 1 mark for recognising that $\underset{\sim}{a} + \underset{\sim}{b}$ is a vector that bisects the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.
- 1 mark for the correct answer.

Question 9a.

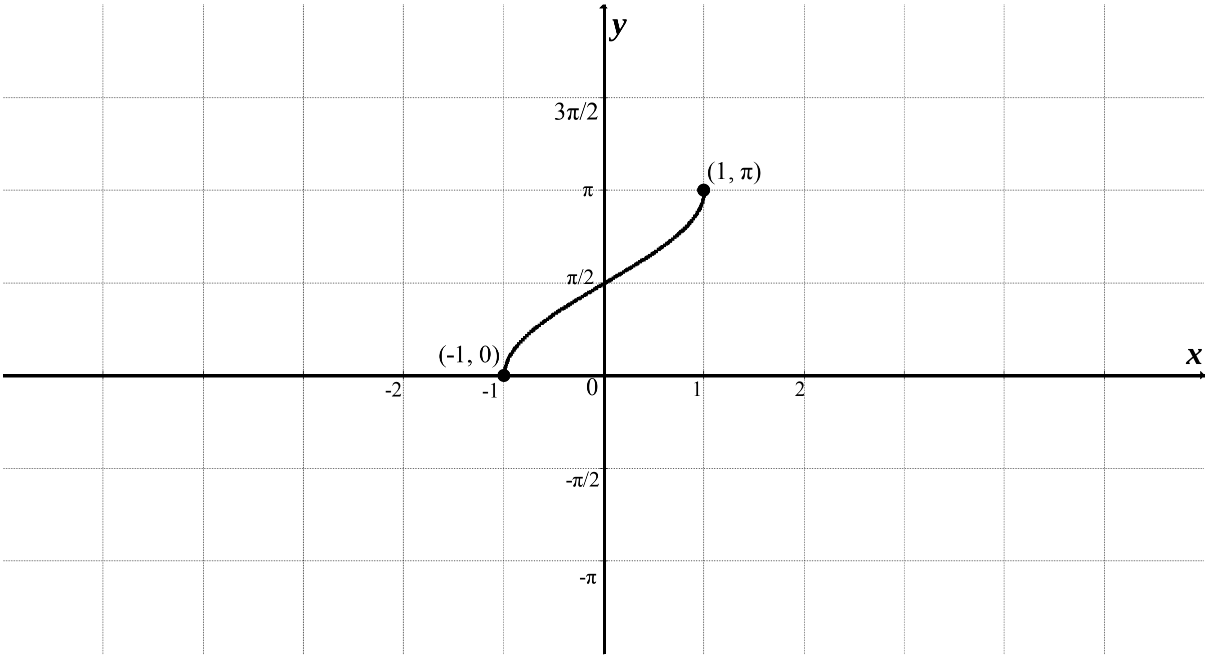
Worked solution

$x = -1,\ y = \frac{\pi}{2} + \sin^{-1}(-1) = \frac{\pi}{2} - \frac{\pi}{2} = 0$

One end point is (–1, 0).

$x = 1,\ y = \frac{\pi}{2} + \sin^{-1}(1) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Other end point is (1, π).



Mark allocation: 2 marks

- 1 mark for the correct end points.
- 1 mark for the correct graph.

Question 9b.

Worked solution

$$y = \frac{\pi}{2} + \sin^{-1}(x)$$

$$y - \frac{\pi}{2} = \sin^{-1}(x)$$

$$\Rightarrow x = \sin\left(y - \frac{\pi}{2}\right)$$

Using symmetry:

$$\text{Volume} = 2\pi\int_0^{\frac{\pi}{2}} \sin^2\left(y - \frac{\pi}{2}\right) dy$$

$$= 2\pi\int_0^{\frac{\pi}{2}} \frac{1-\cos(2y-\pi)}{2}\, dy$$

$$= \pi\int_0^{\frac{\pi}{2}} (1-\cos(2y-\pi)\ dy$$

$$= \pi\left[y - \frac{1}{2}\sin(2y-\pi)\right]_0^{\frac{\pi}{2}}$$

$$= \pi\left[\left(\frac{\pi}{2} - \frac{1}{2}\sin(0)\right) - \left(0 - \frac{1}{2}\sin(-\pi)\right)\right]$$

$$= \pi\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi^2}{2}$$

$\therefore$ The volume is $\frac{\pi^2}{2}$ cubic units.

Mark allocation: 3 marks

- 1 mark for the correct definite integral representing the volume.
- 1 mark for correctly applying the double angle formula to the integrand.
- 1 mark for the correct answer.

END OF SOLUTIONS BOOK