Neap:

Trial Examination 2013

VCE Specialist Mathematics Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

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SECTION 1

Question 1 C

$x^2 - y^2 = 4x - 2y$

$x^2 - 4x + 4 - (y^2 - 2y + 1) = 3$

$(x-2)^2 - (y-1)^2 = 3$

$\frac{(x-2)^2}{3} - \frac{(y-1)^2}{3} = 1$

The equations of the asymptotes are $y - 1 = \pm(x-2)$ i.e. $y = x - 1$ and $y = 3 - x$.

Question 2 D

$x = a\cos(\theta)$ and $y = b\sin(\theta)$

$\frac{dx}{d\theta} = -a\sin(\theta)$ and $\frac{dy}{d\theta} = b\cos(\theta)$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$
$$= -\frac{b\cos(\theta)}{a\sin(\theta)},\ \sin(\theta) \neq 0$$

The equation of the tangent is $y - b\sin(\theta) = -\frac{b\cos(\theta)}{a\sin(\theta)}(x - a\cos(\theta))$

So $y = \frac{b(a - \cos(\theta)x)}{a\sin(\theta)}$.

Question 3 C

We require a vertical asymptote at $x = 1$.

So we can disregard **A**, **B** and **E**.

For $y = g(x)$:

As $x \to 0^+, y \to -\infty$

For $y = \frac{1}{g(x)}$:

As $x \to 0^+, y \to 0^-$

So we can disregard **D**.

Question 4 B

We require $-1 \le 2 - x \le 1$ and so the implied domain is $1 \le x \le 3$.

Question 5 E

From $\tan(x) = \frac{3}{4}$, we obtain $\sin(x) = \frac{3}{5}$ and $\cos(x) = \frac{4}{5}$.

From $\tan(y) = \frac{4}{3}$, we obtain $\sin(y) = \frac{4}{5}$ and $\cos(y) = \frac{3}{5}$.

$$\begin{aligned}\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y)\\ &= \frac{4}{5}\times\frac{3}{5} - \frac{3}{5}\times\frac{4}{5}\\ &= 0\end{aligned}$$

Question 6 A

$v \neq \bar{w}$ i.e. $\bar{w} = -\sqrt{3} + i$ and so **A** is incorrect

The other four alternatives are all correct.

Note that an equilateral triangle is a special case of an isosceles triangle having not just two, but all three sides and angles equal.

Question 7 E

Solving $(x-1)^2 + y^2 = 1$ and $x + y = 1$ simultaneously we obtain $x = \frac{2-\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$ or $x = \frac{2+\sqrt{2}}{2}$ and $y = -\frac{\sqrt{2}}{2}$

So for $T \cap V$ we have $\frac{2-\sqrt{2}}{2} \leq \text{Re}(z) \leq \frac{2+\sqrt{2}}{2}$.

Question 8 C

The cube roots of unity are 1, $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

Let $u = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ and let $v = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

$uv = 1$ and so **A** is incorrect.

$u \neq v$ so **B** is incorrect.

$v^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ and so $u = v^2$ i.e. **C** is correct.

$u = \bar{v}$ and so **D** is incorrect.

$u + v = -1$ and so **E** is incorrect.

Question 9 **A**

If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$ then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$.

Here $x_0 = 0$, $y_0 = 2$ and $h = \frac{1}{5}$.

For $x_0 = 0$:

$$\begin{aligned} y_1 &= y_0 + hf(x_0) \\ &= 2 + \frac{1}{5}\cos^{-1}(0) \\ &= 2 + \frac{\pi}{10} \end{aligned}$$

For $x_1 = \frac{1}{5}$:

$$\begin{aligned} y_2 &= y_1 + hf(x_1) \\ &= 2 + \frac{\pi}{10} + \frac{1}{5}\cos^{-1}\left(\frac{\pi}{10}\right) \end{aligned}$$

Question 10 **A**

There is a repeated linear factor in the denominator so the partial fraction form is $\frac{A}{(x-3)} + \frac{B}{(x-3)^2}$.

Question 11 **D**

The volume, in litres, of the mixture at time t minutes is $100 + 3t$.

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{dt_{\text{in}}} - \frac{dx}{dt_{\text{out}}} \\ &= 30 \times 10 - \left(\frac{x}{100 + 3t}\right) \times 7 \end{aligned}$$

So $\frac{dx}{dt} = 300 - \frac{7x}{100 + 3t}$ (g/min)

Question 12 C

Let V be the volume.

Using $V = \pi \int_a^b ((f(x))^2 - (g(x))^2)dx$ we obtain:

$$V = \pi \int_0^{\sqrt{3}} \left(\left(\frac{4}{\sqrt{1 + x^2}} \right)^2 - 2^2 \right) dx$$

$$= \pi \int_0^{\sqrt{3}} \left(\frac{16}{1 + x^2} - 4 \right) dx$$

Question 13 E

Let $u = 1 + 2\tan(x)$.

$$\frac{du}{dx} = 2\sec^2(x)$$

So $\int \frac{1}{(1 + 2\tan(x))^2 \cos^2(x)} dx = \int \frac{1}{2u^2}\, du$.

Question 14 B

As $\underset{\sim}{u} = 3\underset{\sim}{v} - 2\underset{\sim}{w}$ then each of these vectors can be expressed as a linear combination of the other two vectors. This is the definition for linearly dependent vectors.

Question 15 B

If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular and $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are non-zero vectors, then $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$.

$$\sin(2t)\cos(t) + \sin(t)\cos(2t) - 1 = 0$$

$$\sin(3t) - 1 = 0$$

So $t = \frac{\pi}{6}$ as $0 \le t \le \frac{\pi}{2}$.

Question 16 D

The sum of these two vectors is $\underset{\sim}{u}$

$$\underset{\sim}{u} = (2\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}) + (3\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k})$$

$$= 5\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$$

Question 17 A

$\underset{\sim}{a}(t) = \sin(t)\underset{\sim}{j}$

$\underset{\sim}{v}(t) = -\cos(t)\underset{\sim}{j} + \underset{\sim}{c}$

Using $\underset{\sim}{v}(\pi) = \underset{\sim}{i} + \underset{\sim}{j}$, we obtain $\underset{\sim}{i} + \underset{\sim}{j} = -\cos(\pi)\underset{\sim}{j} + c$

Hence $\underset{\sim}{c} = \underset{\sim}{i}$ and so $\underset{\sim}{v}(t) = \underset{\sim}{i} - \cos(t)\underset{\sim}{j}$

$\underset{\sim}{v}(0) = \underset{\sim}{i} - \cos(0)\underset{\sim}{j}$

$= \underset{\sim}{i} - \underset{\sim}{j}$

Question 18 A

Given $y = e^{pt}$:

$\frac{dy}{dt} = pe^{pt}$ and $\frac{d^2y}{dt^2} = p^2e^{pt}$

Rearranging the differential equation we obtain $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 0$

Substituting for y, $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ into the differential equation we obtain $p^2e^{pt} - 5pe^{pt} + 6e^{pt} = 0$.

Taking out e^{pt} as a common factor we obtain $e^{pt}(p^2 - 5p + 6) = 0$

$p^2 - 5p + 6 = 0$ (as $e^{pt} \neq 0$)

Hence $p = 2$ and 3.

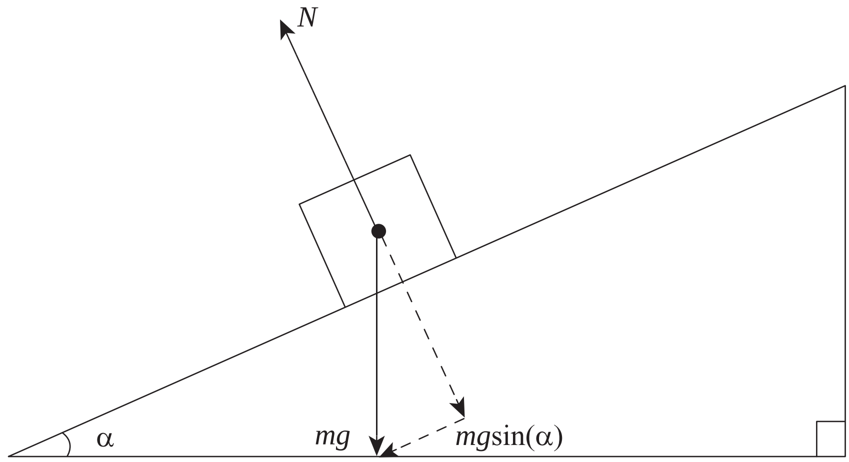
Question 19 C

$\frac{d^2x}{dt^2} = \frac{dx}{dt} \times \frac{dv}{dx}$

$= (x + 2) \times \frac{d}{dx}(x + 2)$

$= x + 2$

Question 20 E



The equation of motion parallel to the plane is $mg\sin(\alpha) = ma$.

So $a = g\sin(\alpha)$.

Question 21 B

The initial momentum (p_i) is 5 × 6 i.e. 30 (kg m/s).

To calculate the final momentum (p_f), we need to find the particle's final velocity.

Given $u = 6$, $s = 20$ and $t = 2$ we can find v by using $s = \frac{1}{2}(u+v)t$.

Solving $20 = \frac{2}{2}(6+v)$ for v we obtain $v = 14$.

The final momentum (p_f) is 5 × 14 i.e. 70 (kg m/s).

Change in momentum $(\Delta p) = p_f - p_i$

$= 40$ (kg m/s)

Question 22 D

The resultant force of $\underset{\sim}{S}$ and $\underset{\sim}{T}$ is $10\sqrt{2}$ newtons in the southwest direction.

If the particle is in equilibrium then the third force must be the negative of this.

So $\underset{\sim}{U}$ has magnitude $10\sqrt{2}$ newtons acting in the northeast direction.

SECTION 2

Question 1 (13 marks)

a. $|\underset{\sim}{r}(t)| = \sqrt{(1-t^2)^2 + (1-t)^2 + t^4}$ A1

b. $\frac{d}{dt}|\underset{\sim}{r}(t)|^2 = 8t^3 - 2t - 2$ M1 A1

Solving $8t^3 - 2t - 2 = 0$ for t with $t \geq 0$ gives $t = 0.76$ (s) (correct to two decimal places) M1 A1

c. $\frac{d^2}{dt^2}|\underset{\sim}{r}(t)| = \frac{8t^6 - 6t^4 - 16t^3 + 24t^2 - 3}{(2t^4 - t^2 - 2t + 2)^{\frac{3}{2}}}$ M1 A1

When $t = 0.76069\ldots$, $\frac{d^2}{dt^2}|\underset{\sim}{r}(t)| = 7.875\ldots$ (> 0), and so $t = 0.76$ (s) is when the particle is closest to the origin. A1

d. The coordinates, correct to two decimal places, are $(0.42, 0.24, 0.58)$ A1

e. $\underset{\sim}{r}(0) = \underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{r}(1) = \underset{\sim}{k}$ and $\underset{\sim}{r}(2) = -3\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$ A1

If the particle travels in a straight line, then $\underset{\sim}{r}(1) - \underset{\sim}{r}(0) = m(\underset{\sim}{r}(2) - \underset{\sim}{r}(1))$ M1

$-\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} = m(-3\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$ M1

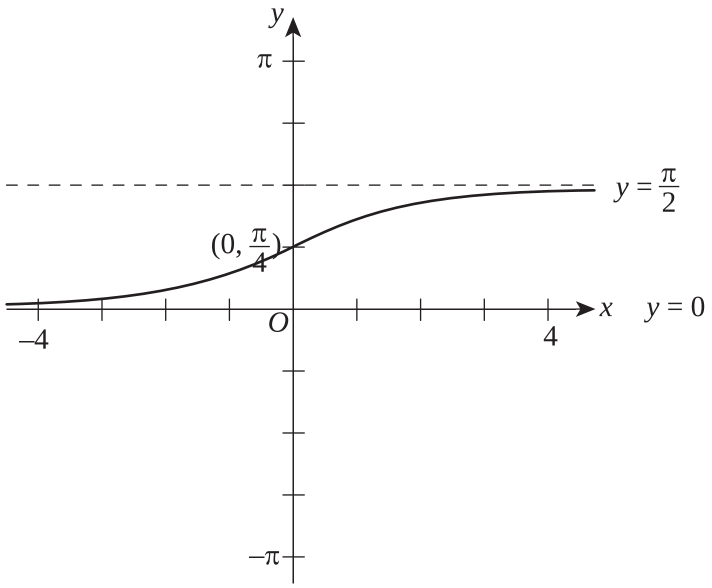
There is no solution for m, i.e. $\underset{\sim}{r}(1) - \underset{\sim}{r}(0)$ and $\underset{\sim}{r}(2) - \underset{\sim}{r}(1)$ are not parallel, and so $\underset{\sim}{r}(0)$, $\underset{\sim}{r}(1)$ and $\underset{\sim}{r}(2)$ are not collinear. The particle does not travel in a straight line. A1

Question 2 (15 marks)

a. Solving $x = \log_e(2\theta)$ for θ gives $\theta = \frac{1}{2}e^x$. M1

Substituting $\theta = \frac{1}{2}e^x$ into $y = \tan^{-1}(2\theta)$ gives $y = \tan^{-1}(e^x)$. A1

b.



Correct shape — A1

Horizontal asymptotes $y = 0$ and $y = \frac{\pi}{2}$; y-axis intercept $\left(0, \frac{\pi}{4}\right)$ — A1

c. Method 1:

$\frac{dx}{d\theta} = \frac{1}{\theta}$ and $\frac{dy}{d\theta} = \frac{2}{1+4\theta^2}$ — M1

So $\frac{dy}{dx} = \frac{2\theta}{1+4\theta^2}$ — A1

When $y = k$, $2\theta = \tan(k)$ (or equivalent)

$\frac{dy}{dx} = \frac{\tan(k)}{1+\tan^2(k)}$ — A1

$= \sin(k)\cos(k)$

So $\frac{dy}{dx} = \frac{1}{2}\sin(2k)$. — A1

Only award the last A1 if the previous line is seen

Method 2:

$\frac{dy}{dx} = \frac{1}{e^x + e^{-x}}$ (or equivalent) — M1

When $y = k$, $\theta = \frac{1}{2}\tan(k)$

So $x = \log_e(\tan(k))$. — A1

$\frac{dy}{dx} = \frac{\tan(k)}{1+\tan^2(k)}$ — A1

$= \sin(k)\cos(k)$

So $\frac{dy}{dx} = \frac{1}{2}\sin(2k)$. — A1

Only award the last A1 if the previous line is seen

d. When $y = \frac{\pi}{4}$, $x = 0$. A1

The equation of the tangent at $\left(0, \frac{\pi}{4}\right)$ is $y = \frac{1}{2}x + \frac{\pi}{4}$ and the tangent cuts the x-axis at $\left(-\frac{\pi}{2}, 0\right)$ M1

The equation of the normal at $\left(0, \frac{\pi}{4}\right)$ is $y = -2x + \frac{\pi}{4}$ and the normal cuts the x-axis at $\left(\frac{\pi}{8}, 0\right)$. M1

Let the area of the triangle be A square units.

$A = \frac{1}{2} \times \frac{\pi}{4} \times \left(\frac{\pi}{8} - \left(-\frac{\pi}{2}\right)\right)$ A1

So $A = \frac{5\pi^2}{64}$ (square units) A1

e. Let V be the volume of the solid.

$V = \pi\int_0^3 (\tan^{-1}(e^x))^2 dx$ M1

So $V = 15.9$ (m^3) (correct to the nearest tenth of a cubic metre) A1

Question 3 (8 marks)

a. $\text{Arg}(z - 2i) = \frac{\pi}{6}$

Substituting $z = x + yi$ we obtain $\text{Arg}(x + (y - 2)i) = \frac{\pi}{6}$

$\frac{y-2}{x} = \tan\left(\frac{\pi}{6}\right)$ M1

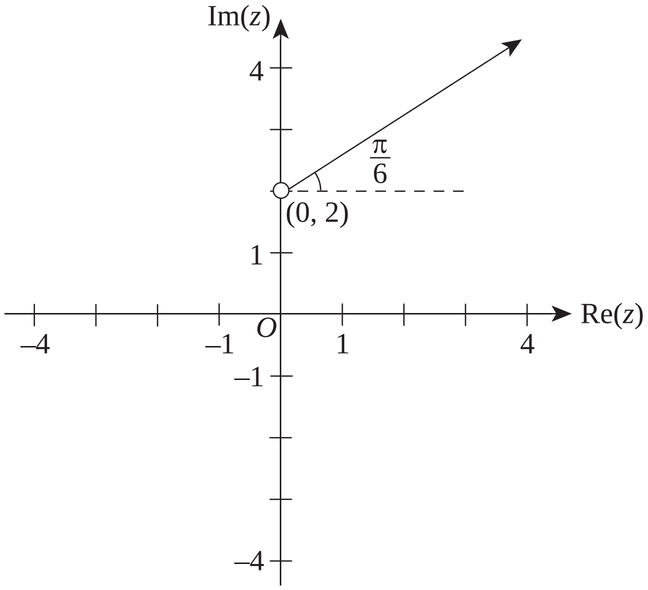
$\frac{y-2}{x} = \frac{1}{\sqrt{3}}$

$y - 2 = \frac{1}{\sqrt{3}}x$

So $y = \frac{1}{\sqrt{3}}x + 2, x > 0.$ A1

Only award the last A1 if one of the two previous lines are seen

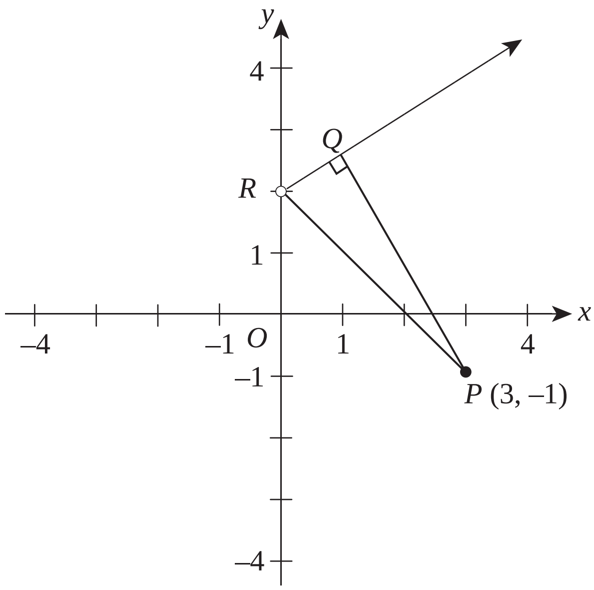
b.



Correct shape A1

Open circle at (0, 2) A1

c.



$$RP = \sqrt{3^2 + 3^2}$$
$$= 3\sqrt{2}$$ A1

$$\angle QRP = \frac{\pi}{4} + \frac{\pi}{6}$$
$$= \frac{5\pi}{12}$$ A1

$$\frac{QP}{3\sqrt{2}} = \sin\left(\frac{5\pi}{12}\right)$$ M1

$$QP = \frac{3(\sqrt{3} + 1)}{2}$$ A1

Question 4 (10 marks)

a. $m\frac{dv}{dt} \propto -v^2$

So $m\frac{dv}{dt} = -kv^2$. A1

Only award the A1 if the first line is seen.

b. Using either integration or a differential equation solver with $v = u$ when $t = 0$. M1

$\frac{1}{u} - \frac{1}{v} = -\frac{kt}{m}$ (or equivalent) A1

Making v the subject we obtain $v = \frac{mu}{kut + m}$. A1

Only award the last A1 if the first line is seen.

c. Using either integration or a differential equation solver with $x = 0$ when $t = 0$. M1

$x = \frac{m}{k}\log_e\left(\frac{kut + m}{m}\right)$ (or equivalent) A1

So $x = \frac{m}{k}\log_e\left(1 + \frac{kut}{m}\right)$. A1

Only award the last A1 if the previous line is seen.

d. $a = \frac{dv}{dt}$ and $v = \frac{mu}{kut + m}$

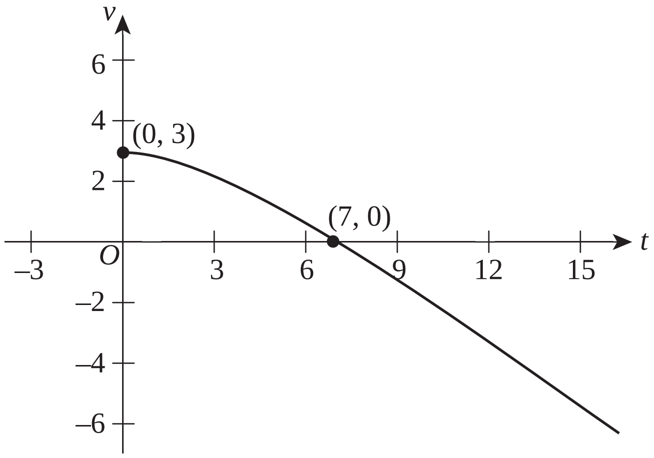
$\frac{d}{dt}\left(\frac{mu}{kut + m}\right) = -\frac{kmu^2}{(kut + m)^2}$ A1

e. Over a long period of time:

- the particle's speed continually decreases A1
- it never comes to rest A1

Question 5 (12 marks)

a.



Correct shape (concavity) for $0 \le t \le 2$ A1

Correct shape (linear) for $t > 2$ A1

Axes intercepts: $(0, 3)$ and $(7, 0)$ A1

b. **i.** 3 m/s A1

ii. $t = 7$ (s) A1

c. Let d be the distance travelled by the particle, d_1 be the distance travelled for $0 \le t \le 2$ and d_2 be the distance travelled for $2 \le t \le 7$.

Attempting to find d_1 and d_2 M1

For $0 \le t \le 2$:

$$d_1 = \int_0^2 \sqrt{9 - t^2}\,dt$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{2}{3}\right) + \sqrt{5}$$ A1

For $2 \le t \le 7$:

Either use of a definite integral $d_2 = \int_2^7 \frac{7 - t}{\sqrt{5}}dt$ or calculating the area of the triangle.

$$d_2 = \frac{5\sqrt{5}}{2}$$ A1

From $d = d_1 + d_2$ we obtain $d = \frac{1}{2}\left(9\sin^{-1}\left(\frac{2}{3}\right) + 7\sqrt{5}\right)$ (m) A1

d. Distance travelled before coming to rest is $d = \frac{1}{2}\left(9\sin^{-1}\left(\frac{2}{3}\right) + 7\sqrt{5}\right)$ (m).

$$\int_{7}^{T} \left|\frac{7-t}{\sqrt{5}}\right| dt = \frac{1}{2}\left(9\sin^{-1}\left(\frac{2}{3}\right) + 7\sqrt{5}\right)$$ A1

Attempting to solve $\int_{7}^{T} \left|\frac{7-t}{\sqrt{5}}\right| dt = \frac{1}{2}\left(9\sin^{-1}\left(\frac{2}{3}\right) + 7\sqrt{5}\right)$ for T M1

$T = 14.0$ (s) A1