2013 Specialist Mathematics Trial Exam 1 Solutions

Q1a $-\frac{\pi}{2} \le \frac{1-y}{2} \le \frac{\pi}{2}$, $-\pi \le 1-y \le \pi$, $-1-\pi \le -y \le -1+\pi$

$1+\pi \ge y \ge 1-\pi$, $1-\pi \le y \le 1+\pi$.

The range is $[1-\pi, 1+\pi]$.

Q1b $Sin\frac{1-y}{2} = 1-2x$, $\frac{1-y}{2} = \sin^{-1}(1-2x)$,

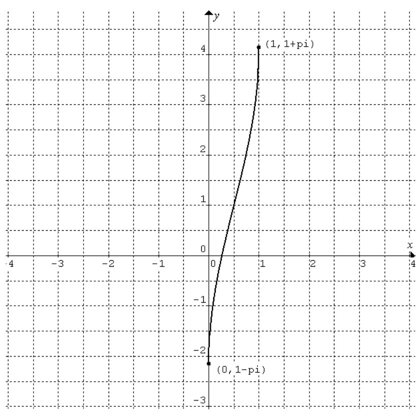
$y = 1-2\sin^{-1}(1-2x)$, .: $-1 \le 1-2x \le 1$, $0 \le x \le 1$

The maximal domain is $[0, 1]$.

Q1c When $x = \frac{1}{4}$, $y = 1-2\sin^{-1}\left(1-2\left(\frac{1}{4}\right)\right)$,

$y = 1-2\sin^{-1}\left(\frac{1}{2}\right) = 1-2\times\frac{\pi}{6} = 1-\frac{\pi}{3}$

Q1d

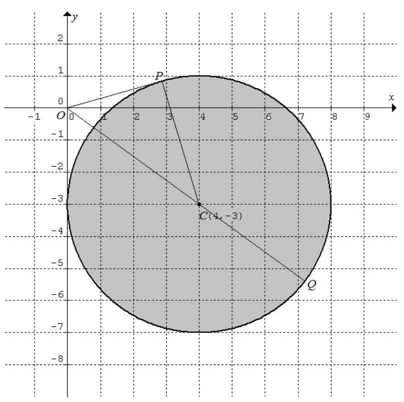


Q2 $2z^3 - iz^2 + 4z - 2i = 0$, $(2z^3 - iz^2) + (4z - 2i) = 0$,

$z^2(2z-i) + 2(2z-i) = 0$, $(z^2+2)(2z-i) = 0$,

.: $z = \pm\sqrt{2}i$, $\frac{i}{2}$

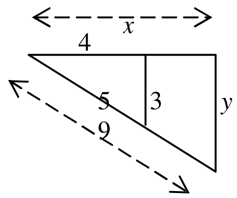
Q3a $S = \{z : 8 \ge |2z + 6i - 8|\}$, i.e. $S = \{z : |z - (4-3i)| \le 4\}$, a circle of radius 4 centred at $(4, -3)$. Set S is the shaded region.



Q3b Maximum $|z|$ occurs at point Q in the diagram.

$|z| = OQ = \sqrt{3^2+4^2} + 4 = 9$

Let $z = x - iy$.



$\frac{y}{9} = \frac{3}{5}$, .: $y = \frac{27}{5}$

$\frac{x}{9} = \frac{4}{5}$, .: $x = \frac{36}{5}$. .: $z = \frac{36}{5} - \frac{27}{5}i$

Q3c Maximum value of $Arg(z)$ occurs at point P (refer to the diagram). OP is a tangent to the circle.

ΔPOC is a 3, 4, 5 right-angled triangle.

$\angle POC = \tan^{-1}\left(\frac{4}{3}\right)$, $\angle xOC = \tan^{-1}\left(\frac{3}{4}\right)$

$Arg(z) = \angle POx = \angle POC - \angle xOC$

$\tan\angle POx = \tan(\angle POC - \angle xOC) = \frac{\tan\angle POC - \tan\angle xOC}{1+\tan\angle POC \times \tan\angle xOC}$

$= \frac{\frac{4}{3}-\frac{3}{4}}{1+\frac{4}{3}\times\frac{3}{4}} = \frac{7}{24}$

.: $Arg(z) = \angle POx = \tan^{-1}\left(\frac{7}{24}\right)$.

Q4a $\overrightarrow{DG} = -\tilde{i} + 3\tilde{j} - 2\tilde{k}$

Q4bi Let θ be the angle between $\overrightarrow{DG}$ and $\overrightarrow{FC}$.

$\overrightarrow{DG} \,.\, \overrightarrow{FC} = |\overrightarrow{DG}|\,|\overrightarrow{FC}|\cos\theta$

$\cos\theta = \frac{\overrightarrow{DG}.\overrightarrow{FC}}{|\overrightarrow{DG}||\overrightarrow{FC}|} = \frac{-1+9-4}{\sqrt{1+9+4}\sqrt{1+9+4}} = \frac{2}{7}$

.: $\theta = \cos^{-1}\left(\frac{2}{7}\right)$

Q4bii $Scalar\ resolute = |DG|\cos\theta = \sqrt{14}\times\frac{2}{7} = \frac{2\sqrt{14}}{7}$

Q5 $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ are linearly ***dependent*** when $a\tilde{p} + b\tilde{q} + \tilde{r} = \tilde{0}$ where $a \ne 0$ and $b \ne 0$.

$a(m\tilde{i} - \tilde{j}) + b(m\tilde{j} + \tilde{k}) + (\tilde{i} - 8m\tilde{k}) = \tilde{0}$

.: $(am+1)\tilde{i} + (bm-a)\tilde{j} + (b-8m)\tilde{k} = \tilde{0}$

.: $am+1 = 0$, $bm - a = 0$ and $b - 8m = 0$

Solve the three equations simultaneously, $8m^3 + 1 = 0$

.: $m = -\frac{1}{2}$

.: $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ are linearly ***independent*** when $m \ne -\frac{1}{2}$,

i.e. $m \in R \setminus \left\{-\frac{1}{2}\right\}$

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Q6a $\tilde{r} = t\tilde{i} + \sqrt{3}t\tilde{j} - 4.9(1-t)^2\tilde{k}$

$\tilde{v} = \frac{d}{dt}\tilde{r} = \tilde{i} + \sqrt{3}\tilde{j} + 9.8(1-t)\tilde{k}$

Horizontal speed $= \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$ m s^{-1}

Q6b *Speed* $= |\tilde{v}| = \sqrt{1^2 + \left(\sqrt{3}\right)^2 + 9.8^2(1-t)^2}$

It is a minimum value at $t = 1$, $\therefore$ *minimum speed* $= 2$ m s^{-1}

Q6c *Acceleration* $= \frac{d}{dt}\tilde{v} = -9.8\tilde{k}$ which is constant.

Q7a $f'(x) = \frac{16\tan^{-1}x}{1+x^2}$, $f(x) = \int \frac{16\tan^{-1}x}{1+x^2}dx$

Let $u = \tan^{-1}x$, $\frac{du}{dx} = \frac{1}{1+x^2}$

$\therefore f(x) = \int 16u\frac{du}{dx}dx = \int 16u\,du = 8u^2 + c = 8\left(\tan^{-1}x\right)^2 + c$

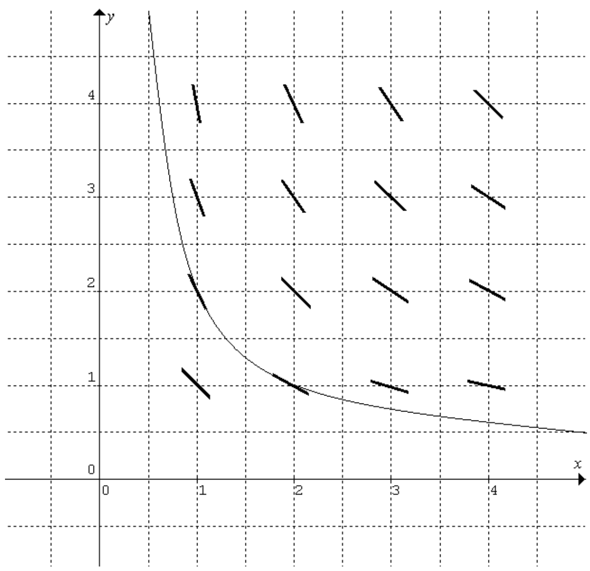
Q7b $\tan^{-1}x$ is an odd function, and $\frac{1}{1+x^2}$ is an even function,

$\therefore f'(x) = \frac{16\tan^{-1}x}{1+x^2}$ is an odd function.

Area $= -\int_{-1}^{0} f'(x)dx + \int_{0}^{1} f'(x)dx = 2\int_{0}^{1} f'(x)dx = 2\left[f(x)\right]_0^1$

$= 2\left[8\left(\tan^{-1}x\right)^2\right]_0^1 = \pi^2$

Q8a Q8b



Q9a $|reaction| = |weight| = mg = 5 \times 9.8 = 49$ N

(Newton's third law)

Q9b $|friction| = 49\sin 30° = 24.5$ N

Q10a $u = {}^{-}5$, $v = {}^{+}10$, $t = 10$, $s = \frac{1}{2}(u+v)t = {}^{+}25$

$\therefore$ *position* $x = {}^{-}15 + {}^{+}25 = {}^{+}10$ m

Q10b Displacement:

$t = 0$ to $t = 10$, $s = {}^{+}25$

$t = 10$ to $t = 20$, $s = \frac{1}{2}\left({}^{+}10 + {}^{+}15\right) \times 10 = {}^{+}125$

$t = 20$ to $t = 30$, $s = \frac{1}{2}\left({}^{+}15\right) \times 10 = {}^{+}75$

Total displacement $= {}^{+}25 + {}^{+}125 + {}^{+}75 = {}^{+}225$ m

Average velocity $= \frac{displacement}{time} = \frac{{}^{+}225}{30} = {}^{+}7.5$ m s^{-1}

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