# 2013 VCAA Specialist Math Exam 2 Solutions



**SECTION 1**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| D | A | E | D | B | D | E | C | B | D | C |

| 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|---|---|---|---|---|---|---|---|---|---|---|
| E | A | B | E | A | B | D | C | A | C | E |

Q1 $-1 \le 3x \le 1$, $-\frac{1}{3} \le x \le \frac{1}{3}$ D

Q2 $x = 2\operatorname{cosec}(t) + 1$, $\operatorname{cosec}(t) = \frac{x-1}{2}$

$y = 3\cot(t) - 1$, $\cot(t) = \frac{y+1}{3}$

$\operatorname{cosec}^2(t) - \cot^2(t) = 1$, .: $\frac{(x-1)^2}{4} - \frac{(y+1)^2}{9} = 1$ A

Q3 $y = \frac{1}{ax^2 + bx + c} = \frac{1}{a(x+5)(x-3)} = \frac{1}{ax^2 + 2ax - 15a}$

Stationary point lies in the middle of $x = -5$ and $x = 3$, i.e.

$x = \frac{-5+3}{2} = -1$, and $y = -\frac{1}{8}$, .: $-\frac{1}{8} = \frac{1}{a - 2a - 15a}$

.: $a = \frac{1}{2}$, $b = 2a = 1$ and $c = -15a = -\frac{15}{2}$ E

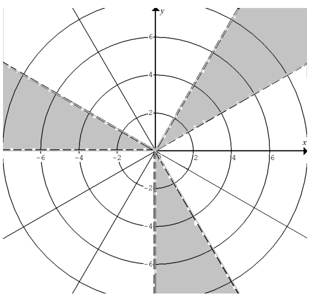
Q4 $y = \tan^{-1}(bx)$, $\frac{dy}{dx} = \frac{b}{1+(bx)^2} = b$ at $x = 0$

To intersect exactly 3 times, gradient of $y = \tan^{-1}(bx)$ at $x = 0$ must be greater than the gradient of $y = ax$ for positive gradients, OR $y = \tan^{-1}(bx)$ at $x = 0$ must be less than the gradient of $y = ax$ for negative gradients.

.: Either $b > a > 0$ or $b < a < 0$ D

Q5 $x^2 + y^2 > b^2$, $|z|^2 > b^2$, $|z| > b$ B

Q6 The complete set is $\left(-\frac{\pi}{2}, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{6}, \frac{\pi}{3}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$.



The best choice is D

Q7 $z = r\,cis(\theta)$, $\frac{z^2}{\bar{z}} = \frac{r^2 cis(2\theta)}{r\,cis(-\theta)} = r\,cis(3\theta)$ E

Q8 $z^2 = 1 + i$, $z^2 = \sqrt{2}\,cis\left(\frac{\pi}{4} + 2k\pi\right)$, $z = 2^{\frac{1}{4}}\,cis\left(\frac{\pi}{8} + k\pi\right)$

Principal arguments are $-\frac{7\pi}{8}$ and $\frac{\pi}{8}$. C

Q9 $u = \log_e(x)$, $\frac{du}{dx} = \frac{1}{x}$. $x = e^3$, $u = 3$; $x = e^4$, $u = 4$

$\int_{e^3}^{e^4} \frac{1}{x\log_e(x)}dx = \int_3^4 \frac{1}{u}du$, .: $a = 3$ and $b = 4$ B

Q10 $V = \int_0^3 \pi x^2 dy = \pi\int_0^3 y^{\frac{3}{2}} dy = \frac{18\pi 3^{\frac{1}{2}}}{5}$ D

Q11 $y_{n+1} \approx y_n + h \times \frac{dy}{dx}$

$x_0 = 0$ $y_0 = 1$ $\frac{dy}{dx} = \frac{1}{3}$

$x_1 = 0.1$ $y_1 \approx 1 + 0.1 \times \frac{1}{3} = \frac{31}{30}$ $\frac{dy}{dx} = 0.302115$

$x_2 = 0.2$ $y_2 \approx \frac{31}{30} + 0.1 \times 0.302115 \approx 1.064$ C

Q12 The solution curves are of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$.

By implicit differentiation, $\frac{2x}{a^2} - \frac{2y}{b^2} \times \frac{dy}{dx} = 0$, .: $\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$ E

Q13 At time *t* min., *amount of salt* $= Q$ grams,

*volume of solution* $= 50 + 4t$ litres,

.: *concentration* $= \frac{Q}{50+4t}$ grams per litre

*Rate of inflow* $= 20$ grams per min

*Rate of outflow* $= \frac{6Q}{50+4t} = \frac{3Q}{25+2t}$ grams per min

.: $\frac{dQ}{dt} = 20 - \frac{3Q}{25+2t}$ A

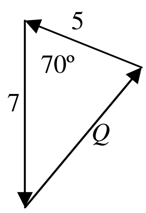
Q14 $\left|\overrightarrow{OP}\right| = \sqrt{7^2 + (-1)^2 + \left(5\sqrt{2}\right)^2} = 10$ B

Q15 $|\tilde{u}| = \sqrt{18}$, $|\tilde{v}| = \sqrt{18}$, $|-\tilde{w}| = \sqrt{18}$, $\tilde{u}.\tilde{v} = 0$

$(\tilde{u} + \tilde{w}).\tilde{v} = 6$ E

Q16

$Q = \sqrt{5^2 + 7^2 - 2(5)(7)\cos 70°}$ A




Q17 Three vectors on the same plane are always linearly dependent. $\tilde{a}$, $\tilde{c}$ and $\ddot{d}$ are on the same plane. B

Q18 $\frac{1}{2}\frac{d(v^2)}{dx}=\sqrt{(v^2)-1}$ , $\frac{dx}{d(v^2)}=\frac{1}{2\sqrt{(v^2)-1}}$ ,

$x=\int\frac{1}{2\sqrt{(v^2)-1}}d(v^2)=\sqrt{v^2-1}+c$

$v=\sqrt{2}$ when $x=0$ , .: $c=-1$

.: $1+x=\sqrt{v^2-1}$ , $v^2=1+(1+x)^2$ , .: $v=\sqrt{1+(1+x)^2}$ D

Q19 $u=+2$ , $a=-9.8$ , $s=-100$ , $s=ut+\frac{1}{2}at^2$ , .: $t\approx 4.7$ s C

Q20 Let $R$ be the reaction force of the floor on the 5kg parcel.
$Resultant\ force=5g-R=5\times 3$ , .: $R=-15+5g$ A

Q21 $u=+20$ , $v=+2$ , $t=4$ , $v=u+at$ , $a=-4.5$ m s$^{-2}$
$|Resultant\ force|=m|a|=9.0$ N C

Q22 $ma=mg-kv^2$ , $mv\frac{dv}{dx}=mg-kv^2$ , .: $\frac{dv}{dx}=\frac{g}{v}-\frac{kv}{m}$ E

**SECTION 2**

Q1a $x=1+3\cos(t)$ , $\cos(t)=\frac{x-1}{3}$

$y=-2+2\sin(t)$, $\sin(t)=\frac{y+2}{2}$

$\sin^2(t)+\cos^2(t)=1$ , .: $\frac{(x-1)^2}{9}+\frac{(y+2)^2}{4}=1$

Q1b $\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$ , $2\cos(t)=-\frac{2\sqrt{3}}{3}\times(-3\sin(t))$

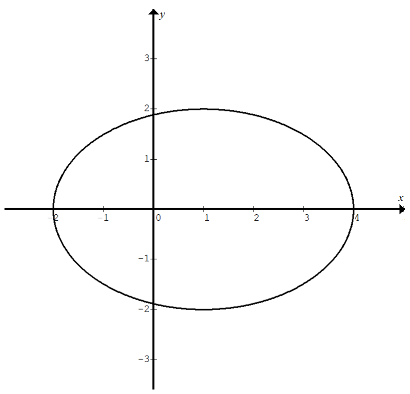
.: $\tan(t)=\frac{1}{\sqrt{3}}$ and $0\le t\le 2\pi$

.: $t=\frac{\pi}{6}$ , $\frac{7\pi}{6}$

Q1c $\frac{(x-1)^2}{9}+\frac{y^2}{4}=1$

$x$-intercepts: $y=0$ , $(x-1)^2=9$ , $x-1=\pm 3$ , .: $x=-2$ , 4

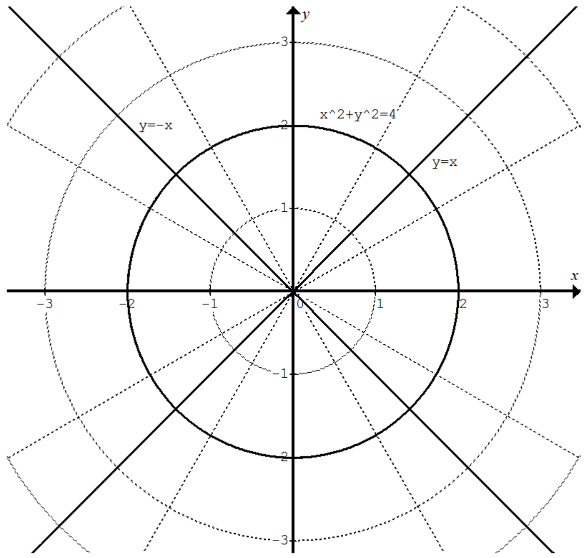
$y$-intercepts: $x=0$ , $y^2=\frac{32}{9}$ , $y=\pm\frac{4\sqrt{2}}{3}$



Q1di $Volume=\int_1^3\pi y^2dx=\int_1^3 4\pi\left(1-\frac{(x-1)^2}{9}\right)dx$

Q1dii $Volume=4\pi\left[x-\frac{(x-1)^3}{27}\right]_1^3=\frac{184\pi}{27}$ cubic units

Q2a $z\bar{z}=4$ , $x^2+y^2=4$ ; $|z-\bar{z}|=|z+\bar{z}|$ , $|y|=|x|$



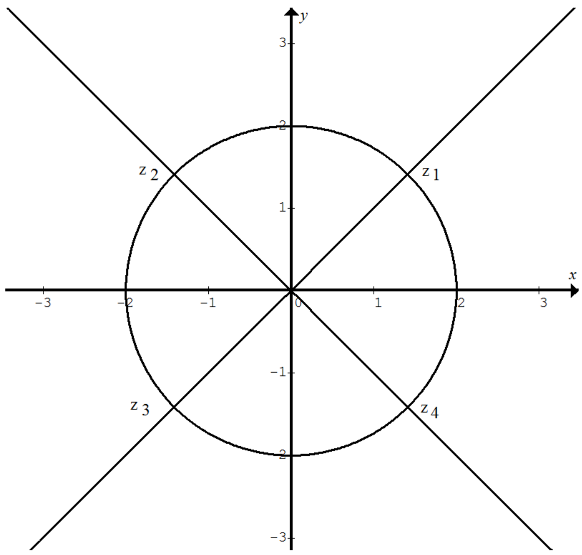

Q2b In the first quadrant: $x=2\cos\frac{\pi}{4}=\sqrt{2}$ , $y=2\sin\frac{\pi}{4}=\sqrt{2}$

.: $z=\sqrt{2}+i\sqrt{2}$

The others are: $z=-\sqrt{2}+i\sqrt{2}$ , $z=-\sqrt{2}-i\sqrt{2}$ , $z=\sqrt{2}-i\sqrt{2}$

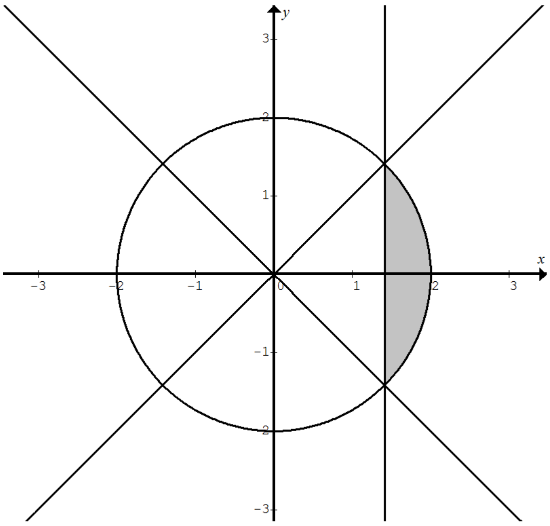
Q2c $z_1=\sqrt{2}+i\sqrt{2}$

$z_2=-\sqrt{2}+i\sqrt{2}$ , $z_3=-\sqrt{2}-i\sqrt{2}$ , $z_4=\sqrt{2}-i\sqrt{2}$



Q2d $z^4+16=(z-z_1)(z-z_2)(z-z_3)(z-z_4)$
$=\left(z-\sqrt{2}-i\sqrt{2}\right)\left(z+\sqrt{2}-i\sqrt{2}\right)\left(z+\sqrt{2}+i\sqrt{2}\right)\left(z-\sqrt{2}+i\sqrt{2}\right)$

Q2e



Q2f *Area of shaded segment* $=$ *area of sector* $-$ *area of triangle*
$=$ *area of quarter circle* $-$ *area of triangle*
$=\frac{1}{4}\times\pi 2^2-\frac{1}{2}\times 2\times 2=\pi-2$ square units

Q3a $\log_e(N)=6-3e^{-0.4t}$, $t\geq 0$

$\frac{d}{dt}\log_e(N)=\frac{d}{dt}\left(6-3e^{-0.4t}\right)$, $\frac{d}{dN}\log_e(N)\times\frac{dN}{dt}=\frac{d}{dt}\left(6-3e^{-0.4t}\right)$

$\therefore\ \frac{1}{N}\frac{dN}{dt}=1.2e^{-0.4t}$

$\therefore\ \frac{1}{N}\frac{dN}{dt}+0.4\log_e(N)-2.4=1.2e^{-0.4t}+0.4\left(6-3e^{-0.4t}\right)-2.4=0$

Q3b When $t=0$, $\log_e(N)=3$, $N=e^3\approx 20$

Q3c When $t\to\infty$, $\log_e(N)\to 6$, $N\to e^6\approx 403$

Q3di $\frac{dN}{dt}=0.4N\left(6-\log_e(N)\right)$

$\frac{d^2N}{dt^2}=0.4\frac{dN}{dt}\left(6-\log_e(N)\right)+0.4N\left(-\frac{1}{N}\right)\frac{dN}{dt}$

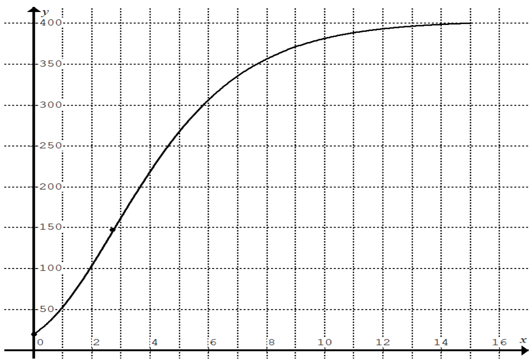
$=\frac{dN}{dt}\left(2-0.4\log_e(N)\right)=0.4\frac{dN}{dt}\left(5-\log_e(N)\right)$

$=0.16N\left(6-\log_e(N)\right)\left(5-\log_e(N)\right)$

Q3dii Let $\frac{d^2N}{dt^2}=0$ to find the point of inflection.

$5-\log_e(N)=0$, $N=e^5\approx 148$ when $t\approx 2.7$

Q3e




Q4a $\tilde{b}=\tilde{i}+\sqrt{3}\tilde{j}+2\sqrt{3}\tilde{k}$, $\left|\tilde{b}\right|=\sqrt{1^2+\left(\sqrt{3}\right)^2+\left(2\sqrt{3}\right)^2}=4$

$\hat{b}=\frac{\tilde{b}}{\left|\tilde{b}\right|}=\frac{1}{4}\left(\tilde{i}+\sqrt{3}\tilde{j}+2\sqrt{3}\tilde{k}\right)$

Q4b Component of $\tilde{a}$ parallel to $\tilde{b}$:

$\left(\tilde{a}.\tilde{b}\right)\hat{b}=\left(-\frac{4\sqrt{3}}{3}\right)\frac{1}{4}\left(\tilde{i}+\sqrt{3}\tilde{j}+2\sqrt{3}\tilde{k}\right)$

$=-\frac{\sqrt{3}}{3}\left(\tilde{i}+\sqrt{3}\tilde{j}+2\sqrt{3}\tilde{k}\right)=-\left(\frac{\sqrt{3}}{3}\tilde{i}+\tilde{j}+2\tilde{k}\right)$

Component of $\tilde{a}$ perpendicular to $\tilde{b}$:

$\tilde{a}-\left(\tilde{a}.\tilde{b}\right)\hat{b}=\left(-\frac{7\sqrt{3}}{3}\tilde{i}+\tilde{j}-2\tilde{k}\right)+\left(\frac{\sqrt{3}}{3}\tilde{i}+\tilde{j}+2\tilde{k}\right)$

$=-2\sqrt{3}\tilde{i}+2\tilde{j}$

Q4c $\tilde{c}.\tilde{b}=\left|\tilde{c}\right|\left|\tilde{b}\right|\cos\frac{2\pi}{3}$

$m+\sqrt{3}-4\sqrt{3}=\sqrt{m^2+1+4}\times 4\times\left(-\frac{1}{2}\right)$

$\therefore\ 3m^2+6\sqrt{3}m-7=0$ and given $\tilde{c}\neq\tilde{a}$, i.e. $m\neq-\frac{7\sqrt{3}}{3}$

$\therefore\ m=\frac{\sqrt{3}}{3}$

Q4d $\tilde{c}.\tilde{a}=\left|\tilde{c}\right|\left|\tilde{a}\right|\cos\theta$, $\cos\theta=\frac{\tilde{c}.\tilde{a}}{\left|\tilde{c}\right|\left|\tilde{a}\right|}=\frac{1}{4}$, $\therefore\ \theta\approx 75.5°$

Q4ei $\overrightarrow{AN}=\tilde{u}+\frac{1}{2}\tilde{v}$

Q4eii $\overrightarrow{CM}=\overrightarrow{CA}+\frac{1}{2}\overrightarrow{AB}=-\tilde{u}+\frac{1}{2}\left(\tilde{u}+\tilde{v}\right)=\frac{1}{2}\tilde{v}-\frac{1}{2}\tilde{u}$

$\overrightarrow{BP}=-\frac{1}{2}\tilde{u}-\tilde{v}$

Q4eiii

$\overrightarrow{AN}+\overrightarrow{CN}+\overrightarrow{BP}=\left(\tilde{u}+\frac{1}{2}\tilde{v}\right)+\left(\frac{1}{2}\tilde{v}-\frac{1}{2}\tilde{u}\right)+\left(-\frac{1}{2}\tilde{u}-\tilde{v}\right)=\tilde{0}$

Q5a

$\tilde{r}(t) = 7.5t\tilde{i} + \left(50 - 10\sin\left(\frac{\pi t}{6}\right)\right)\tilde{j}$ , $\dot{r}(t) = 7.5\tilde{i} - \frac{5\pi}{3}\cos\left(\frac{\pi t}{6}\right)\tilde{j}$

$Speed = |\dot{r}(t)| = \sqrt{7.5^2 + \left(-\frac{5\pi}{3}\cos\left(\frac{\pi t}{6}\right)\right)^2} \approx \sqrt{56.25 + 27.42\cos^2\left(\frac{\pi t}{6}\right)}$

*Minimum speed* $= \sqrt{56.25} = 7.5$ m s$^{-1}$

*Maximum speed* $\approx \sqrt{56.25 + 27.42} \approx 9.1$ m s$^{-1}$

Q5b Let $\ddot{r}(t) = \frac{5\pi^2}{18}\sin\left(\frac{\pi t}{6}\right)\tilde{j} = \tilde{0}$ , .: $\frac{\pi t}{6} = 0$ , $\pi$ , $2\pi$ , $3\pi$ ……

.: $t = 0$ , 6, 12, 18, …… i.e. $t = 6n$ where $n$ is an integer $\geq 0$ .

Q5c *Vertical component of initial velocity* $= 15\sin 30° = 7.5$

*Height of ramp* $= 10\sin 30° = 5$

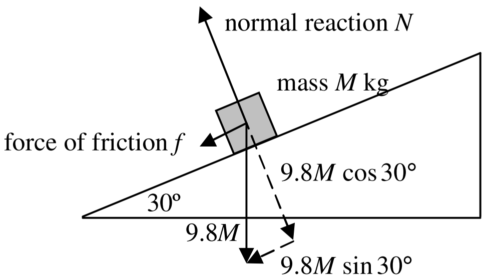
$u = +7.5$ , $s = -5$ , $a = -9.8$ , $s = ut + \frac{1}{2}at^2$

.: $4.9t^2 - 7.5t - 5 = 0$ , $t \approx 2.03$ s

Q5d *Horizontal component of initial velocity* $= 15\cos 30° \approx 12.99$

*Distance* $\approx 12.99 \times 2.03 \approx 26$ m

Q5e




$N = 9.8M\cos 30°$ , $f = \mu N = \frac{1}{8\sqrt{3}} \times 9.8M\cos 30° = \frac{9.8M}{16}$

$a = \frac{net.force}{mass} = \frac{-9.8M\sin 30° - \frac{9.8M}{16}}{M} = -5.5125$

$u = +10$ , $v = 0$ , $a = -5.5125$ , $v^2 = u^2 + 2as$ , $s \approx +9.1$ m

.: *distance up the plane* $= 9.1$ m

Q5f *Friction required* $= 9.8M\sin 30° = 4.9M$

$f = \mu N$ , $4.9M = \mu \times 9.8M\cos 30°$ , $4.9M = \mu \times 9.8M \times \frac{\sqrt{3}}{2}$

$\mu = \frac{1}{\sqrt{3}}$ which is $\mu = \tan 30°$ .

*Please inform mathline@itute.com re conceptual and/or mathematical errors*