INSIGHT

YEAR 12 *Trial Exam Paper*

2013

Specialist Mathematics

Written examination 2

Solutions book

This book presents:

- correct solutions with full working
- explanatory notes
- mark allocations
- tips and guidelines.

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SECTION 1

Question 1

Answer is A

Worked solution

For a hyperbola with general equation $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, the equations of the asymptotes are $y-k=\pm\frac{b}{a}(x-h)$.

So the equations of the asymptotes are for a hyperbola with centre (0, –1), b = 4 and a = 3 are

$$y+1=\pm\frac{4}{3}(x-0)$$

$$y=\pm\frac{4x}{3}-1$$

Question 2

Answer is C

Worked solution

By dividing, only alternatives **B** and **C** give the correct oblique asymptote $y = ax$, $a > 0$ and alternative **C** gives the correct vertical asymptote $x = 0$. Alternative **B** is actually a straight line $y = ax + b$, which has a maximal domain $R \setminus \{0\}$.

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Question 3

Answer is D

Worked solution

Use the double angle formula for $\tan(2x)$.

$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$$

So, $\dfrac{-3}{4} = \dfrac{2\tan(x)}{1-\tan^2(x)}$.

Let $y = \tan(x)$, then

$$\frac{-3}{4} = \frac{2y}{1-y^2}$$

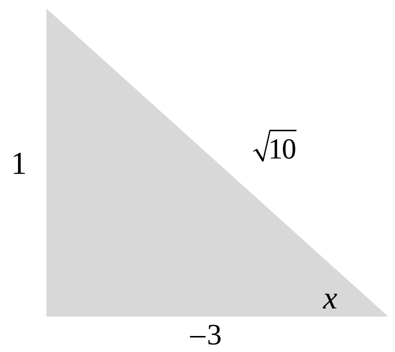
$$3y^2 - 8y - 3 = 0$$

$$(3y+1)(y-3) = 0$$

So, $y = \dfrac{-1}{3}$ or $y = 3$.

$\tan(x) = \dfrac{-1}{3}$ or $\tan(x) = 3$

Since $\dfrac{\pi}{2} < x < \pi,\ \tan(x) = \dfrac{-1}{3}$.



$\sin(x) = \dfrac{1}{\sqrt{10}}$ and $\text{cosec}(x) = \sqrt{10}$, as $\dfrac{\pi}{2} < x < \pi$.

Tip

- *Due to time constraints, this question is best done using CAS, although it is worthwhile that the student understands the method employed in the solution above, as this type of question appears frequently on Examination 1.*
- *Alternatively, using CAS:*
  - *Press menu algebra solve enter*, giving

    $\text{solve}(\tan(2x) = \dfrac{-3}{4}, x) \mid \dfrac{\pi}{2} < x < \pi$.
  - *Press ctrl enter and csc ctrl ans enter*, giving $x = 3.16227\ldots$

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Question 4

Answer is C

Worked solution

Maximal domain is found by letting

$$-1 \le 2x - k \le 1$$

$$k - 1 \le 2x \le k + 1$$

$$\frac{k-1}{2} \le x \le \frac{k+1}{2}$$

And if $x = \frac{k-1}{2}$, then

$$y = a - \pi \sin^{-1}(-1)$$

$$= a - \left(\frac{-\pi^2}{2}\right)$$

$$= \frac{2a + \pi^2}{2}$$

If $x = \frac{k+1}{2}$, then $y = a - \pi \sin^{-1}(1)$

$$= a - \left(\frac{\pi^2}{2}\right)$$

$$= \frac{2a - \pi^2}{2}$$

Hence, the range is $\left[\frac{2a-\pi^2}{2}, \frac{2a+\pi^2}{2}\right]$, since $a > 0$.

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Question 5

Answer is E

Worked solution

If $z = 1 - i$, then $\text{Arg}\left(\frac{i^3}{z}\right) = \text{Arg}(i^3) - \text{Arg}(z)$

$$= \text{Arg}(-i) - \text{Arg}(z)$$

$$= \frac{-\pi}{2} - \left(\frac{-\pi}{4}\right)$$

$$= \frac{-\pi}{4}$$

Tip

- *Use the principal argument* $-\pi < \text{Arg}(z) \leq \pi$.

Question 6

Answer is E

Worked solution

Let the solutions be represented by z_1, z_2 and z_3.

$z_1 = \text{r}\,cis\,\frac{\pi}{3}$

$z_2 = \text{r}\,cis\,\frac{-\pi}{3}$

$z_3 = \text{r}\,cis\,\pi$, where $\text{r} > 0$.

If any one of these solutions is cubed, then use de Moivres' theorem:

$z_1^{\,3} = \left(\text{r}\,cis\,\frac{\pi}{3}\right)^3$

$z_1^{\,3} = \text{r}^3 cis\,\pi$

$z_1^{\,3} = \text{r}^3(\cos\pi + i\sin\pi)$

$z_1^{\,3} = -\text{r}^3 + 0i$

Hence, $a = -\text{r}^3$ and $b = 0$, so $a < 0$ and $b = 0$ since $\text{r} > 0$.

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Question 7

Answer is E

Worked solution

As all coefficients of the cubic polynomial are real, the solutions are either real or occur as conjugate complex numbers.
If $-ki$ is a solution, then ki is also a solution. Hence:

$$(z+ki)(z-ki)\left(z-\frac{m}{n}\right)=0$$

$$(z^2+k^2)\left(z-\frac{m}{n}\right)=0$$

$$z^3-\frac{m}{n}z^2+k^2z-\frac{m}{n}k^2=0$$

As none of these coefficients is shown in the answers, then multiplying all terms by n, where $n \neq 0$ gives
$nz^3-mz^2+nk^2z-mk^2=0$
Hence, $a=n, b=-m, c=nk^2$ and $d=-mk^2$.

Question 8

Answer is E

Worked solution

Convert the alternatives to Cartesian form.
Doing this, alternative **E** then becomes $x^2+y^2 \leq 4,$ which is the area inside a circle with centre (0, 0) and radius of 2 units. $\text{Im}(z) \geq 0$ is the area above and including the line $y = 0$.
The intersection of these areas results in the area, including the boundaries, shown on the Argand diagram.

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Question 9

Answer is B

Worked solution

$$\frac{dx}{dy} = \sqrt{9-4x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{9-4x^2}}$$

$$y = \int \frac{1}{\sqrt{4\left(\frac{9}{4}-x^2\right)}}\,dx$$

$$y = \int \frac{1}{2\sqrt{\frac{9}{4}-x^2}}\,dx$$

$$y = \frac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx$$

Tip

- *You need to write* $\frac{1}{\sqrt{9-4x^2}}$ *as* $\frac{1}{\sqrt{4\left(\frac{9}{4}-x^2\right)}}$.

Question 10

Answer is B

Worked solution

For $0 < x < x_1$, $f'(x)$ is increasing, so $f''(x) > 0$.
At x_1, $f'(x)$ is a maximum and $f''(x) = 0$.
For $x_1 < x < x_3$, $f'(x)$ is decreasing, so $f''(x) < 0$.
At x_3, $f'(x)$ is a minimum and $f''(x) = 0$.
For $x_3 < x < x_5$, $f'(x)$ is increasing, so $f''(x) > 0$.
At x_5, $f'(x)$ is a maximum and $f''(x) = 0$.
So $f''(x) > 0$ for $0 < x < x_1$ and $x_3 < x < x_5$.

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Question 11

Answer is A

Worked solution

$$\frac{d}{dx}(x\sin(y)) - \frac{d}{dx}(x^2) = \frac{d}{dx}(5)$$

$$\left[\frac{d}{dx}(x).\sin(y) + \frac{d}{dx}(\sin(y)).x\right] - 2x = 0$$

$$\sin(y) + \frac{d}{dy}(\sin(y))\frac{dy}{dx}.x = 2x$$

$$x\cos(y)\frac{dy}{dx} = 2x - \sin(y)$$

$$\frac{dy}{dx} = \frac{2x - \sin(y)}{x\cos(y)}$$

Tip

- *Students need to recognise* $x\sin(y)$ *as a product when differentiating implicitly.*

Question 12

Answer is C

Worked solution

Now

$$\frac{3x-2}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

$$= \frac{A(x-1)+B}{(x-1)^2}$$

$$= \frac{Ax - A + B}{(x-1)^2}$$

$\Rightarrow A = 3$

And $-A + B = -2$

$\Rightarrow B = 1$

So, $\int \frac{3x-2}{(x-1)^2}\,dx = \int \frac{3}{(x-1)} + \frac{1}{(x-1)^2}\,dx.$

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Question 13

Answer is D

Worked solution

Since the rate of outflow (litres/min) $\neq$ rate of inflow (litres/min), the volume of the solution increases by 3 litres every minute.

$\frac{dx}{dt} = X_\text{i} - X_\text{o}$, where X_i = rate of inflow of salt (kg/min)

X_o = rate of outflow of salt (kg/min)

$$\frac{dx}{dt} = (10\text{ L/min} \times 0\text{ kg/L}) - \left(7\text{ L/min} \times \frac{x}{100+3t}\text{ kg/L}\right)$$

$$\frac{dx}{dt} = 0 - \frac{7x}{100+3t}$$

$$\frac{dx}{dt} = \frac{-7x}{100+3t}$$

Question 14

Answer is C

Worked solution

$$x = \frac{e^{2t}+1}{e^t},\ t \geq 0$$

$$x = e^t + e^{-t}$$

$$v = e^t - e^{-t}$$

$$a = e^t + e^{-t}$$

When t = 2 seconds, then

$$a = e^2 + e^{-2}$$

$$a = e^2 + \frac{1}{e^2}$$

$$a = \frac{e^4+1}{e^2}$$

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Question 15

Answer is D

Worked solution

$$v = \sin(x)\cos(x)$$

$$v = \frac{1}{2}\sin(2x)$$

Now, $a = v\dfrac{dv}{dx}$

$$a = \frac{1}{2}\sin(2x)\left[\frac{1}{2}\cos(2x).(2)\right]$$

$$a = \frac{1}{2}\sin(2x)\cos(2x)$$

$$a = \frac{\sin(4x)}{4}$$

Question 16

Answer is B

Worked solution

$$(\underset{\sim}{b}.\hat{\underset{\sim}{a}})\cdot\hat{\underset{\sim}{a}}$$

$$= \left[\left(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)\cdot\frac{1}{\sqrt{30}}\left(5\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}\right)\right]\frac{1}{\sqrt{30}}\left(5\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

$$= \frac{9}{\sqrt{30}}\cdot\frac{1}{\sqrt{30}}\left(5\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

$$= \frac{3}{10}\left(5\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

$$= \frac{3}{2}\underset{\sim}{i} - \frac{3}{10}\underset{\sim}{j} + \frac{3}{5}\underset{\sim}{k}$$

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Question 17

Answer is E

Worked solution

$x = \cos(2t)$ and $y = \sin^2(t)$

Now, $\cos(2t) = 1 - 2\sin^2(t)$

$x = 1 - 2y^2$

$2y^2 = 1 - x$

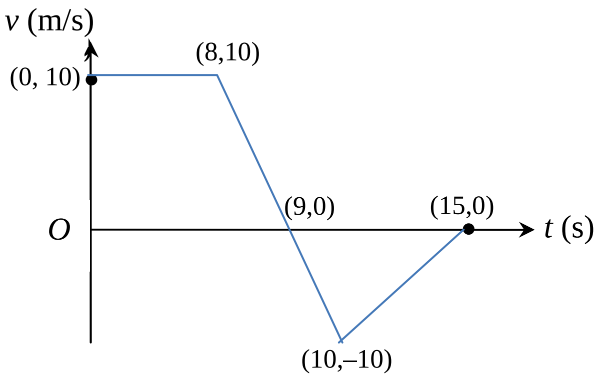
$y^2 = \frac{1}{2}(1 - x)$ and if $t \geq 0$, then $-1 \leq x \leq 1$.

$y^2 = \frac{1}{2}(1 - x), \ -1 \leq x \leq 1$

Question 18

Answer is A

Worked solution



Intercept is 9.

Area under trapezium representing westerly displacement $= \frac{1}{2} \times (8 + 9) \times 10 = 85$

Area under triangle representing easterly displacement $= \frac{1}{2} \times 6 \times 10 = 30$

Resultant displacement is 85 – 30 = 55 m west of *O*.

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Question 19

Answer is B

Worked solution

$y_{n+1} = y_n + hf(x_n)$ and $x_{n+1} = x_n + h$

$$y_1 = y_0 + 0.1f(x_0)$$
$$y_1 = 0 + 0.1(\log_e 1)$$
$$y_1 = 0$$

$y_2 = y_1 + 0.1f(x_1)$ and $x_1 = x_0 + h = 0.1$

$$y_2 = 0 + 0.1(\log_e 1.2)$$
$$y_2 = \frac{\log_e 1.2}{10}$$

Question 20

Answer is C

Worked solution

Let X = mass of larger object.

$$0.6X = \frac{Y}{4}$$
$$X = \frac{5Y}{12} \text{ kg}$$

Equations of motion on each mass are

$$\frac{5Y}{12}g - T = \frac{5Y}{12}a \quad (1)$$
$$T - \frac{Y}{4}g = \frac{Y}{4}a \quad (2)$$

Adding equations (1) and (2) gives

$$g\left(\frac{5Y}{12} - \frac{Y}{4}\right) = a\left(\frac{5Y}{12} + \frac{Y}{4}\right)$$
$$\frac{gY}{6} = \frac{2aY}{3}$$
$$a = \frac{g}{4} \text{ m/s}^2$$

Tip

- *For connected particles, the equations of motion need to be written for each particle separately, knowing that the magnitude of acceleration is the same in both equations.*

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Question 21

Answer is C

Worked solution
The forces acting on the object are in equilibrium; therefore, opposing horizontal and vertical forces sum to zero.

Resolving horizontal forces
$T\sin\theta = 60 \quad (1)$

Resolving vertical forces
$T\cos\theta = 10g \quad (2)$

Dividing equation 1 by equation 2 gives

$$\tan\theta = \frac{60}{10g}$$

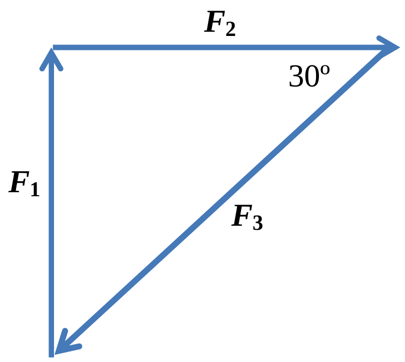
$$\theta = \tan^{-1}\left(\frac{60}{10g}\right)$$

$$\theta = 31.48°$$

Question 22

Answer is A

Worked solution



$$\frac{F_1}{\sin 30°} = \frac{F_2}{\sin 60°} = \frac{F_3}{\sin 90°} \quad \text{(Lami's theorem)}$$

$$2F_1 = \frac{2F_2}{\sqrt{3}} = F_3$$

$$2\sqrt{3}F_1 = 2F_2 = \sqrt{3}F_3$$

END OF SECTION 1

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SECTION 2

Question 1a.

Worked solution

P is the y-intercept of $y = \frac{a}{x^2 + b}$ and is found by letting $x = 0$.

If $x = 0$, $y = \frac{a}{b}$, then P is $\left(0, \frac{a}{b}\right)$.

Q is the intersection of $y = \frac{a}{x^2 + b}$ and $y = c$. Solving, this gives:

$$\frac{a}{x^2 + b} = c$$

$$x^2 + b = \frac{a}{c}$$

$$x^2 = \frac{a - bc}{c}$$

$$x = \sqrt{\frac{a - bc}{c}}\text{, since } x > 0$$

So Q is $\left(\sqrt{\frac{a - bc}{c}}, c\right)$.

Mark allocation: 3 marks

- 1 mark for finding the coordinates of P are $\left(0, \frac{a}{b}\right)$.
- 1 mark for solving $\frac{a}{x^2 + b} = c$ to give $x = \sqrt{\frac{a - bc}{c}}$.
- 1 mark for finding the coordinates of Q are $\left(\sqrt{\frac{a - bc}{c}}, c\right)$.

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Question 1b.

Worked solution

i. $A = \int_0^{\sqrt{\frac{a-bc}{c}}} \left(\frac{a}{x^2+b} - c \right) dx$

Mark allocation: 1 mark

- 1 mark for the definite integral correctly written as $A = \int_0^{\sqrt{\frac{a-bc}{c}}} \left(\frac{a}{x^2+b} - c \right) dx$.

ii. Rotation around the *y*-axis is found by $V = \pi \int_{y_1}^{y_2} x^2 dy$.

Now, $y = \frac{a}{x^2+b}$

So $x^2 + b = \frac{a}{y}$

$x^2 = \frac{a}{y} - b$

and $V = \pi \int_c^{\frac{a}{b}} \left(\frac{a}{y} - b \right) dy$.

Mark allocation: 2 marks

- 1 mark for transposing correctly to find $x^2 = \frac{a}{y} - b$.
- 1 mark for definite integral correctly written as $V = \pi \int_c^{\frac{a}{b}} \left(\frac{a}{y} - b \right) dy$.

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Question 1c.

Worked solution

$$A=\int_0^{\sqrt{\frac{a-bc}{c}}}\left(\frac{a}{x^2+b}-c\right)\,dx$$

$$A=\int_0^{\sqrt{\frac{30-6.4}{1.6}}}\left(\frac{30}{x^2+4}-1.6\right)\,dx$$

Using CAS, $A=10.2$

$$V=\pi\int_c^{\frac{a}{b}}\left(\frac{a}{y}-b\right)\,dy$$

$$V=\pi\int_{1.6}^{\frac{30}{4}}\left(\frac{30}{y}-4\right)\,dy$$

Using CAS, $V=71.5$

Mark allocation: 2 marks

- 1 mark for A = 10.2.
- 1 mark for V = 71.5.

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Question 2a.

Worked solution

$\tan\theta = \frac{x}{2}$

$\theta = \tan^{-1}\left(\frac{x}{2}\right)$

Mark allocation

- 1 mark for $\theta = \tan^{-1}\left(\frac{x}{2}\right)$.

Question 2b.

Worked solution

$\frac{d\theta}{dx} = \frac{2}{4+x^2}$

$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{2}{4+x^2} \cdot \frac{3x}{2}$

$\frac{d\theta}{dt} = \frac{3x}{4+x^2}$

When $x = 1$, $\frac{d\theta}{dt} = \frac{3}{5}$ rad/s.

Mark allocation: 2 marks

- 1 mark for using the chain rule $\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}$.
- 1 mark for $\frac{d\theta}{dt} = \frac{3}{5}$ rad/s, including correct unit.

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Question 2c.

Worked solution

$\frac{d\theta}{dt} = \frac{3x}{4+x^2}$ and $x = 2\tan(\theta)$

$\frac{d\theta}{dt} = \frac{3(2\tan(\theta))}{4+(2\tan(\theta))^2}$

$\frac{d\theta}{dt} = \frac{6\tan(\theta)}{4(1+\tan^2(\theta))}$

$\frac{d\theta}{dt} = \frac{\frac{3}{2}\tan(\theta)}{\sec^2(\theta)}$

$\frac{d\theta}{dt} = \frac{3}{2}.\frac{\sin(\theta)}{\cos(\theta)}.\cos^2(\theta)$

$\frac{d\theta}{dt} = \frac{3}{2}\sin(\theta)\cos(\theta)$

$\frac{d\theta}{dt} = \frac{3}{4}\cdot 2\sin(\theta)\cos(\theta)$

$\frac{d\theta}{dt} = \frac{3\sin(2\theta)}{4}$

So, a = 3, b = 2 and c = 4.

Mark allocation: 3 marks

- 1 mark for substituting $x = 2\tan(\theta)$ into $\frac{d\theta}{dt} = \frac{3x}{4+x^2}$.
- 1 mark for using the identity $\tan^2(\theta)+1=\sec^2(\theta)$.
- 1 mark for cancelling $\cos(\theta)$ in $\frac{d\theta}{dt} = \frac{3}{2}.\frac{\sin(\theta)}{\cos(\theta)}\cdot\cos^2(\theta)$ and then showing $\frac{d\theta}{dt} = \frac{3\sin(2\theta)}{4}$, stating a = 3, b = 2 and c = 4.

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Question 2d.

Worked solution

i. $\cos(\theta) = \dfrac{2}{L}$

$$L = \frac{2}{\cos(\theta)}$$

Mark allocation

- 1 mark for $L = \dfrac{2}{\cos(\theta)}$ or $L = 2\sec(\theta)$.

Worked solution

ii. $\dfrac{dL}{d\theta} = -2(\cos(\theta))^{-2}(-\sin(\theta))$

$$\frac{dL}{d\theta} = \frac{2\sin(\theta)}{\cos^2(\theta)}$$

$$\frac{dL}{dt} = \frac{dL}{d\theta} . \frac{d\theta}{dt}$$

$$\frac{dL}{dt} = \frac{2\sin(\theta)}{\cos^2(\theta)} . \frac{3}{2}\sin(\theta)\cos(\theta) \text{ as } \frac{d\theta}{dt} = \frac{3}{2}\sin(\theta)\cos(\theta) \text{ from part } \mathbf{c}.$$

$$\frac{dL}{dt} = \frac{3\sin^2(\theta)}{\cos(\theta)} \text{ or } 3\tan(\theta)\sec(\theta)$$

Mark allocation: 2 marks

- 1 mark for $\dfrac{dL}{d\theta} = \dfrac{2\sin(\theta)}{\cos^2(\theta)}$.
- 1 mark for $\dfrac{dL}{dt} = \dfrac{3\sin^2(\theta)}{\cos(\theta)}$, using the chain rule.

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Worked solution

iii.

$$\frac{dL}{dt} = \frac{3\sin^2(\theta)}{\cos(\theta)}$$

$$\frac{dL}{dt} = \frac{3(1-\cos^2(\theta))}{\cos(\theta)}$$

$$\frac{dL}{dt} = \frac{3\left(1-\frac{4}{L^2}\right)}{\frac{2}{L}}$$

$$\frac{dL}{dt} = \frac{3(L^2-4)}{L^2}.\frac{L}{2}$$

$$\frac{dL}{dt} = \frac{3(L^2-4)}{2L}$$

Hence, $\dfrac{dt}{dL} = \dfrac{2L}{3(L^2-4)}$

$$t = \int \frac{2L}{3(L^2-4)}\, dL$$

$$t = \frac{2}{3}\int \frac{L}{(L^2-4)}\, dL$$

Now $\dfrac{L}{L^2-4} = \dfrac{A}{L+2} + \dfrac{B}{L-2}$

Solving $A = \dfrac{1}{2}$ and $B = \dfrac{1}{2}$ gives

$$t = \frac{2}{3}\int \frac{\frac{1}{2}}{L+2} + \frac{\frac{1}{2}}{L-2}\, dL$$

$$t = \frac{1}{3}\int \frac{1}{L+2} + \frac{1}{L-2}\, dL$$

$$t = \frac{1}{3}[\log_e|L+2| + \log_e|L-2|] + c \quad L \in \left[\sqrt{5}, \sqrt{29}\right]$$

$$t = \frac{1}{3}\log_e|L^2-4| + c \quad L \in \left[\sqrt{5}, \sqrt{29}\right]$$

When $t = 0$, $L = \sqrt{5}$

$\Rightarrow c = 0$

$$t = \frac{1}{3}\log_e|L^2-4|$$

$$L^2 - 4 = e^{3t}$$

$$L^2 = e^{3t} + 4$$

$$L = \sqrt{e^{3t}+4} \text{ since } L\left[\sqrt{5}, \sqrt{29}\right].$$

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Mark allocation: 5 marks

- 1 mark for writing $\frac{dL}{dt}$ in terms of $\cos(\theta)$, $\frac{dL}{dt}=\frac{3(1-\cos^2(\theta))}{\cos(\theta)}$.
- 1 mark for using $\cos\theta=\frac{2}{L}$ and simplifying to $\frac{dL}{dt}=\frac{3(L^2-4)}{2L}$.
- 1 mark for inverting $\frac{dL}{dt}$ and using an appropriate integration technique to solve for t in terms of L.
- 1 mark for using initial values t = 0, $L=\sqrt{5}$ to find the constant of integration.
- 1 mark for $L=\sqrt{e^{3t}+4}$, giving reasons for selecting the positive root.

Tip

- *The integration technique used in the solution is to split the integrand using partial fractions. Alternatively, the first form of the change of variable rule could be used to solve the integral* $t=\int\frac{2L}{3(L^2-4)}\,dL$.

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Question 3a.

Worked solution

M is $\frac{2x}{3}$ kg.

L is $\frac{x}{3}$ kg.

Mark allocation

- 1 mark for M = $\frac{2x}{3}$ kg and L = $\frac{x}{3}$ kg.

Question 3b.

Worked solution

Amount of M that remains is $8-\frac{2x}{3}$ kg.

Amount of L that remains is $5-\frac{x}{3}$ kg.

Mark allocation

- 1 mark for $\text{M} = 8-\frac{2x}{3}$ kg and $\text{L} = 5-\frac{x}{3}$ kg.

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Question 3c.

Worked solution

$$\frac{dx}{dt} \propto \left(8 - \frac{2x}{3}\right)\left(5 - \frac{x}{3}\right)$$

$$\frac{dx}{dt} \propto \frac{1}{9}(24 - 2x)(15 - x)$$

$$\frac{dx}{dt} \propto \frac{2}{9}(12 - x)(15 - x)$$

$$\frac{dx}{dt} = k(12 - x)(15 - x)$$

Mark allocation: 2 marks

- 1 mark for $\frac{dx}{dt} \propto \left(8 - \frac{2x}{3}\right)\left(5 - \frac{x}{3}\right)$.
- 1 mark for $\frac{dx}{dt} = k(12 - x)(15 - x)$.

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Question 3d.

Worked solution

$$\frac{dx}{dt} = k(12-x)(15-x)$$

$$\frac{dt}{dx} = \frac{1}{k(12-x)(15-x)}$$

$$t = \frac{1}{k}\int \frac{1}{(12-x)(15-x)}\,dx$$

Now $\dfrac{1}{(12-x)(15-x)} = \dfrac{A}{12-x} + \dfrac{B}{15-x}$

$$= \frac{A(15-x)+B(12-x)}{(12-x)(15-x)}$$

$$= \frac{15A - Ax + 12B - Bx}{(12-x)(15-x)}$$

$\Rightarrow -A - B = 0 \quad (1)$

$15A + 12B = 1 \quad (2)$

Solving equations (1) and (2) gives $A = \dfrac{1}{3}$ and $B = \dfrac{-1}{3}$.

$$t = \frac{1}{k}\int \frac{\frac{1}{3}}{12-x} - \frac{\frac{1}{3}}{15-x}\,dx$$

$$t = \frac{1}{3k}\int \frac{1}{12-x} - \frac{1}{15-x}\,dx$$

$$t = \frac{1}{3k}[\log_e|15-x| - \log_e|12-x|] + c, \quad 0 \le x < 12$$

$$t = \frac{1}{3k}\log_e\left|\frac{15-x}{12-x}\right| + c, \quad 0 \le x < 12$$

Now $x = 0$ when $t = 0$

$$0 = \frac{1}{3k}\log_e\left|\frac{5}{4}\right| + c$$

$$\Rightarrow c = \frac{-1}{3k}\log_e\left|\frac{5}{4}\right|$$

$$t = \frac{1}{3k}\log_e\left|\frac{15-x}{12-x}\right| - \frac{1}{3k}\log_e\left|\frac{5}{4}\right|$$

$$t = \frac{1}{3k}\log_e\left|\frac{60-4x}{60-5x}\right|$$

Now $x = 2$ when $t = 1$

$$1 = \frac{1}{3k}\log_e\left|\frac{26}{25}\right|$$

$$k = \frac{1}{3}\log_e\left|\frac{26}{25}\right|$$

$$t = \frac{1}{\log_e\left|\frac{26}{25}\right|}\log_e\left|\frac{60-4x}{60-5x}\right|$$

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Mark allocation: 7 marks

- 1 mark for inverting $\frac{dx}{dt} = k(12-x)(15-x)$.
- 1 method mark for using correct partial fraction technique.
- 1 mark for finding $A = \frac{1}{3}$ and $B = \frac{-1}{3}$.
- 1 mark for $t = \frac{1}{3k}[\log_e|15-x| - \log_e|12-x|] + c, \quad 0 \le x < 12$.
- 1 mark for $c = \frac{-1}{3k}\log_e\left|\frac{5}{4}\right|$.
- 1 mark for $k = \frac{1}{3}\log_e\left|\frac{26}{25}\right|$.
- 1 mark for writing t in terms of x.

Question 3e.

Worked solution

When x = 10, then

$$t = \frac{1}{\log_e\left|\frac{26}{25}\right|}\log_e\left|\frac{20}{10}\right|$$

$t = 18$ minutes

Mark allocation

- 1 mark for t = 18 minutes.

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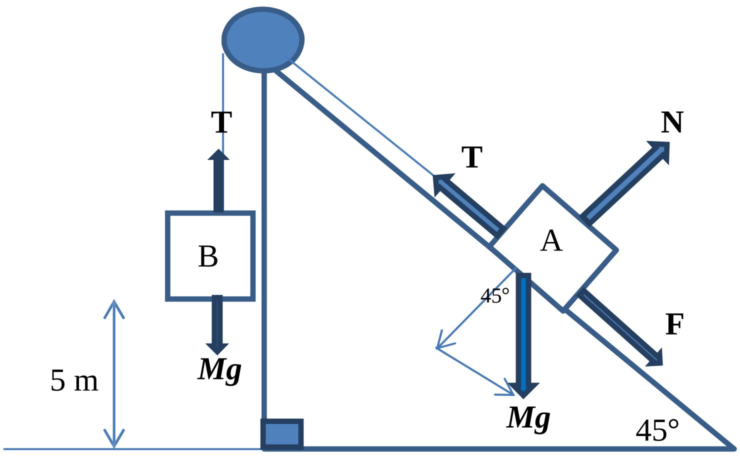
Question 4a.

Worked solution

If A is on the point of moving up the plane, then the forces acting on both loads A and B are as shown in the diagram below.



Resolving forces acting on B

$T = Mg \quad (1)$

Resolving forces acting on A

$T = F + Mg\sin 45° \quad (2)$, where $F = \mu N$ and $N = Mg\cos 45°$.

Solving equations (1) and (2) gives

$$Mg = \mu Mg\cos 45° + Mg\sin 45°$$

$$1 = \mu\cos 45° + \sin 45°$$

$$1 = \frac{\mu}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$\sqrt{2} = \mu + 1$$

$$\mu = \sqrt{2} - 1$$

Mark allocation: 3 marks

- 1 mark for $T = Mg$ (equation 1) and $T = F + Mg\sin 45°$ (equation 2).
- 1 mark for $F = \mu N$ and $N = Mg\cos 45°$.
- 1 mark for solving equations (1) and (2) to show $\mu = \sqrt{2} - 1$.

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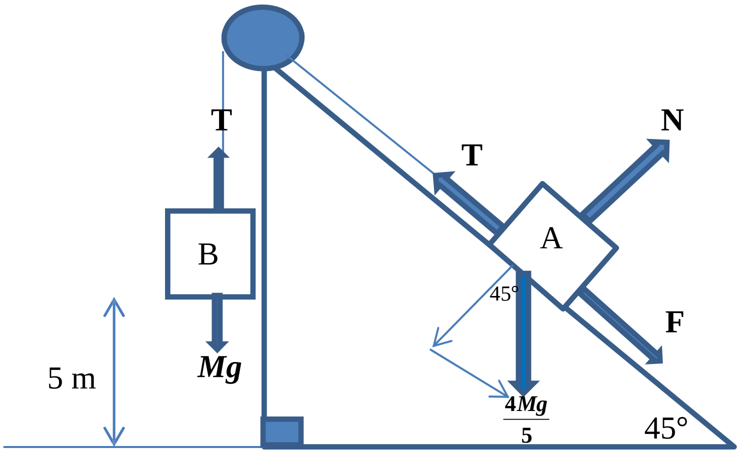
Question 4b.

Worked solution



Mass of load A $= \dfrac{4M}{5}$ kg.

Resolving forces acting on B

$Mg - T = Ma$ (1), where a is acceleration of load B.

Resolving forces acting on A

$$T - \left(F + \frac{4Mg}{5}\sin 45°\right) = \frac{4Ma}{5} \quad (2),$$

where a is acceleration of load A and $F = \mu N$ and $N = \dfrac{4Mg}{5}\cos 45°$.

$$T - \left(\frac{4\mu Mg}{5}\cos 45° + \frac{4Mg}{5}\sin 45°\right) = \frac{4Ma}{5} \quad (2)$$

Solving equations (1) and (2) gives

$$Mg - Ma - \left(\frac{4\mu Mg}{5}\cos 45° + \frac{4Mg}{5}\sin 45°\right) = \frac{4Ma}{5}$$

$$g - a - \left(\frac{4\mu g}{5}\cos 45° + \frac{4g}{5}\sin 45°\right) = \frac{4a}{5}$$

$$\frac{2\sqrt{2}\mu g}{5} + \frac{2\sqrt{2}g}{5} = g - a - \frac{4a}{5}$$

Since $\mu = \sqrt{2} - 1,$ then

$$2g\sqrt{2}(\sqrt{2}-1) + 2g\sqrt{2} = 5g - 9a$$

$$4g = 5g - 9a$$

$$a = \frac{g}{9}$$

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Mark allocation: 4 marks

- 1 mark for mass of load A $= \dfrac{4M}{5}$ kg.
- 1 mark for $Mg - T = Ma$.
- 1 mark for $T - \left(\dfrac{4\mu Mg}{5}\cos 45° + \dfrac{4Mg}{5}\sin 45°\right) = \dfrac{4Ma}{5}$.
- 1 mark for solving equations simultaneously to show $a = \dfrac{g}{9}$.

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Question 4c.

Worked solution

Before the rope breaks

When $u = 0,\ a = \frac{g}{9}$ and $t = 2,$ then

$$s = ut + \frac{1}{2}at^2 \qquad \text{and} \qquad v = u + at$$

$$s = 0 + \left(\frac{1}{2}.\frac{g}{9}.2^2\right) \qquad v = 0 + \left(\frac{g}{9}.2\right)$$

$$s = \frac{2g}{9} \qquad v = \frac{2g}{9}$$

After the rope breaks

$$u = \frac{2g}{9},\ a = g,\ s = 5 - \frac{2g}{9}$$

$$v^2 = u^2 + 2as$$

$$v^2 = \frac{4g^2}{81} + 2g\left(5 - \frac{2g}{9}\right)$$

$$v^2 = \frac{4g^2}{81} + 10g - \frac{4g^2}{9}$$

$$v^2 = 10g - \frac{32g^2}{81}$$

$$v^2 = \frac{810g - 32g^2}{81}$$

$$v = \frac{\sqrt{810g - 32g^2}}{9}$$

$$a = 810,\ b = -32,\ c = 9$$

Mark allocation: 3 marks

- 2 marks for obtaining $displacement = \frac{2g}{9}$ and $final\ velocity = \frac{2g}{9}$ for the first 2 seconds.
- 1 mark for obtaining $final\ velocity = \frac{\sqrt{810g - 32g^2}}{9}$ and the values *a* = 810, *b* = –32, *c* = 9.

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Question 5a.

Worked solution

i. $\bar{u} = -1 - \sqrt{3}i$

$$\bar{u} = 2\text{cis}\left(\frac{-2\pi}{3}\right) \text{ in polar form}$$

So, $z^2 = 2\text{cis}\left(\frac{-2\pi}{3}\right)$.

Let $z = r\text{cis}\theta$, then

$$(r\text{cis}\theta)^2 = 2\text{cis}\left(\left(\frac{-2\pi}{3}\right) + 2k\pi\right)$$

$$r^2\text{cis}(2\theta) = 2\text{cis}\left(\left(\frac{-2\pi}{3}\right) + 2k\pi\right)$$

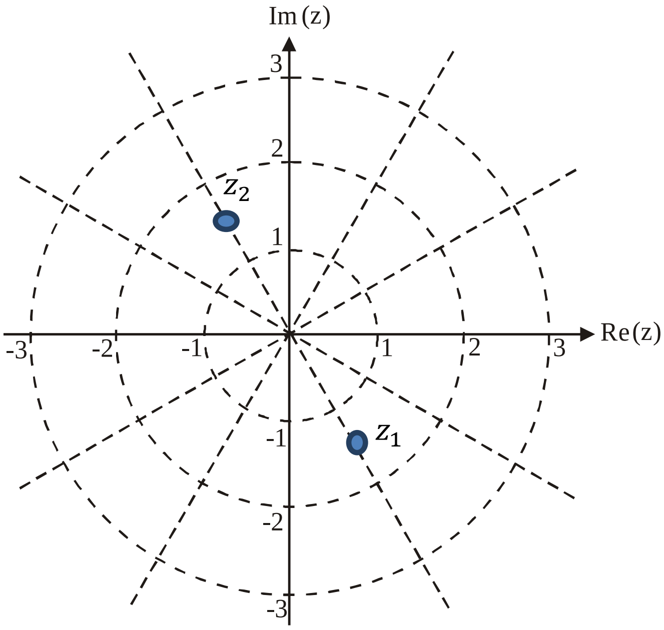
$$r = \sqrt{2} \text{ since } r > 0 \quad \text{and}$$

$$2\theta = \frac{-2\pi}{3} + 2k\pi,\ k \in Z$$

$$\theta = \frac{-\pi}{3} + k\pi,\ k \in Z$$

$$\theta = \frac{-\pi}{3} \text{ and } \frac{2\pi}{3}$$

Hence, $z_1 = \sqrt{2}\text{cis}\left(\frac{-\pi}{3}\right)$ and $z_2 = \sqrt{2}\text{cis}\left(\frac{2\pi}{3}\right)$, using the principal argument $-\pi < \text{Arg } z \leq \pi$.



Mark allocation: 3 marks

- 1 method mark for using either a valid polar or Cartesian process to solve $z^2 = \bar{u}$.
- 1 mark for two correct solutions.
- 1 mark for correctly plotting solutions on an Argand diagram.

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Worked solution

ii. $|z-z_1|=2|z-z_2|$

Let $z=x+yi$

$$\left|(x+yi)-\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}\right)i\right|=2\left|(x+yi)-\left(\frac{-\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\right)i\right|$$

$$\left|\left(x-\frac{\sqrt{2}}{2}\right)+\left(y+\frac{\sqrt{6}}{2}\right)i\right|=2\left|\left(x+\frac{\sqrt{2}}{2}\right)+\left(y-\frac{\sqrt{6}}{2}\right)i\right|$$

$$\sqrt{\left(x-\frac{\sqrt{2}}{2}\right)^2+\left(y+\frac{\sqrt{6}}{2}\right)^2}=2\sqrt{\left(x+\frac{\sqrt{2}}{2}\right)^2+\left(y-\frac{\sqrt{6}}{2}\right)^2}$$

$$x^2-\sqrt{2}x+\frac{1}{2}+y^2+\sqrt{6}y+\frac{3}{2}=4\left(x^2+\sqrt{2}x+\frac{1}{2}+y^2-\sqrt{6}y+\frac{3}{2}\right)$$

$$3x^2+5\sqrt{2}x+3y^2-5\sqrt{6}y+6=0$$

$$3\left(x^2+\frac{5\sqrt{2}}{3}x+\frac{25}{18}\right)+3\left(y^2-\frac{5\sqrt{6}}{3}y+\frac{25}{6}\right)=-6+\frac{25}{6}+\frac{25}{2}$$

$$3\left(x+\frac{5\sqrt{2}}{6}\right)^2+3\left(y-\frac{5\sqrt{6}}{6}\right)^2=\frac{-36}{6}+\frac{25}{6}+\frac{75}{6}$$

$$3\left(x+\frac{5\sqrt{2}}{6}\right)^2+3\left(y-\frac{5\sqrt{6}}{6}\right)^2=\frac{32}{3}$$

$$\left(x+\frac{5\sqrt{2}}{6}\right)^2+\left(y-\frac{5\sqrt{6}}{6}\right)^2=\frac{32}{9}$$

Centre is $\left(-\frac{5\sqrt{2}}{6},\frac{5\sqrt{6}}{6}\right)$ and radius is $\frac{4\sqrt{2}}{3}$.

Mark allocation: 4 marks

- 1 mark for $\left|(x+yi)-\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}\right)i\right|=2\left|(x+yi)-\left(\frac{-\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\right)i\right|$.
- 1 mark for $\sqrt{\left(x-\frac{\sqrt{2}}{2}\right)^2+\left(y+\frac{\sqrt{6}}{2}\right)^2}=2\sqrt{\left(x+\frac{\sqrt{2}}{2}\right)^2+\left(y-\frac{\sqrt{6}}{2}\right)^2}$.
- 1 method mark for completing square on x and y.
- 1 mark for $\left(x+\frac{5\sqrt{2}}{6}\right)^2+\left(y-\frac{5\sqrt{6}}{6}\right)^2=\frac{32}{9}$, with centre $\left(-\frac{5\sqrt{2}}{6},\frac{5\sqrt{6}}{6}\right)$

  and radius $\frac{4\sqrt{2}}{3}$.

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Question 5b.

Worked solution

i. If $z = \text{cis}\theta$, then $z^3 = 1^3\text{cis}(3\theta)$ and $\frac{1}{z^3} = z^{-3} = 1^{-3}\text{cis}(-3\theta)$ (using DeMoivre's theorem).

So, $z^3 + \frac{1}{z^3} = 1\text{cis}(3\theta) + 1\text{cis}(-3\theta)$ since $1^3 = 1$ and $1^{-3} = 1$

$= \cos(3\theta) + i\sin(3\theta) + \cos(-3\theta) + i\sin(-3\theta)$

Now, $\cos(-3\theta) = \cos(3\theta)$ and $\sin(-3\theta) = -\sin(3\theta)$

So, $\cos(3\theta) + i\sin(3\theta) + \cos(-3\theta) + i\sin(-3\theta)$

$= \cos(3\theta) + i\sin(3\theta) + \cos(3\theta) - i\sin(3\theta)$

$= 2\cos(3\theta)$

Mark allocation: 2 marks

- 1 mark for using de Moivre's theorem for $z^3 = 1^3\text{cis}(3\theta)$ and $\frac{1}{z^3} = z^{-3} = 1^{-3}\text{cis}(-3\theta)$.
- 1 mark for using identities for negative angles $\cos(-3\theta) = \cos(3\theta)$ and $\sin(-3\theta) = -\sin(3\theta)$.

Worked solution

ii. If $z = \text{cis}\theta$, then $z = 1\text{cis}(\theta)$ and $\frac{1}{z} = z^{-1} = 1^{-1}\text{cis}(-\theta)$ (using de Moivre's theorem).

So, $z + \frac{1}{z}$

$= 1\text{cis}(\theta) + 1\text{cis}(-\theta)$ since $1^{-1} = 1$

$= \cos(\theta) + i\sin(\theta) + \cos(-\theta) + i\sin(-\theta)$

Now, $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$

So, $\cos(\theta) + i\sin(\theta) + \cos(-\theta) + i\sin(-\theta)$

$= \cos(\theta) + i\sin(\theta) + \cos(\theta) - i\sin(\theta)$

$= 2\cos(\theta)$

Mark allocation

- 1 mark for showing $z + \frac{1}{z} = 2\cos(\theta)$.

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Worked solution

iii. From part **ii**

$$2\cos(\theta) = z + \frac{1}{z}$$

$$(2\cos(\theta))^3 = \left(z + \frac{1}{z}\right)^3$$

$$8\cos^3(\theta) = z^3 + 3z^2.\frac{1}{z} + 3z\left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3$$

$$8\cos^3(\theta) = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$$

$$8\cos^3(\theta) = \left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)$$

$8\cos^3(\theta) = 2\cos(3\theta) + 3\times 2\cos(\theta)$ (from parts **i** and **ii**)

$$8\cos^3(\theta) = 2\cos(3\theta) + 6\cos(\theta)$$

$$4\cos^3(\theta) = \cos(3\theta) + 3\cos(\theta)$$

Alternatively, the expression could be shown using compound and double angle formulae to write $\cos(3\theta)$ in terms of $\cos(\theta)$ only.

Mark allocation: 4 marks

- 1 mark for cubing both sides of $2\cos(\theta) = z + \frac{1}{z}$.
- 1 mark for expanding correctly $8\cos^3(\theta) = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$.
- 1 mark for grouping correctly $z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} = \left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)$.
- 1 mark for using parts **i** and **ii** to show $4\cos^3(\theta) = \cos(3\theta) + 3\cos(\theta)$.

END OF SECTION 2

END OF SOLUTIONS BOOK

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