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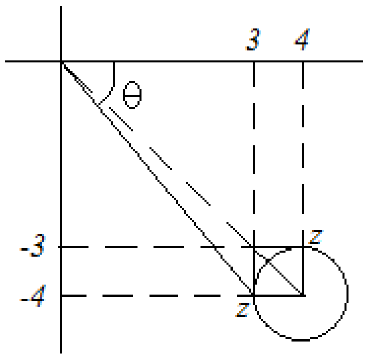
2013 Specialist Maths Trial Exam 2 Solutions

Section 1

1	2	3	4	5	6	7	8	9	10	11
B	E	C	D	B	D	C	E	B	D	A

12	13	14	15	16	17	18	19	20	21	22
C	E	D	D	A	A	C	B	D	C	C

Q1 $\tan\theta = -\frac{4}{3}$, (another possible answer is $\tan\theta = -\frac{3}{4}$) B



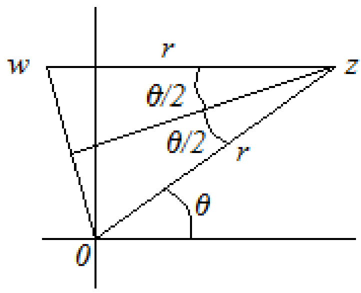
Q2 $i^7 z = \frac{\pi}{2} cis\left(-\frac{2\pi}{3}\right)$, $i^8 z = i\frac{\pi}{2} cis\left(-\frac{2\pi}{3}\right)$,

$z = i\frac{\pi}{2} cis\left(-\frac{2\pi}{3}\right)$, $z = \frac{\pi}{2} cis\left(-\frac{2\pi}{3}+\frac{\pi}{2}\right) = \frac{\pi}{2} cis\left(-\frac{\pi}{6}\right)$ E

Q3 $2\,\mathrm{Re}(z) = \mathrm{Im}(z)$ forms a straight line through the origin.

$a|z|^2 + b|z| - c = 0$, .: $|z| = \frac{-b+\sqrt{b^2+4ac}}{2a}$ forms a circle centred at the origin. The two sets have two intersections. C

Q4 Refer to the diagram below. $|w| = 2r\sin\frac{\theta}{2}$ D



Q5 $\frac{b}{a} = \tan 60°$, $\frac{b}{a} = \frac{\sqrt{3}}{1}$, $\frac{b^2}{a^2} = \frac{3}{1}$ B

Q6 $y = \frac{1}{x^2 - px + q}$ has a turning point when $x^2 - px + q$ is ***not*** a perfect square, i.e. $\Delta \neq 0$, $p^2 - 4q \neq 0$, $p^2 \neq 4q$ D

Q7 $\frac{(x-2)^2}{4} + 4y^2 = 1 \rightarrow \frac{x^2}{4} + 4y^2 = 1$, $\left(\frac{x}{2}\right)^2 + (2y)^2 = 1$

$\rightarrow x^2 + y^2 = 1$ C

Q8 $\sec(a+b) = -\operatorname{cosec}(a-b)$, $\frac{1}{\cos(a+b)} = -\frac{1}{\sin(a-b)}$,

$\cos(a+b) = -\sin(a-b)$,

.: $\sin\left(\frac{\pi}{2} - (a+b)\right) = \sin(-(a-b))$ or

$\sin\left(\frac{\pi}{2} - (a+b)\right) = \sin(\pi - (a-b))$

.: $\frac{\pi}{2} - (a+b) = -(a-b)$ or $\frac{\pi}{2} - (a+b) = \pi - (a-b)$

.: $b = \frac{\pi}{4}$ or $-\frac{\pi}{4}$ E

Q9 $f(x)$ is a decreasing function. $f\left(\frac{1}{a}\right) = \frac{2}{3}$, $f(0) = \frac{4}{3}$.

The range of f is $\left[\frac{2}{3}, \frac{4}{3}\right)$ and it is the domain of f^{-1}. B

Q10 Let $\frac{x-c}{b} = \theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

.: $\frac{x-c}{a} = \frac{b}{a}\theta$ and the equation is $\frac{b}{a}\tan^{-1}\left(\frac{b}{a}\theta\right) - \tan\theta = 0$.

More than one solution when $\frac{b}{a} > 1$, .: $a < b$ D

Q11 The addition of position vectors is undefined in kinematics. A

Q12 The vectors $3\tilde{k} - a\tilde{i}$, $\tilde{i} - b\tilde{j}$ and $2\tilde{j} - c\tilde{k}$ are linearly *dependent* if

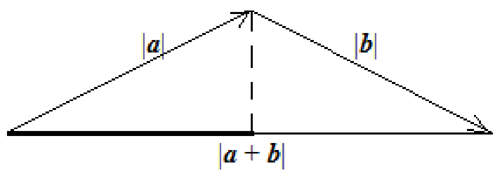
$3\tilde{k} - a\tilde{i} = m(\tilde{i} - b\tilde{j}) + n(2\tilde{j} - c\tilde{k}) = m\tilde{i} + (2n - bm)\tilde{j} - cn\tilde{k}$

.: $m = -a$, $n = -\frac{3}{c}$ and $2n - bm = 0$,

.: $-\frac{6}{c} + ab = 0$, .: $abc = 6$

.: the vectors $3\tilde{k} - a\tilde{i}$, $\tilde{i} - b\tilde{j}$ and $2\tilde{j} - c\tilde{k}$ are linearly *independent* if $abc \neq 6$. C

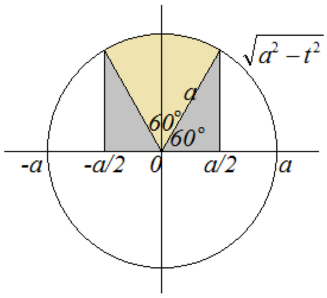
Q13 Refer to the following diagram. The scalar resolute of $\tilde{a}$ in the direction of $\tilde{a} + \tilde{b}$ is the darker line segment which is half as long as vector $\tilde{a} + \tilde{b}$, i.e. $\frac{1}{2}|\tilde{a} + \tilde{b}|$. E



Q14 $\int_{-\frac{a}{2}}^{\frac{a}{2}} \sqrt{a^2-t^2}\,dt =$ *area of the shaded regions*

$= \textit{area of the sector} + \textit{total area of the 2 triangles}$

$= \left(\dfrac{\pi}{6}+\dfrac{\sqrt{3}}{4}\right)a^2$ D



Q15 Find the intersections: $\left(\tan^{-1}x\right)^2 = \dfrac{\pi^2}{16}$, $x = \pm 1$

$Area = \int_{-1}^{1}\left(\dfrac{\pi^2}{16}-\left(\tan^{-1}x\right)^2\right)dx \approx 0.7431$ by CAS D

Q16 $t = \tan^{-1}x$, $\dfrac{dt}{dx} = \dfrac{1}{1+x^2}$

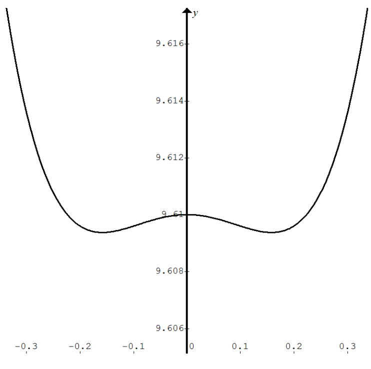
$\dfrac{dy}{dx} = \dfrac{dt}{dx}\times\dfrac{dy}{dt}$, .: $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}\times\dfrac{dy}{dt}$

$\left(\tan^{-1}x\right)^2\dfrac{dy}{dx} - \dfrac{1}{1+x^2} = 0$, $t^2\times\dfrac{1}{1+x^2}\times\dfrac{dy}{dt} - \dfrac{1}{1+x^2} = 0$

.: $\dfrac{1}{1+x^2}\left(t^2\dfrac{dy}{dt}-1\right) = 0$

Since $\dfrac{1}{1+x^2} \neq 0$, .: $t^2\dfrac{dy}{dt} - 1 = 0$ A

Q17 A



Q18 $v = \dfrac{dx}{dt} = 2e^{-x} - 1$, $\dfrac{dx}{dt} = \dfrac{2-e^x}{e^x}$, $\dfrac{dt}{dx} = \dfrac{e^x}{2-e^x}$

.: $t = \int \dfrac{e^x}{2-e^x}dx = -\log_e\left(2-e^x\right)$ satisfying $x = 0$ initially.

.: $e^{-t} = 2 - e^x$, .: $e^{-x} = \dfrac{1}{2-e^{-t}} = \dfrac{e^t}{2e^t-1}$

.: $v = 2e^{-x} - 1 = \dfrac{2e^t}{2e^t-1} - 1 = \dfrac{1}{2e^t-1}$ C

Q19 $\Delta\tilde{p} = m\Delta\tilde{v}$, $\Delta\tilde{v} = \dfrac{\Delta\tilde{p}}{m} = \dfrac{-3\tilde{i}+3\tilde{j}-1.5\tilde{k}}{m}$

$\tilde{a} = \tilde{a}_{average} = \dfrac{\Delta\tilde{v}}{\Delta t} = \dfrac{-3\tilde{i}+3\tilde{j}-1.5\tilde{k}}{5.0m} = \dfrac{-0.6\tilde{i}+0.6\tilde{j}-0.3\tilde{k}}{m}$

$\tilde{R} = m\tilde{a} = -0.6\tilde{i}+0.6\tilde{j}-0.3\tilde{k}$

.: $\left|\tilde{R}\right| = \sqrt{(-0.6)^2+0.6^2+(-0.3)^2} = 0.9$ N B

Q20 *Total distance travelled in the first 40 seconds*

$= \dfrac{1}{2}\times(10+30)\times5 + \dfrac{1}{2}\times(5+10)\times5 = 137.5$ m

Average speed $= \dfrac{137.5}{40} \approx 3.4$ m s^{-1} D

Q21 Let P be the reaction force of the crate on the machine.

$R = ma$, $1500\times9.8 - P = 1500\times0.8$

.: $P = 13500$ N C

Q22 For one crate: $99 - \mu M\times9.8 = M\times1.00$ ……… (1)

For two crates: $99 - \mu(2M)\times9.8 = (2M)\times0.010$ …… (2)

$(1)-(2)$: $\mu M\times9.8 = M\times0.98$, .: $\mu = 0.10$ ……….. (3)

Substitute (3) in (1): $M = 50$ C

Section 2

Q1a $\dfrac{z}{2} = \sqrt{a-\sqrt{a+\dfrac{z}{2}}}$, $\dfrac{z^2}{4} = a - \sqrt{a+\dfrac{z}{2}}$, $a - \dfrac{z^2}{4} = \sqrt{a+\dfrac{z}{2}}$

$\left(a-\dfrac{z^2}{4}\right)^2 = a+\dfrac{z}{2}$, $a^2 - \dfrac{az^2}{2} + \dfrac{z^4}{16} = a + \dfrac{z}{2}$

.: $\dfrac{z^4}{16} - \dfrac{az^2}{2} - \dfrac{z}{2} + a^2 - a = 0$

.: $z^4 - 8az^2 - 8z + 16\left(a^2-a\right) = 0$

.: $l = -8a$, $m = -8$ and $n = 16\left(a^2-a\right)$

Q1bi $z^4 - 8az^2 - 8z + 16\left(a^2-a\right) = \left(z^2+2z+p\right)\left(z^2+rz+q\right)$

$= z^4 + (r+2)z^3 + (p+q+2r)z^2 + (pr+2q)z + pq$

.: $r+2=0$, $p+q+2r = -8a$, $pr+2q = -8$ and

$pq = 16\left(a^2-a\right)$

.: $r = -2$, $p+q = 4-8a$, $q-p = -4$

.: $2q = -8a$, $q = -4a$ and $p = 4+q = 4-4a$

Q1bii $z^4 - 8az^2 - 8z + 16\left(a^2-a\right) = 0$

$\left(z^2+2z+(4-4a)\right)\left(z^2-2z-4a\right) = 0$

$z^2+2z+(4-4a) = 0$ or $z^2-2z-4a = 0$

By quadratic formula: $z = -1\pm\sqrt{4a-3}$ or $z = 1\pm\sqrt{4a+1}$

Q1ci All real solutions: $4a-3\geq0$ AND $4a+1\geq0$

i.e. $a\geq\dfrac{3}{4}$ AND $a\geq-\dfrac{1}{4}$, .: $a\geq\dfrac{3}{4}$

Q1cii All solutions has imaginary part:

$4a-3<0$ AND $4a+1<0$,

i.e. $a<\dfrac{3}{4}$ AND $a<-\dfrac{1}{4}$, .: $a<-\dfrac{1}{4}$

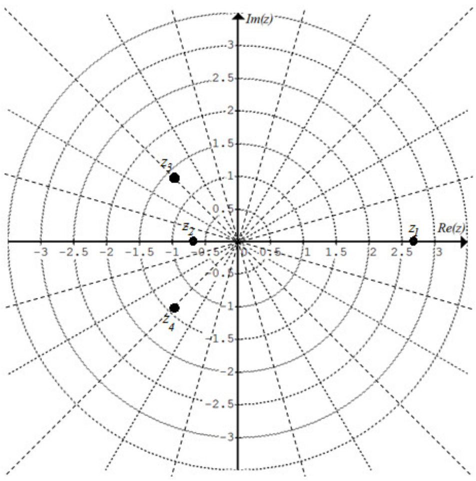
Q1ciii To have both real solutions and solutions with imaginary part: $a<\frac{3}{4}$ AND $a\ge -\frac{1}{4}$, .: $-\frac{1}{4}\le a<\frac{3}{4}$

Q1di $z=-1\pm\sqrt{4a-3}$ OR $z=1\pm\sqrt{4a+1}$

When $a=\frac{1}{2}$, $z=-1\pm i=\sqrt{2}cis\left(\pm\frac{3\pi}{4}\right)$

OR $z=1\pm\sqrt{3}$ $=\left(1+\sqrt{3}\right)cis0$ or $\left(\sqrt{3}-1\right)cis\pi$

Q1dii $z_1=\left(1+\sqrt{3}\right)cis0$, $z_2=\left(\sqrt{3}-1\right)cis\pi$, $z_3=\sqrt{2}cis\left(\frac{3\pi}{4}\right)$, $z_4=\sqrt{2}cis\left(-\frac{3\pi}{4}\right)$

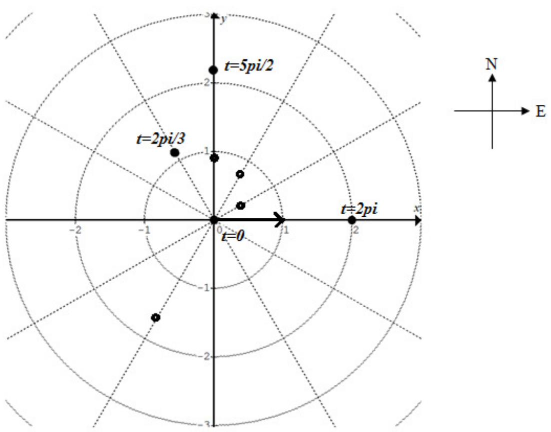


Q2ai $\tilde{r}(0)=\log_e(1)\left[\cos(0)\tilde{i}+\sin(0)\tilde{j}\right]=\tilde{0}$

Q2aii

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	2π	$\frac{5\pi}{2}$
$\lvert\tilde{r}\rvert$	0.00	***0.42***	0.72	0.94	1.13	***1.65***	1.99	2.18

Q2aiii, Q2aiv and Q2biii



Q2bi $\tilde{r}(t)=\log_e(t+1)\left[\cos(t)\tilde{i}+\sin(t)\tilde{j}\right]$

$\tilde{v}(t)=\frac{1}{t+1}\left[\cos(t)\tilde{i}+\sin(t)\tilde{j}\right]+\log_e(t+1)\left[-\sin(t)\tilde{i}+\cos(t)\tilde{j}\right]$

$=\left(\frac{\cos(t)}{t+1}-\log_e(t+1)\sin(t)\right)\tilde{i}+\left(\frac{\sin(t)}{t+1}+\log_e(t+1)\cos(t)\right)\tilde{j}$

Q2bii $\tilde{v}(0)=\tilde{i}$

Q2biv Let $\frac{\cos(t)}{t+1}-\log_e(t+1)\sin(t)=0$, by CAS $t\approx 0.78$ s, heading north; $t=3.30$ s, heading south.

Q2bv $\tilde{v}(3.3)\approx 0\tilde{i}-1.48\tilde{j}$, $speed\approx 1.48$ m s^{-1}

Q3a $\overrightarrow{OM}=\frac{1}{2}\left(\tilde{b}+\tilde{c}\right)$, $\overrightarrow{ON}=\frac{1}{2}\left(\tilde{c}+\tilde{a}\right)$

Q3bi

$2\overrightarrow{OM}=\tilde{b}+\tilde{c}$, .: $-2\overrightarrow{OM}+\tilde{b}+\tilde{c}=\tilde{0}$, $2m\tilde{a}+\tilde{b}+\tilde{c}=\tilde{0}$ ... (1)

$2\overrightarrow{ON}=\tilde{c}+\tilde{a}$, .: $-2\overrightarrow{ON}+\tilde{c}+\tilde{a}=\tilde{0}$, $\tilde{a}+2n\tilde{b}+\tilde{c}=\tilde{0}$ ….. (2)

Q3bii (2) – (1): $\tilde{a}-2m\tilde{a}-\tilde{b}+2n\tilde{b}=\tilde{0}$

$(1-2m)\tilde{a}-(1-2n)\tilde{b}=\tilde{0}$

Q3biii Since $\tilde{a}$ and $\tilde{b}$ are independent (non-parallel), .: $1-2m=0$ and $1-2n=0$

.: $m=\frac{1}{2}$ and $n=\frac{1}{2}$

Q3biv Since $2m\tilde{a}+\tilde{b}+\tilde{c}=\tilde{0}$, .: $\tilde{a}+\tilde{b}+\tilde{c}=\tilde{0}$, .: $\tilde{b}=-\tilde{a}-\tilde{c}$

Q3bv Since $\overrightarrow{AB}=\tilde{b}-\tilde{a}$, .: $\overrightarrow{AB}=\left(-\tilde{a}-\tilde{c}\right)-\tilde{a}=-2\tilde{a}-\tilde{c}$

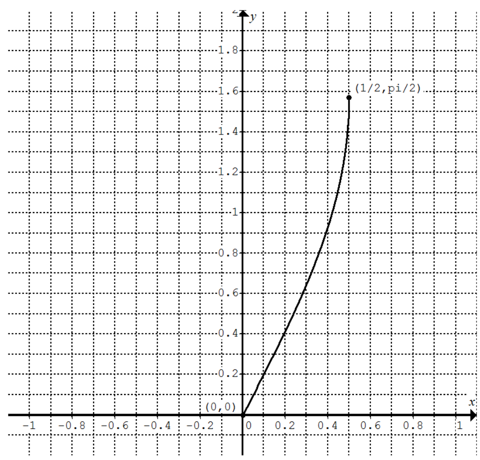
Q3ci $\overrightarrow{AP}=k\overrightarrow{AB}$, $-\tilde{a}-p\tilde{c}=k\left(-2\tilde{a}-\tilde{c}\right)$,

$2k\tilde{a}-\tilde{a}+k\tilde{c}-p\tilde{c}=\tilde{0}$, .: $(2k-1)\tilde{a}+(k-p)\tilde{c}=\tilde{0}$

Q3cii Since $\tilde{a}$ and $\tilde{c}$ are independent, .: $2k-1=0$ and $k=p$, .: $k=\frac{1}{2}$ and $p=\frac{1}{2}$

Q3ciii $\overrightarrow{AP}=k\overrightarrow{AB}$, .: $\overrightarrow{AP}=\frac{1}{2}\overrightarrow{AB}$, .: P is the midpoint of $\overrightarrow{AB}$, .: $\overline{CP}$ is a median of ΔABC.

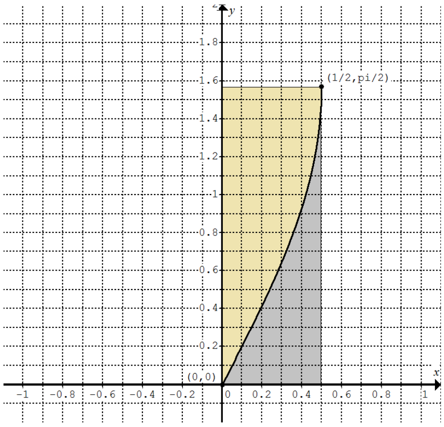
Q4a $x=0$, $y=0$; $x=\frac{1}{2}$, $y=\frac{\pi}{2}$



Q4b $\frac{dy}{dx}=\frac{2}{\sqrt{1-4x^2}}=4$, $1-4x^2=\frac{1}{4}$, $x=\frac{\sqrt{3}}{4}$,

.: $y=\sin^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{3}$, .: the point is $\left(\frac{\sqrt{3}}{4},\frac{\pi}{3}\right)$.

Q4c



$y=\sin^{-1}2x$, .: $x=\frac{1}{2}\sin y$

$Area=\int_0^{\frac{1}{2}}\sin^{-1}(2x)dx=\frac{1}{2}\times\frac{\pi}{2}-\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin y dy$

$=\frac{\pi}{4}+\frac{1}{2}\int_0^{\frac{\pi}{2}}(-\sin y)dy=\frac{\pi}{4}+\frac{1}{2}[\cos y]_0^{\frac{\pi}{2}}=\frac{\pi}{4}-\frac{1}{2}$

Q4d $V_{top}=\int_0^{\frac{\pi}{2}}\pi x^2 dy=\int_0^{\frac{\pi}{2}}\frac{\pi}{4}\sin^2 y dy=\int_0^{\frac{\pi}{2}}\frac{\pi}{8}(1-\cos 2y)dy$

$=\frac{\pi}{8}\left[y-\frac{\sin 2y}{2}\right]_0^{\frac{\pi}{2}}=\frac{\pi^2}{16}$ unit^3

Volume of wood cut out from the block

$=1\times1\times2-\frac{\pi^2}{16}=2-\frac{\pi^2}{16}$ unit^3

Q4e $y=\frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-x^2}}$, $y^2=\frac{1}{\frac{1}{4}-x^2}$, $x^2=\frac{1}{4}-\frac{1}{y^2}$

When $x=0$, $y=10$

$V_{dowel}=\int_2^{10}\pi x^2 dy=\pi\int_2^{10}\left(\frac{1}{4}-\frac{1}{y^2}\right)dy=\pi\left[\frac{y}{4}+\frac{1}{y}\right]_2^{10}=\frac{8\pi}{5}$ unit^3

Q5a Let $\tilde{i}$ and $\tilde{j}$ be unit vectors pointing to the east and to the north respectively.

Resultant $\tilde{R}$ of the 3 pulling forces

$=(16\sin120°+24\sin(-135°))\tilde{i}+(28+16\cos120°+24\cos135°)\tilde{j}$

$=-3.1142\tilde{i}+3.0294\tilde{j}$

$\left|\tilde{R}\right|=4.3446\approx4.3$ N

$\tan\theta=\frac{3.114}{3.029}$, $\theta\approx45.8°$, i.e. $N45.8°W$

Q5b Approximately 4.3 N $N45.8°W$

Q5c *Limiting friction* $=\mu N$, $4.3446\approx0.25\times m\times9.8$, $m\approx1.8$ kg

Q5d *Resultant $\tilde{R}$ of the 3 pulling forces*

$=\left(16\sin120°+20\sqrt{2}\sin(-135°)\right)\tilde{i}+\left(28+16\cos120°+20\sqrt{2}\cos135°\right)\tilde{j}$

$=-6.1436\tilde{i}$

Resultant force on the box $=ma$

.: $-6.1436\tilde{i}+4.3446\tilde{i}\approx1.8\tilde{a}$

.: $\tilde{a}\approx-1.0\tilde{i}$ m s^{-2}, i.e. 1.0 m s^{-2} west

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