Neap:

Trial Examination 2013

VCE Specialist Mathematics Units 3 & 4

Written Examination 2

Question and Answer Booklet

Reading time: 15 minutes
Writing time: 2 hours

Student's Name: ______________________________

Teacher's Name: ______________________________

Structure of Booklet

Section	Number of questions	Number of questions to be answered	Number of marks
1	22	22	22
2	5	5	58
			Total 80

Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved graphics calculator or approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.

Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
Question and answer booklet of 20 pages. Formula sheet of miscellaneous formulas.

Answer sheet for multiple-choice questions.

Instructions

Write **your name** and your **teacher's name** in the space provided above on this page and in the space provided on the answer sheet for multiple-choice questions.
All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1

The equations of the asymptotes of the hyperbola $x^2 - y^2 = 4x - 2y$ are

A. $y = x$ and $y = -x$

B. $y = x + 1$ and $y = -3 - x$

C. $y = x - 1$ and $y = 3 - x$

D. $y = \sqrt{3}x - 1$ and $y = 3 - \sqrt{3}x$

E. $y = 1 - x$ and $y = x - 3$

Question 2

The equation of the tangent to the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(a\cos(\theta), b\sin(\theta))$ is

A. $y = -\frac{b(a - \sin(\theta)x)}{a\cos(\theta)}$

B. $y = \frac{b(a - \sin(\theta)x)}{a\cos(\theta)}$

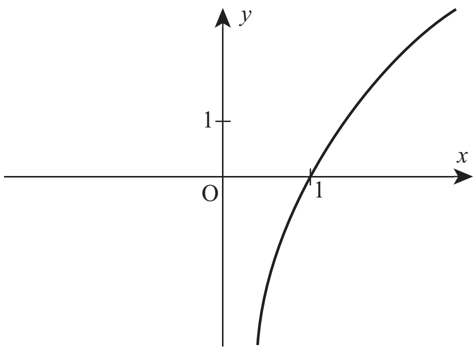
C. $y = \frac{b(a + \cos(\theta)x)}{a\sin(\theta)}$

D. $y = \frac{b(a - \cos(\theta)x)}{a\sin(\theta)}$

E. $y = -\frac{b(a - \cos(\theta)x)}{a\sin(\theta)}$

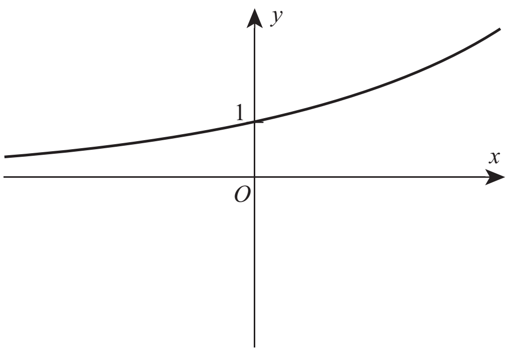
Question 3

The graph of $y = g(x)$ is shown below.

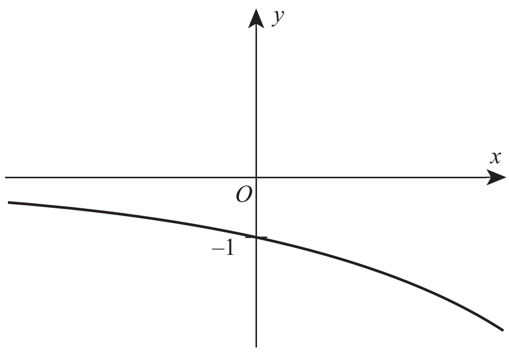


Which one of the following graphs best represents the graph of $y = \dfrac{1}{g(x)}$?

A.



B.



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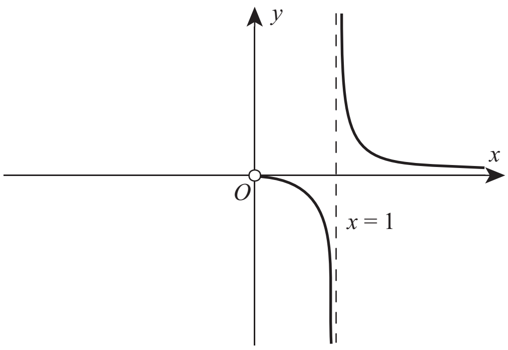

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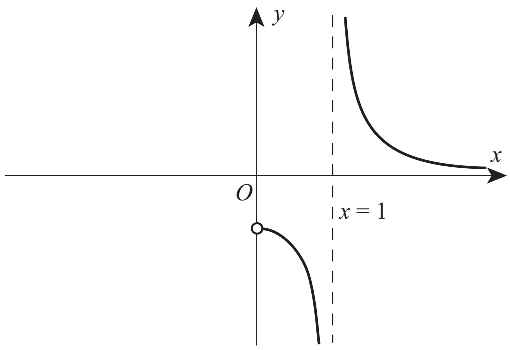
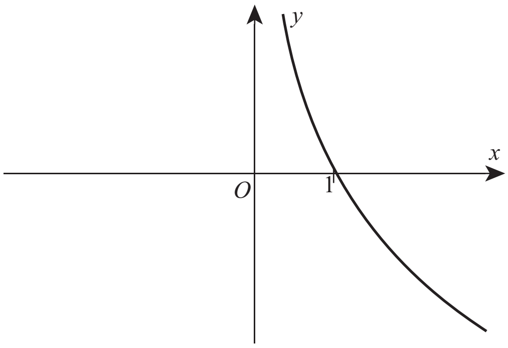


E.



Question 4

The implied domain of the function $y = -\cos^{-1}(2 - x)$ is

A. $1 < x < 3$

B. $1 \leq x \leq 3$

C. $-1 \leq x \leq 1$

D. $-\pi \leq x \leq 0$

E. $-3 \leq x \leq -1$

Question 5

Given $\tan(x) = \frac{3}{4}$ and $\tan(y) = \frac{4}{3}$, where x and y are both acute, then $\cos(x + y)$ is equal to

A. $\frac{24}{25}$

B. $\frac{7}{25}$

C. $\frac{7}{5}$

D. 1

E. 0

Question 6

The points U, V and W correspond to the complex numbers $u = 2i$, $v = \sqrt{3} - i$ and $w = -\sqrt{3} - i$ respectively.

Which one of the following statements is incorrect?

A. $v = \overline{w}$

B. UVW is an isosceles triangle

C. $UV = 2\sqrt{3}$

D. u, v and w are roots of the equation $z^3 + 8i = 0$

E. $\angle UVW = \frac{\pi}{3}$

Question 7

$T = \{z: |z - 1| \leq 1\}$ and $V = \{z: \text{Re}(z) + \text{Im}(z) = 1\}$ are two subsets of the complex plane.

The values of $\text{Re}(z)$ for which z belongs to $T \cap V$ are

A. $\frac{-\sqrt{2}}{2} \leq \text{Re}(z) \leq \frac{\sqrt{2}}{2}$

B. $-1 \leq \text{Re}(z) \leq 1$

C. $-\sqrt{2} \leq \text{Re}(z) \leq \sqrt{2}$

D. $-\left(\frac{2 - \sqrt{2}}{2}\right) \leq \text{Re}(z) \leq \frac{2 + \sqrt{2}}{2}$

E. $\frac{2 - \sqrt{2}}{2} \leq \text{Re}(z) \leq \frac{2 + \sqrt{2}}{2}$

Question 8

If 1, u and v are the cube roots of unity then

A. $uv = -1$

B. $u = v$

C. $u = v^2$

D. $u = -\bar{v}$

E. $u + v = 1$

Question 9

Euler's method, with a step size of $\frac{1}{5}$, is used to solve the differential equation $\frac{dy}{dx} = \cos^{-1}\left(\frac{\pi x}{2}\right)$, with initial condition $y = 2$ when $x = 0$.

The approximation obtained for y when $x = \frac{2}{5}$ is given by

A. $2 + \frac{\pi}{10} + \frac{1}{5}\cos^{-1}\left(\frac{\pi}{10}\right)$

B. $2 + \frac{2}{5}\cos^{-1}\left(\frac{\pi}{10}\right)$

C. $2 + \frac{2}{5}\cos^{-1}\left(\frac{\pi}{5}\right)$

D. $2 + \frac{\pi}{10} + \frac{1}{5}\cos^{-1}\left(\frac{\pi}{5}\right)$

E. $2 + \frac{1}{5}\cos^{-1}\left(\frac{\pi}{10}\right) + \frac{1}{5}\cos^{-1}\left(\frac{\pi}{5}\right)$

Question 10

$\frac{2x+3}{(x-3)^2}$ expressed in partial fractions has the form

A. $\frac{A}{(x-3)} + \frac{B}{(x-3)^2}$

B. $\frac{A}{(x+3)} + \frac{B}{(x-3)}$

C. $\frac{A}{(x-3)} + \frac{B}{(x-3)}$

D. $\frac{A}{(x-3)} + \frac{Bx+C}{(x-3)^2}, B \neq 0$

E. $\frac{A}{(x-3)^2} + \frac{B}{(x-3)^2}$

Question 11

A tank initially contains 100 litres of pure water. A salt solution of concentration 30 grams per litre is added to the tank at a rate of 10 litres per minute. The concentration of salt in the tank is kept uniform by stirring and the mixture flows out of the tank at a rate of 7 litres per minute.

If x grams is the amount of salt in the tank t minutes after the water begins to flow, then $\frac{dx}{dt}$ is equal to

A. $30 - \frac{7x}{100 - 3t}$

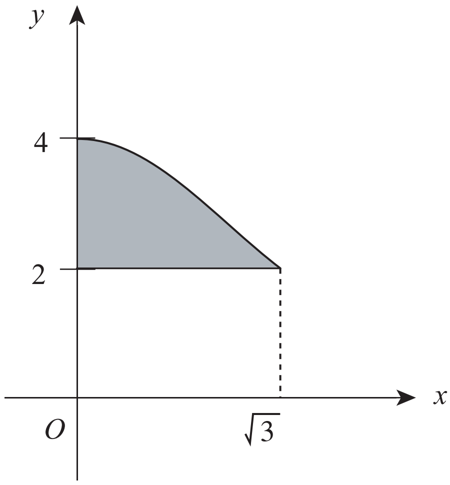
B. $300 - \frac{7x}{100 + 7t}$

C. $300 - \frac{7x}{100 - 3t}$

D. $300 - \frac{7x}{100 + 3t}$

E. $30 - \frac{7x}{100 + 3t}$

Question 12



The shaded region above is enclosed by the curve $y = \frac{4}{\sqrt{1+x^2}}$, the line $y = 2$ and the y-axis.

This region is rotated $360°$ about the x-axis to form a solid of revolution. The volume of this solid, in cubic units, is given by

A. $\pi[8\tan^{-1}(x) - 2]_0^{\sqrt{3}}$

B. $\int_0^{\sqrt{3}} \left(\frac{4}{\sqrt{1+x^2}} - 2\right) dx$

C. $\pi\int_0^{\sqrt{3}} \left(\frac{16}{1+x^2} - 4\right) dx$

D. $\pi\int_0^{\sqrt{3}} \left(\frac{4}{\sqrt{1+x^2}}\right)^2 dx$

E. $\pi\int_2^{4} \left(\frac{4}{\sqrt{1+x^2}} - 2\right)^2 dx$

Question 13

Using a suitable substitution, $\int \frac{1}{(1+2\tan(x))^2\cos^2(x)}\,dx$ can be expressed in terms of u as

A. $\int \frac{1}{2u}\,du$

B. $\int \frac{2}{u^2}\,du$

C. $\int \frac{1}{u^2}\,du$

D. $\int 2u^2\,du$

E. $\int \frac{1}{2u^2}\,du$

Question 14

If $\underset{\sim}{u} = 3\underset{\sim}{v} - 2\underset{\sim}{w}$, where $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are non-zero vectors, which one of the following statements must be true?

A. $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are linearly dependent

B. $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are linearly dependent

C. $\underset{\sim}{v}$ is parallel to $\underset{\sim}{w}$

D. $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are linearly independent

E. $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are linearly independent

Question 15

The vectors $\underset{\sim}{u} = \sin(2t)\underset{\sim}{i} - \cos(2t)\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = \cos(t)\underset{\sim}{i} - \sin(t)\underset{\sim}{j} - \underset{\sim}{k}$ where $0 \le t \le \frac{\pi}{2}$ are perpendicular when

A. $t = \frac{\pi}{12}$

B. $t = \frac{\pi}{6}$

C. $t = \frac{\pi}{3}$

D. $t = \frac{\pi}{2}$

E. $t = \frac{5\pi}{6}$

Question 16

The vector resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$ is $2\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$.

The vector resolute of $\underset{\sim}{u}$ perpendicular to $\underset{\sim}{v}$ is $3\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}$.

$\underset{\sim}{u}$ is equal to

A. $6\underset{\sim}{i} - 12\underset{\sim}{j} + 4\underset{\sim}{k}$

B. $\underset{\sim}{i} + 7\underset{\sim}{j}$

C. $5\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$

D. $5\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$

E. $-5\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}$

Question 17

The acceleration of a particle at time t, $t \geq 0$, is given by $\underset{\sim}{a}(t) = \sin(t)\underset{\sim}{j}$.

The velocity of the particle when $t = \pi$ is $\underset{\sim}{i} + \underset{\sim}{j}$.

The initial velocity of the particle is

A. $\underset{\sim}{i} - \underset{\sim}{j}$

B. $\underset{\sim}{i} + \underset{\sim}{j}$

C. $-\underset{\sim}{j}$

D. $-\underset{\sim}{i} - \underset{\sim}{j}$

E. $3\underset{\sim}{i} + \underset{\sim}{j}$

Question 18

If $y = e^{pt}$ satisfies the differential equation $\frac{1}{5}\left(\frac{d^2y}{dt^2} + 6y\right) = \frac{dy}{dt}$, the possible values for p are

A. 2 and 3
B. –3 and –2
C. –5 and –6
D. –1 and –6
E. –6 and 1

Question 19

A particle moves in a straight line such that at time t, its displacement from a fixed origin is x.

If $\frac{dx}{dt} = x + 2$, then $\frac{d^2x}{dt^2}$ is equal to

A. $\frac{1}{2}x^2 + 2x$

B. $(x + 2)^2$

C. $x + 2$

D. 1

E. $\frac{1}{x + 2}$

Question 20

A body of mass m kg slides from rest down a smooth plane inclined at an angle $\alpha°$ to the horizontal.

The acceleration, in m/s^2, of the body down the plane has magnitude

A. 0

B. $mg\cos(\alpha)$

C. $g\cos(\alpha)$

D. $mg\sin(\alpha)$

E. $g\sin(\alpha)$

Question 21

A particle of mass 5 kg travels in a straight line with constant acceleration. Its initial velocity is 6 m/s and it travels a distance of 20 metres in 2 seconds.

The change of momentum of the particle in kg m/s, in the direction of its motion, is

A. 20

B. 40

C. 60

D. 80

E. 100

Question 22

A particle is held in equilibrium by three concurrent coplanar forces $\underset{\sim}{S}$, $\underset{\sim}{T}$ and $\underset{\sim}{U}$.

$\underset{\sim}{S}$ has magnitude 10 newtons and acts in the west direction.

$\underset{\sim}{T}$ has magnitude 10 newtons and acts in the south direction.

The magnitude and direction of $\underset{\sim}{U}$ are, respectively

A. 10 newtons, southwest

B. 10 newtons, northeast

C. $10\sqrt{2}$ newtons, southwest

D. $10\sqrt{2}$ newtons, northeast

E. 10 newtons, northwest

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this booklet are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1 (13 marks)

At time t seconds, a particle's position vector as a function of time relative to a fixed origin O is given by $\underset{\sim}{r}(t) = (1 - t^2)\underset{\sim}{i} + (1 - t)\underset{\sim}{j} + t^2\underset{\sim}{k},\ t \geq 0$, where the components are measured in metres.

a. Write down an expression for $|\underset{\sim}{r}(t)|$, the particle's distance from the origin at time t. 1 mark

b. Use calculus to find, correct to two decimal places, the value of t for which the particle is closest to the origin. 4 marks

c. Find $\frac{d^2}{dt^2}(|\underset{\sim}{r}(t)|)$ and use it to justify that the particle is closest to the origin for the value of t found in part **b.** 3 marks

d. Find, correct to two decimal places, the coordinates of the particle at the instant it is closest to the origin. 1 mark

e. Show that the particle does not travel in a straight line. 4 marks

Question 2 (15 marks)

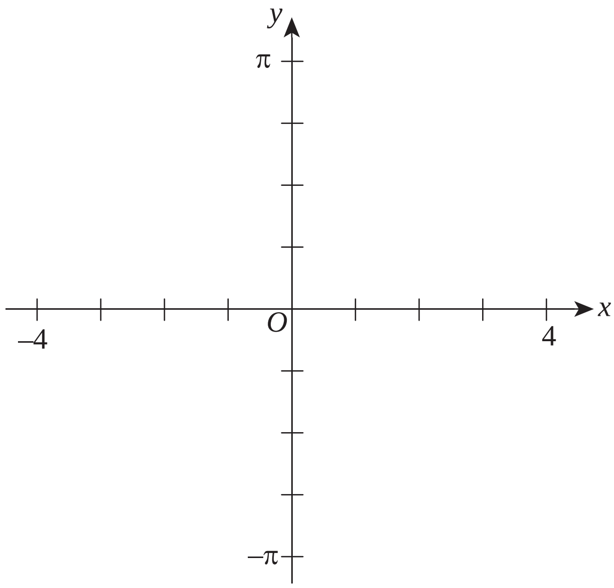
A curve C is defined by the parametric equations

$$x = \log_e(2\theta)$$

$$y = \tan^{-1}(2\theta) \text{ where } \theta > 0$$

a. Show that the cartesian equation of C is given by $y = \tan^{-1}(e^x)$. 2 marks

b. On the set of axes below, sketch C, clearly stating the coordinates of any axes intercepts and the equations of any asymptotes. 2 marks



c. Show that the gradient of C at $y = k$ is $\frac{1}{2}\sin(2k)$. 4 marks

d. Find the area of the triangle bounded by the x-axis, the tangent to C at $y = \frac{\pi}{4}$ and the normal to C at $y = \frac{\pi}{4}$. 5 marks

A plastics company is commissioned to make a type of large plastic container.

The design involves taking the region bounded by C and the vertical lines $x = 0$ and $x = 3$ and rotating it 360° about the x-axis to form a solid of revolution.

One unit of length on the x-axis represents 1 metre.

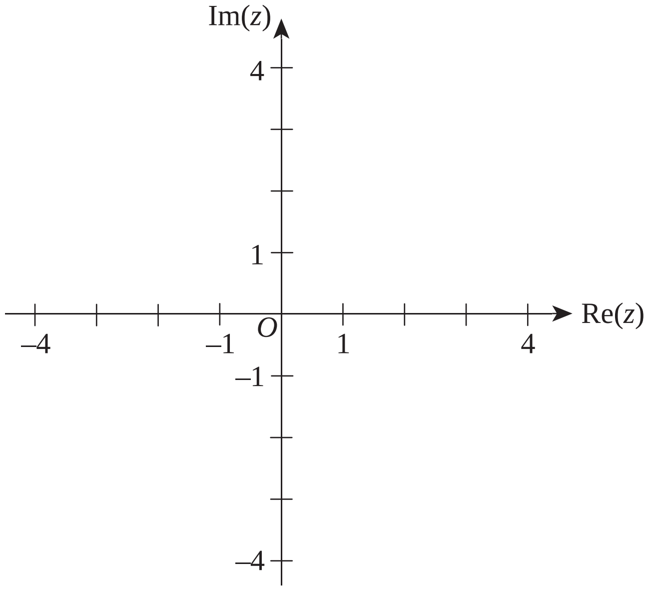
e. Find the volume of this plastic container. Express your answer correct to the nearest tenth of a cubic metre. 2 marks

Question 3 (8 marks)

In the complex plane, L is the half-line with equation $\text{Arg}(z - 2i) = \frac{\pi}{6}$, $z \in C$.

a. Show that the cartesian equation of L is given by $y = \frac{1}{\sqrt{3}}x + 2$, $x > 0$. 2 marks

b. Sketch L on the Argand diagram below. 2 marks



c. Hence find the least value of $|z - 3 + i|$. Express your answer in exact form. 4 marks

Question 4 (10 marks)

A particle of mass m kg moves in a straight horizontal line such that at time t seconds its displacement from a fixed origin is x metres and its speed is v m/s.

During its motion, the particle is acted upon by a force which opposes the motion and has magnitude proportional to the square of its speed.

The particle starts its motion from the fixed origin with a speed of u m/s.

a. Show that $m\frac{dv}{dt} = -kv^2$. 1 mark

b. Show that the speed of the particle is given by $v = \frac{mu}{kut + m}$. 3 marks

c. Show that the displacement of the particle is given by $x = \frac{m}{k}\log_e\left(1 + \frac{kut}{m}\right)$. 3 marks

d. Express the acceleration of the particle as a function of t. 1 mark

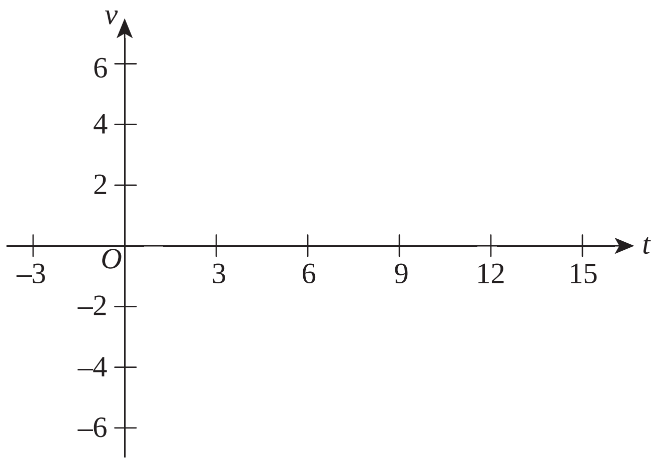
e. According to this model, describe the particle's speed over a long period of time. 2 marks

Question 5 (12 marks)

A particle moves in a straight line such that its velocity, v m/s, at time t seconds is given by

$$v(t) = \begin{cases} \sqrt{9 - t^2} & 0 \le t \le 2 \\ \dfrac{7 - t}{\sqrt{5}} & t > 2 \end{cases}$$

a. Sketch a velocity–time graph for $t \ge 0$, clearly indicating all intercepts with the coordinate axes. 3 marks



b. **i.** State the particle's initial velocity. 1 mark

ii. State the time when the particle changes its direction of motion. 1 mark

c. Find the exact distance travelled by the particle from its initial position until the instant it comes to rest. 4 marks

The particle returns to its initial position at $t = T$.

d. Find, correct to one decimal place, the value of *T*. 3 marks

END OF QUESTION AND ANSWER BOOKLET