NAME: ____________________________

VCE SPECIALIST MATHEMATICS

Practice Written Examination 1

Reading time: 15 minutes **Writing time: 1 hour**

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
8	8	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
- Students are **not** permitted to bring into the examination room: notes of any kind, a calculator of any type, blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book of 10 pages.
- Formula booklet of 5 pages.
- Working space is provided throughout the book.

Instructions

- All written responses must be in English.
- Write your student name in the space provided above on this page.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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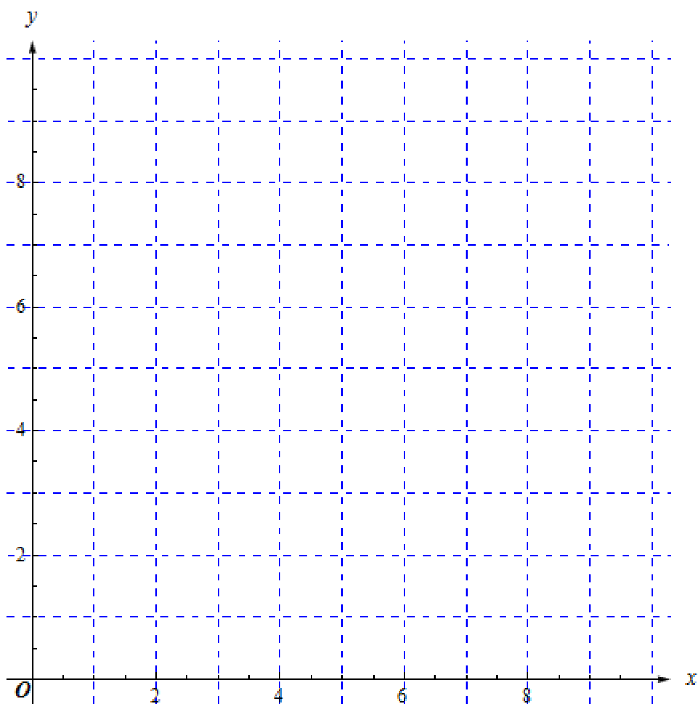
Instructions

Answer **all** questions in the spaces provided.
Unless otherwise specified an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude $g\ m/s^2$, where $g = 9.8$.

Question 1

a. Sketch the graphs of $f(x) = x + \frac{3}{x}$ and $g(x) = 4$, on the axes below, showing any relevant asymptotes, turning points and/or points of intersection between $f(x)$ and $g(x)$.



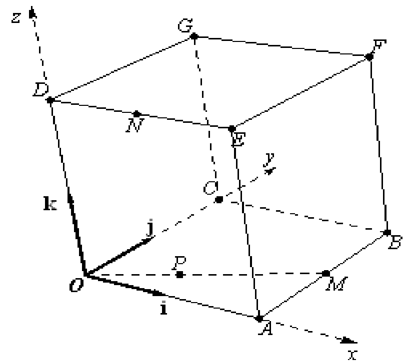
3 marks

b. Find the volume when the area bounded between $f(x)$ and $g(x)$ is rotated about the x −axis.

2 marks

Question 2

In the diagram below, $OABCDEFG$ is a cube of side length 2 units.



Unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are parallel to $\overrightarrow{OA}, \overrightarrow{OC}, \overrightarrow{OD}$ respectively. Points M and N are the midpoints of AB and DE respectively, and point P is the location on the OM that is closest to point N.

Express the following vectors in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

a. $\overrightarrow{OM}$ and $\overrightarrow{ON}$

2 marks

b. $\overrightarrow{OP}$

2 marks

c. $\overrightarrow{PN}$

1 mark

Question 3

Consider the polynomial $P(z) = z^3 - 6z - 9,\ z \in \mathbb{C}$.

a. If $\omega = -\frac{3}{2} + \frac{i\sqrt{3}}{2}$, find $|\omega|$ and $Arg(\omega)$.

2 marks

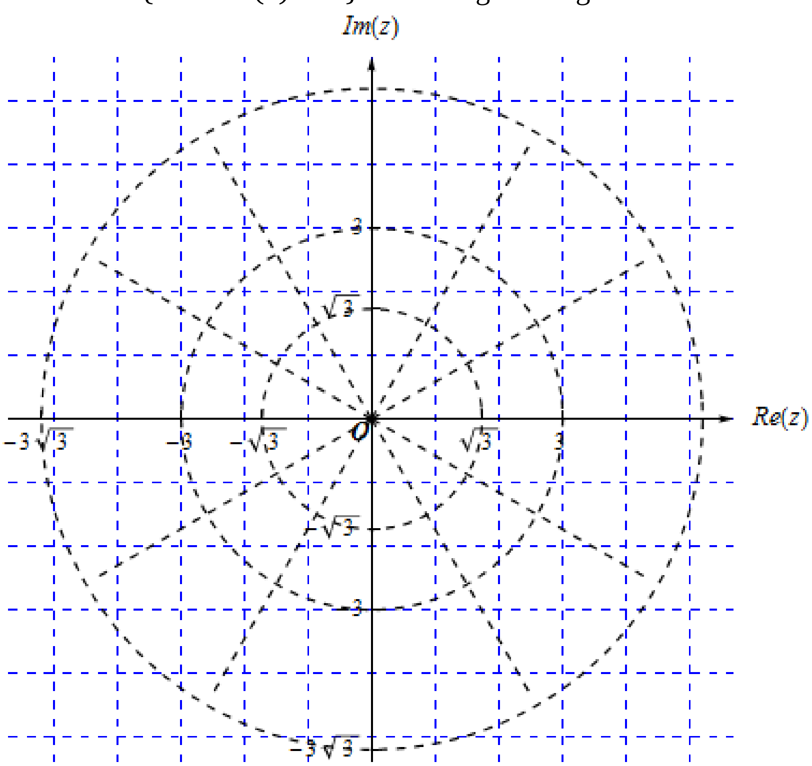
b. Given that $P(\omega) = 0$, show that $(z^2 + 3z + 3)$ is also factor of $P(z)$.

1 mark

c. Hence find a, in $P(z) = (z - a)(z^3 - 6z - 9)$.

1 mark

d. Sketch $S = \{z \in \mathbb{C}: P(z) = 0\}$ on the Argand diagram below.



3 marks

Question 4

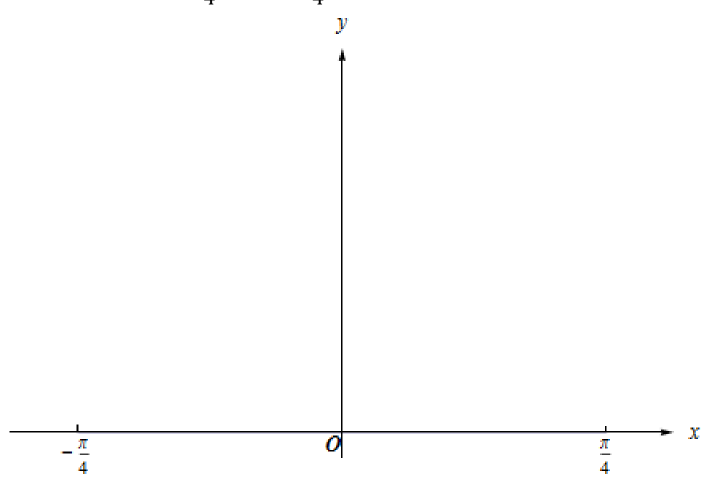
a. Show that $\frac{2}{1+\cos(4\theta)} = \sec^2(2\theta)$.

1 mark

b. Hence, find $\int_0^{\pi/12} \frac{2}{1+\cos(4\theta)}\,d\theta$.

2 marks

c. Sketch the graph of $y = \sec^2(2\theta)$, $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$



2 marks

d. Hence solve the equation $\frac{1}{1+\cos(4\,\theta)} = 2$ for $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

2 marks

Question 5

Find the gradient of the tangent to the curve $xy + y^2 + \arctan(x) = 6 + \frac{\pi}{4}$ at the point $(1,2)$.

3 marks

Question 6

Let $y = x \arcsin(e^c x), x \geq 0$

a. Assuming $x \geq 0$, show that $\tan(\arcsin(e^c x)) = \frac{e^c x}{\sqrt{1 - e^{2c} x^2}}$

1 mark

b. Hence, find the value of a given that $\frac{dy}{dx} = a\frac{y}{x} + \tan\left(\frac{y}{x}\right)$, where a is a real constant.

2 marks

c. If $y(1) = \frac{\pi}{4}$, find the value of c in the form $\frac{\log_e\left(\frac{1}{a}\right)}{b}$ where a, b are positive integers.

1 mark

Question 7

The position of a particle at time t is given by

$$\mathbf{r}(t) = \frac{6t}{1+t^3}\mathbf{i} + \frac{6t^2}{1+t^3}\mathbf{j},\ \ t \in \mathbb{R}$$

a. Find velocity vector **v.**

2 marks

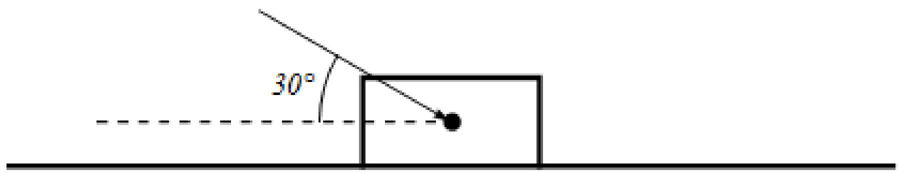
b. For points on the curve, find the greatest value of x.

2 marks

Question 8

A box of mass 4 kg is resting on a rough horizontal plane. The box is pushed downwards with a force of magnitude P newtons acting at an angle of 30° to the horizontal. The box is in equilibrium. The coefficient of friction between the box and the plane is 0.5.

a. On the diagram below, show all forces acting on the box and label them. (You may assume that all forces, including contact forces, act through the centre of mass).



1 mark

b. Find, in terms of P, the normal component and frictional component of the contact force acting between the box and the plane.

2 marks

c. Find the value of P when the box is just about to slide.

2 marks

SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.

Specialist Mathematics Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
cosine rule:	$c^2 = a^2 + b^2 - 2\,ab\sin C$

Coordinate geometry

ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	hyperbola: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Circular (trigonometric) functions

$cos^2(x) + sin^2(x) = 1$	$\sin(2x) = 2\sin(x)\cos(x)$
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$	$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$

$\cos(2x) = \cos^2(x) - \sin^2(x)$ $= 2\cos^2(x) - 1$ $= 1 - 2\sin^2(x)$	$\tan(2x) = \dfrac{2\tan(x)}{1-\tan^2(x)}$

function	sin^{-1}	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1,1]$	$[-1,1]$	$\mathbb{R}$
range	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0,\pi]$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + iy = r(\cos\theta + i\sin\theta) = r\text{ cis }\theta$	
$\lvert z\rvert = \sqrt{x^2+y^2} = r$	$-\pi < Arg(z) \le \pi$
$z_1z_2 = r_1r_2\text{cis }(\theta_1+\theta_2)$	$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}cis\ (\theta_1-\theta_2)$
$z^n = r^n\text{cis }(n\theta)$ (de Moivre's theorem)	

Calculus

$\dfrac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \dfrac{1}{n+1}x^{n+1} + c, n \ne -1$
$\dfrac{d}{dx}(e^{ax}) = ae^{ax}$	$\int e^{ax}dx = \dfrac{1}{a}e^{ax} + c$
$\dfrac{d}{dx}(\log_e(x)) = \dfrac{1}{x}$	$\int \dfrac{1}{x}\,dx = \log_e\lvert x\rvert + c$
$\dfrac{d}{dx}(\sin(ax)) = a\cos(ax)$	$\int \sin(ax)\,dx = \dfrac{1}{a}\cos(ax) + c$
$\dfrac{d}{dx}(\cos(ax)) = -a\sin(ax)$	$\int \cos(ax)\,dx = -\dfrac{1}{a}\sin(ax) + c$
$\dfrac{d}{dx}(\tan(ax)) = a\sec^2(ax)$	$\int \sec^2(ax)\,dx = \dfrac{1}{a}\tan(ax) + c$

$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}(ax) + c, a > 0$
$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \cos^{-1}(ax) + c, a > 0$
$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1}(x)) = \frac{1}{1+x^2}$	$\int \frac{a}{a^2+x^2}\,dx = sin^{-1}(ax) + c$

product rule:	$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}}{\mathrm{d}x}(v) + v\frac{\mathrm{d}}{\mathrm{d}x}(u)$
quotient rule:	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$
chain rule:	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Euler's method:	$\frac{dy}{dx} = f(x), x_0 = a \text{ and } y_0 = b$ $\Rightarrow x_{n+1} = x_n + h,\ \ y_{n+1} = y_n + hf(x_n)$
acceleration:	$a = \frac{d^2x}{dx^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
constant (uniform) acceleration:	$v = u + at, s = ut + \frac{1}{2}at^2, v^2 = u^2 + 2as, s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$	
$\lvert\mathbf{r}\rvert = \sqrt{x^2+y^2+z^2} = r$	$\mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2 \cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$
$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$	

Mechanics

momentum:	$\mathbf{p} = m\mathbf{v}$
equation of motion:	$\mathbf{R} = m\mathbf{a}$
friction:	$F \leq \mu N$

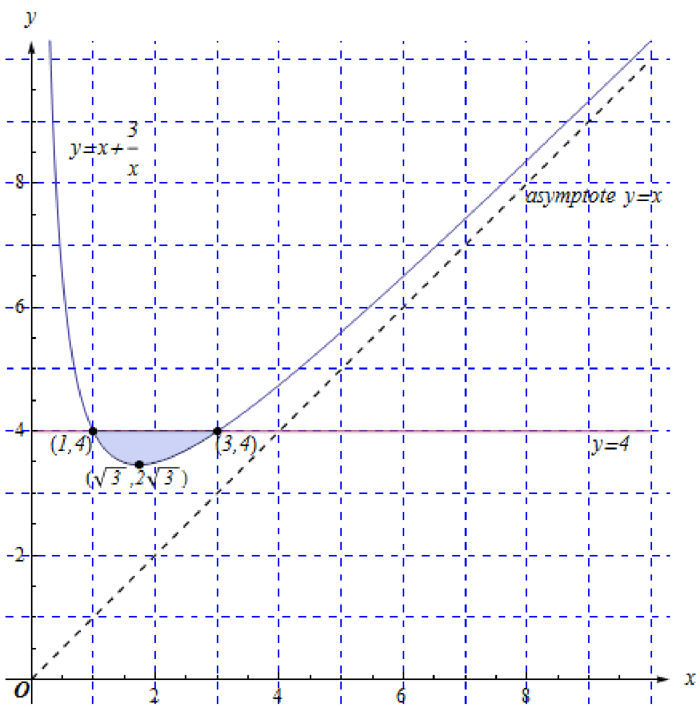
Solution Pathway

Below are sample answers. Please consider the merit of alternative responses.

Specialist Mathematics Exam 1: SOLUTIONS

Question 1

a.



turning point (A1)
Asymptote (A1)
points of intersection (A1)

3 marks

b.

$$\begin{aligned} V &= \pi \int_1^3 g^2(x) - f^2(x)\,dx \\ &= \pi \int_1^3 10 - \frac{9}{x^2} - x^2\,dx \quad (M1) \\ &= \frac{16\pi}{3} \text{ cubic units} \quad (A1) \end{aligned}$$

2 marks

Question 2

a.

$\overrightarrow{OM} = 2\mathbf{i} + \mathbf{j},\qquad (A1)$

$\overrightarrow{ON} = \mathbf{i} + 2\mathbf{k}\qquad (A1)$

2 marks

b.

$$\begin{aligned}\overrightarrow{OP} &= proj_{\overrightarrow{ON}}(\overrightarrow{OM})\\ &= \left(\frac{\overrightarrow{ON}.\overrightarrow{OM}}{\overrightarrow{OM}\cdot\overrightarrow{OM}}\right)\overrightarrow{OM} \quad mostly\ correct(M1)\\ &= \frac{4}{5}\mathbf{i} + \frac{2}{5}\mathbf{j} \qquad (A1)\end{aligned}$$

2 marks

c.

$$\begin{aligned}\overrightarrow{PN} &= \overrightarrow{ON} - \overrightarrow{OP}\\ &= \frac{14}{5}\mathbf{i} - \frac{2}{5}\mathbf{j} + 2\mathbf{k} \quad (A1)\end{aligned}$$

1 mark

Question 3

a.

$|\omega| = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{3},\qquad (A1)$

$Arg(\omega) = \arctan\left(\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(-\frac{3}{2}\right)}\right) + \pi = \frac{5\pi}{6}\qquad (A1)$

2 marks

b.

$Coefficients\ of\ P(z) are\ real, so\ conjugate\ root\ theorem\ applies \Rightarrow P(\overline{\omega}) = 0$

$\therefore (z-\omega)(z-\overline{\omega}) = (z^2 + 3z + 3)$ *is also a factor.* $(R1)$

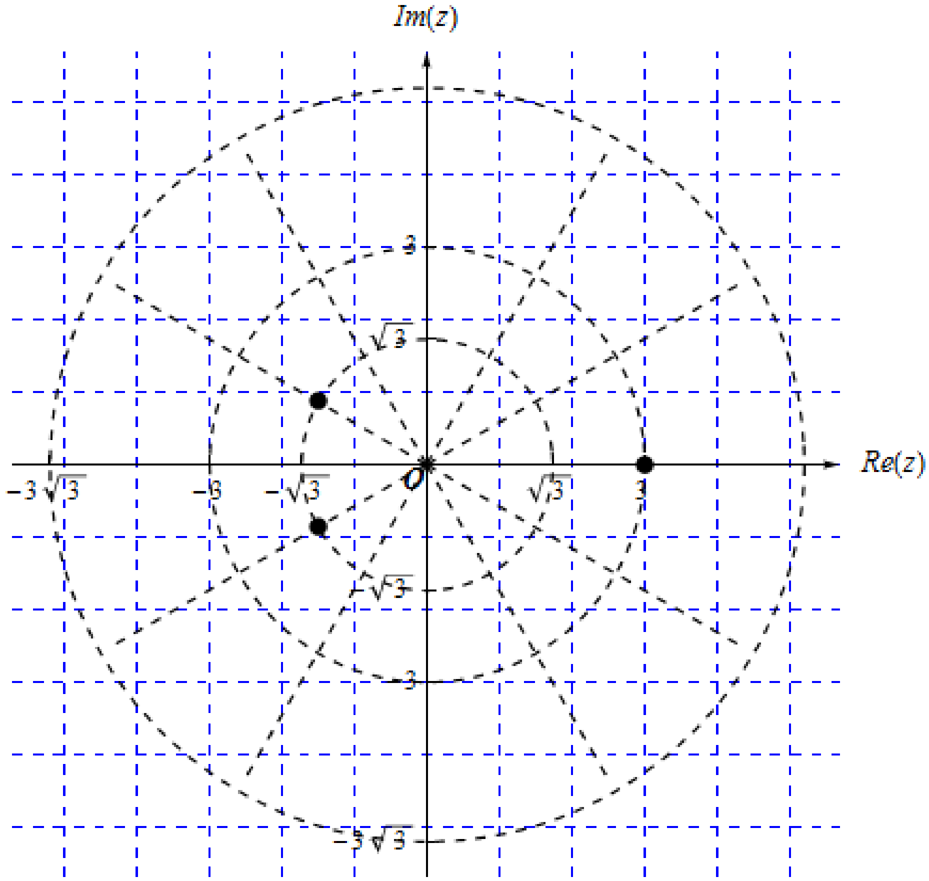
1 mark

c.

$$\begin{aligned}P(z) &= z^3 - 6z - 9\\ &= (z-a)(z^2 + 3z + 3) \qquad expanding\ and\ equating\ like\ terms\\ &= z^3 + (3-a)z^2 + (3-3a)z - 3a\\ \therefore a &= 3 \qquad (A1)\end{aligned}$$

1 mark

d.



3 marks

Question 4

a.

$$\begin{aligned} \cos(4\theta) &= 2\cos^2(2\theta) - 1 \\ \Rightarrow \frac{\cos(4\theta)+1}{2} &= \cos^2(2\theta) \qquad \textit{Attempt to use double angle cos}(M1) \\ \therefore \frac{2}{1+\cos(4\theta)} &= \sec^2(2\theta) \end{aligned}$$

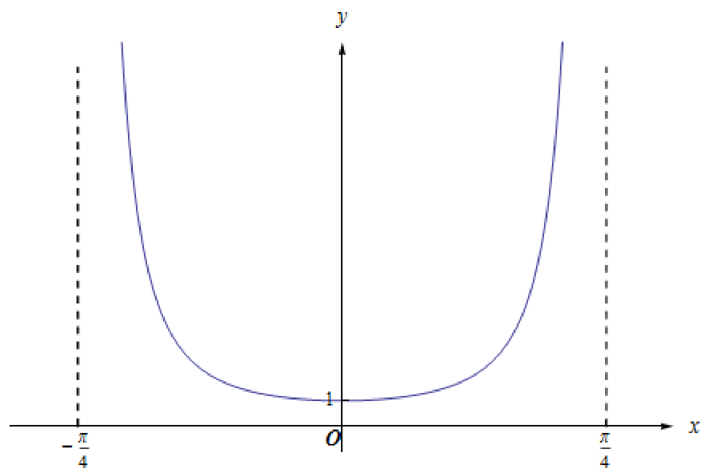
1 mark

b.

$$\begin{aligned} \int_0^{\pi/12} \frac{2}{1+\cos(4\,\theta)}\,d\theta &= \int_0^{\pi/12} \sec^2(2\theta)\,d\theta \\ \Rightarrow \quad &= \frac{1}{2}\int_0^{\pi/12} \frac{d}{d\theta}(\tan(2\theta))\,d\theta \quad \textit{mostly correct } (M1) \\ &= \frac{1}{2}[\tan(2\theta)]_0^{\pi/12} \\ \therefore \quad &= \frac{1}{2\sqrt{3}} \quad (A1) \end{aligned}$$

2 marks

c.



turning point $(A1)$
Asymptotes $(A1)$

2 marks

d.

$$\begin{aligned} \frac{1}{1+\cos(4\,\theta)} &= 2 \\ \Rightarrow \quad \sec^2(2\theta) &= 4 \quad \textit{mostly correct } (M1) \\ \Rightarrow \quad \cos(2\theta) &= \pm\frac{1}{2} \\ \Rightarrow \quad 2\theta &\in \{-\frac{\pi}{3}, \frac{\pi}{3}\} \\ \therefore \quad \theta &\in \{-\frac{\pi}{6}, \frac{\pi}{6}\} \quad (A1) \end{aligned}$$

2 marks

Question 5

$$\text{differentiate both sides wrt } x$$

$$y + x\frac{d}{dx}(y) + 2y\frac{d}{dx}(y) + \frac{1}{1+x^2} = 0 \qquad \text{mostly correct } (M1)$$

$$\frac{d}{dx}(y) = \frac{-y - \frac{1}{(1+x^2)}}{x+2y} \qquad (A1)$$

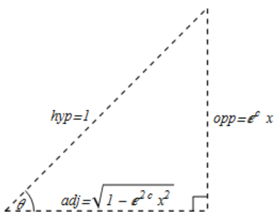
$$\left.\frac{d}{dx}(y)\right|_{(1,2)} = -\frac{1}{2} \qquad (A1)$$

3 marks

Question 6

a. Let $\theta = \arcsin(e^c x) \Rightarrow \sin(\theta) = \frac{e^c x}{1} = \frac{opp}{hyp}$

Using Pythagoras' Theorem



So,

$$\tan(\arcsin(e^c x)) = \tan(\theta) = \frac{opp}{adj} = \frac{e^c x}{\sqrt{1-e^{2c}x^2}} \qquad \text{mostly correct } (M1)$$

1 mark

b.

$$\frac{dy}{dx} = \frac{d(x\arcsin(e^c x))}{dx} = \arcsin(e^c x) + x\frac{e^c}{\sqrt{1-e^{2c}x^2}}$$

$$= \arcsin(e^c x) + \tan(\arcsin(e^c x)) \qquad (M1)$$

$$= \frac{y}{x} + \tan\left(\frac{y}{x}\right)$$

$$\therefore a = 1 \qquad (A1)$$

2 marks

c.

$$y(1) = \frac{\pi}{4} \Rightarrow \arcsin(e^c) = \frac{\pi}{4}$$

$$e^c = \sin\left(\frac{\pi}{4}\right)$$

$$\therefore c = \ln\left(sin\left(\frac{\pi}{4}\right)\right) = \frac{\log_e\left(\frac{1}{2}\right)}{2} \qquad (A1)$$

1 mark

Question 7

a.

$$\begin{aligned} \mathbf{v} &= \frac{d}{dt}\left(\frac{6t}{1+t^3}\right)\mathbf{i} + \frac{d}{dt}\left(\frac{6t^2}{1+t^3}\right)\mathbf{j} && (M1) \\ &= \frac{6\,(1-2t^3)}{(1+t^3)^2}\mathbf{i} + \frac{6t(2-t^3)}{(1+t^3)^2}\mathbf{j} && (A1) \end{aligned}$$

2 marks

b.

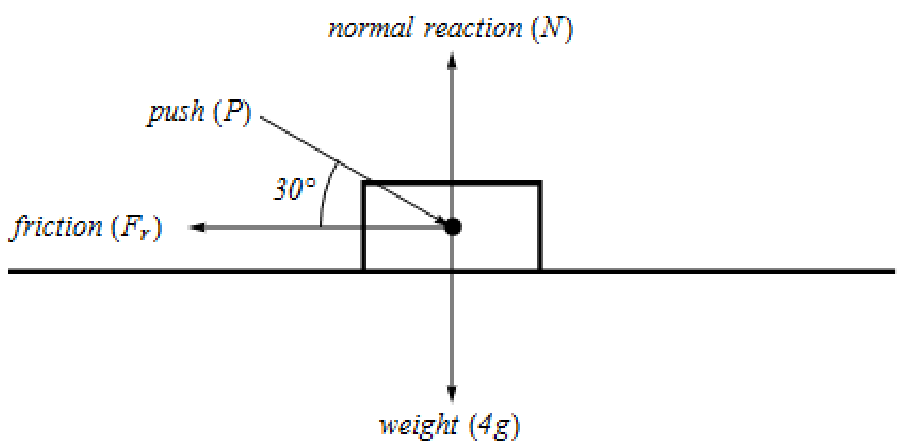
maximum $x \Rightarrow \mathbf{i}\cdot\mathbf{v} = 0$

$$\begin{aligned} \frac{dx}{dt} &= 0 && (M1) \\ \Rightarrow\ (1-2t^3) &= 0 \\ t &= 1/\sqrt[3]{2} \\ \therefore\ x(1/\sqrt[3]{2}) &= 2\sqrt[3]{4} && (A1) \end{aligned}$$

2 marks

Question 8

a.



1 mark

b.

Resolving along the plane

$$F_r = P\cos(30°)$$
$$= \frac{\sqrt{3}}{2}P \qquad (A1)$$

Resolving perpendicular to the plane

$$N = P\sin(30°) + 4g$$
$$= \frac{P}{2} + 4g \qquad (A1)$$

2 marks

c.

Limiting Friction

$$F_r = \frac{1}{2}N$$
$$\frac{\sqrt{3}}{2}P = \frac{1}{2}\left(\frac{P}{2} + 4g\right) \qquad (M1)$$
$$\therefore \quad P = \frac{8g}{2\sqrt{3} - 1} \qquad (A1)$$

2 marks

