SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2

(TSSM's *2013 trial exam updated for the current study design)*

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: C

Explanation

$$y = \frac{2x^3 - 8x^2 + 8x - 3}{x^2 - 4x + 4}$$
$$= 2x - \frac{3}{(x-2)^2}$$

Question 2

Answer: B

Explanation

Equation required is $(x+2)^2 + (y-3)^2 = 25$

Expanding brackets gives:

$x^2 + 4x + y^2 - 6y - 12 = 0$

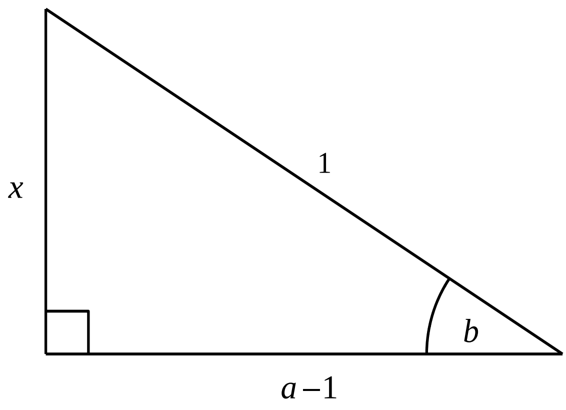
Question 3

Answer: E

Explanation

$\text{Cos}^{-1}(a-1) = b \Rightarrow \cos b = a-1$

From Pythagoras' theorem,



$$x = \sqrt{1-(a-1)^2}$$
$$= \sqrt{2a-a^2}$$
$$= \sqrt{a(2-a)}$$

Therefore,

$$\cot\left(b-\frac{\pi}{2}\right) = \cot\left(-\left(\frac{\pi}{2}-b\right)\right)$$
$$= -\cot\left(\frac{\pi}{2}-b\right)$$
$$= \frac{\sqrt{a(2-a)}}{1-a}$$

Question 4

Answer: C

Explanation

$-1 \le 2x-1 \le 1 \Rightarrow 0 \le x \le 1 \Rightarrow$ the domain is $[0, 1]$

$$-\frac{\pi}{2} \le \text{Sin}^{-1}(2x-1) \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{4} \le \frac{1}{2}\text{Sin}^{-1}(2x-1) \le \frac{\pi}{4} \Rightarrow 2-\frac{\pi}{4} \le 2-\frac{1}{2}\text{Sin}^{-1}(2x-1) \le 2+\frac{\pi}{4}$$

Therefore the range of $y = \frac{1}{2}f(x)$ is $\left[1-\frac{\pi}{8}, 1+\frac{\pi}{8}\right]$ or $\left[\frac{8-\pi}{8}, \frac{8+\pi}{8}\right]$

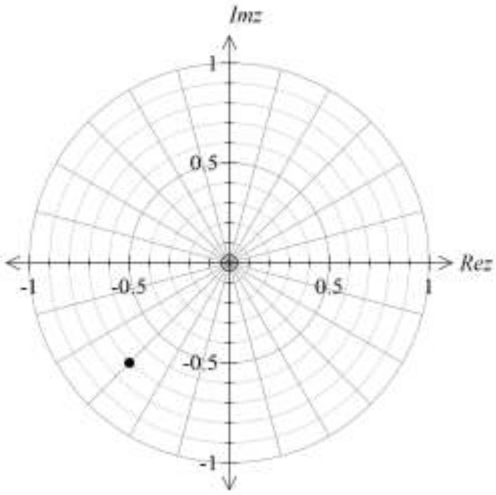
Question 5

Answer: B

Explanation

$z = \frac{1+i}{(1-i)^2} \Rightarrow z = -\frac{1}{2} + \frac{1}{2}i \Rightarrow \overline{z} = -\frac{1}{2} - \frac{1}{2}i$

In polar form, $\overline{z} = \frac{\sqrt{2}}{2}\text{cis}\left(-\frac{3\pi}{4}\right)$



Question 6

Answer: D

Explanation

$$z^3 + bz^2 + (2+i)z - 2 = (z+i)(z^2 + mz + 2i)$$
$$= z^3 + mz^2 + 2iz + z^2 i + miz - 2$$
$$= z^3 + z^2(m+i) + z(mi + 2i) - 2$$

Equating coefficients,

$m + i = b$ and $mi + 2i = 2 + i$

It follows, $mi = 2 - i \Rightarrow m = \frac{2-i}{i} \times \frac{-i}{-i} = -1 - 2i$

Therefore, $b = -1 - 2i + i = -1 - i$

Question 7

Answer: B

Explanation

$1 + ax = 0 \;\; gives \;\; x = -\frac{1}{a}$

$\frac{1}{2} = -\frac{1}{a} \rightarrow a = -2$

Question 8

Answer: A

Explanation

METHOD 1: Non CAS

$$z^3 - 64i = 0$$

$$z^3 = 64i$$

$$r^3 \text{cis} 3\theta = 64 \text{cis} \frac{\pi}{2}$$

$$r^3 = 64 \Rightarrow r = 4 \qquad \text{and} \qquad 3\theta = \frac{\pi}{2} + 2k\pi, \; k = 0, 1, \; 2$$

It follows,

$$3\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$$

The roots are $z_1 = 4\text{cis}\frac{\pi}{6}, \; z_2 = 4\text{cis}\frac{5\pi}{6}, \; z_3 = 4\text{cis}\left(-\frac{\pi}{2}\right)$

It follows that $\frac{2z_1z_3}{z_2} = 8\text{cis}\left(\frac{\pi}{6} - \frac{\pi}{2} - \frac{5\pi}{6}\right) = 8\text{cis}\left(-\frac{7\pi}{6}\right) = 8\text{cis}\left(\frac{5\pi}{6}\right)$

Therefore $\text{Arg}\left(\frac{2z_1z_3}{z_2}\right) = \frac{5\pi}{6}$

<u>METHOD 2</u>: Using CAS

The CAS calculator can be used to solve the expression $z^3 - 64i = 0$ and subsequently find the complex roots. The expression $\frac{2z_1z_3}{z_2}$ can then be calculated using CAS and the principal argument determined.

Question 9

Answer: E

Explanation

$$\begin{aligned}
\sec^2(2\pi x) &= 2\tan(2\pi x)\\
\sec^2(2\pi x) - 2\tan(2\pi x) &= 0\\
1 + \tan^2(2\pi x) - 2\tan(2\pi x) &= 0\\
\tan^2(2\pi x) - 2\tan(2\pi x) + 1 &= 0\\
(\tan(2\pi x) - 1)^2 &= 0\\
\tan(2\pi x) - 1 &= 0\\
\tan(2\pi x) &= 1
\end{aligned}$$

Question 10

Answer: D

Explanation

$$\begin{aligned}
\underset{\sim}{c} &= \underset{\sim}{b} - \underset{\sim}{a}\\
\underset{\sim}{c}.\underset{\sim}{c} &= (\underset{\sim}{b} - \underset{\sim}{a}).(\underset{\sim}{b} - \underset{\sim}{a})\\
|\underset{\sim}{c}|^2 &= (\underset{\sim}{b} - \underset{\sim}{a}).(\underset{\sim}{b} - \underset{\sim}{a})
\end{aligned}$$

Question 11

Answer: B

Explanation

Since the object is in equilibrium, then

$$\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = \underset{\sim}{0}$$

$$-5\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{i} + 7\underset{\sim}{j} + a\underset{\sim}{i} + b\underset{\sim}{j} = \underset{\sim}{0}$$

$$(-5+3+a)\underset{\sim}{i} + (4+7+b)\underset{\sim}{j} = \underset{\sim}{0}$$

$$(a-2)\underset{\sim}{i} + (b+11)\underset{\sim}{j} = \underset{\sim}{0}$$

$$a = 2 \text{ and } b = -11$$

And so $F_3 = 2\underset{\sim}{i} - 11\underset{\sim}{j}$

If the angle between $\underset{\sim}{F}_1$ and F_3 is represented by θ, then

$$\theta = \cos^{-1}\left(\frac{\left(-5\underset{\sim}{i} + 4\underset{\sim}{j}\right).\left(2\underset{\sim}{i} - 11\underset{\sim}{j}\right)}{\left|-5\underset{\sim}{i} + 4\underset{\sim}{j}\right|\left|2\underset{\sim}{i} - 11\underset{\sim}{j}\right|}\right)$$

$$= \cos^{-1}\left(\frac{-10-44}{\sqrt{41\times125}}\right)$$

$$= \cos^{-1}\left(\frac{-54}{\sqrt{5125}}\right)$$

$$= 139^\circ$$

Question 12

Answer: C

Explanation

$$\sqrt{a^2+b^2} = 3 \Rightarrow a^2 + b^2 = 9$$

$$\left(a\underset{\sim}{i} + b\underset{\sim}{j}\right)\left(2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}\right) = 0$$

$$2a - b = 0$$

$$b = 2a$$

Therefore,

$$a^2 + (2a)^2 = 9$$

$$5a^2 = 9$$

$$a = \pm\frac{3\sqrt{5}}{5} \Rightarrow b = \pm\frac{6\sqrt{5}}{5}$$

Question 13

Answer: B

Explanation

$$\underset{\sim}{r}(t) = \left(t^2 - 3\right)\underset{\sim}{i} + \sqrt{3t^2 + 1}\,\underset{\sim}{j}$$

$$\dot{\underset{\sim}{r}}(t) = 2t\underset{\sim}{i} + \frac{3t}{\sqrt{3t^2 + 1}}\underset{\sim}{j}$$

$$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{4t^2 + \frac{9t^2}{3t^2 + 1}}$$

$$\left|\dot{\underset{\sim}{r}}(4)\right| = 8.2\,\text{ms}^{-1}$$

Question 14

Answer: A

Explanation

$$x = 2\sin t \Rightarrow \frac{dx}{dt} = 2\cos t$$

$$y = 3\tan t \Rightarrow \frac{dy}{dt} = \frac{3}{\cos^2 t}$$

It follows,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} \\ &= \frac{3}{\cos^2 t} \times \frac{1}{2\cos t} \\ &= \frac{3}{2\cos^3 t}\end{aligned}$$

When $t = \frac{5\pi}{6}$, $\frac{dy}{dx} = \frac{3}{2\left(-\frac{\sqrt{3}}{2}\right)^3} = -\frac{4\sqrt{3}}{3}$

Question 15

Answer: D

Explanation

$$u = \log_e(4x) \qquad\qquad x = 4 \quad u = \log_e 16 = 4\log_e 2$$

$$\frac{du}{dx} = \frac{1}{x} \qquad\qquad x = 8 \quad u = \log_e 32 = 5\log_e 2$$

It follows,

$$\int_4^8 \frac{2}{x\log_e(4x)}\,dx = \int_4^8 2\times\frac{1}{x}\times\frac{1}{\log_e(4x)}\,dx$$

$$= \int_{4\log_e 2}^{5\log_e 2} 2\times\frac{du}{dx}\times\frac{1}{u}\,dx$$

$$= \int_{4\log_e 2}^{5\log_e 2} \frac{2}{u}\,du$$

Question 16

Answer: E

Explanation

$$\frac{dV}{dt} = \frac{k}{\sqrt{V}}$$

$$\frac{dt}{dV} = \frac{\sqrt{V}}{k}$$

$$t = \frac{1}{k}\int V^{\frac{1}{2}}dV$$

$$= \frac{2k}{3}V^{\frac{3}{2}} + c \qquad\qquad t = 0,\ V = 9 \Rightarrow c = -\frac{18}{k}$$

$$= \frac{2}{3k}V^{\frac{3}{2}} - \frac{18}{k}$$

$$= \frac{2}{k}\left(\frac{V^{\frac{3}{2}}}{3} - 9\right)$$

$$V = \left(\frac{3kt + 54}{2}\right)^{\frac{2}{3}}$$

When $t = 5,\ \ V = \left(\frac{15k + 54}{2}\right)^{\frac{2}{3}}$

Question 17

Answer: A

Explanation

The solution could be $y = \log_e |x-2| + c \Rightarrow \frac{dy}{dx} = \frac{1}{x-2}$

Question 18

Answer: B

Explanation

Let $g(x_n) = x_n \log_e x_n$

Tabulating, we have

n	x_n	y_n	$g(x_n)$
0	2	1	1.386294
1	2.2	1.277259	1.734606
2	2.4	1.62418	

Therefore $y(2.4) = 1.624$

Question 19

Answer: B

Explanation

During the third second of motion

$$20 = (u + 2a) + \frac{a}{2}$$

$$40 = 2u + 5a$$

During the fifth second of motion

$$10 = (u + 4a) + \frac{a}{2}$$

$$20 = 2u + 9a$$

Solving for a and u, $a = -5\,\text{ms}^{-2}$ and $u = 32.5\,\text{ms}^{-1}$

Question 20

Answer: C

Explanation

$a = 24 - 6t$

Therefore,

$$v = \int (24 - 6t)\,dt$$
$$= 24t - 3t^2 + c \qquad t = 0,\ v = 24 \Rightarrow c = 24$$
$$= 24t - 3t^2 + 24$$

It follows, the distance, s is

$$s = \int_0^6 (24t - 3t^2 + 24)\,dt$$
$$= 360 \text{ metres}$$

Question 21

Answer: C

Explanation

$$2a = 2v^2 - 1$$
$$a = \frac{2v^2 - 1}{2}$$
$$v\frac{dv}{dx} = \frac{2v^2 - 1}{2}$$
$$\frac{dx}{dv} = \frac{2v}{2v^2 - 1}$$
$$x = \int \frac{2v}{2v^2 - 1}\,dv$$
$$u = 2v^2 - 1$$
$$= \frac{1}{2}\int \frac{1}{u}\,du \qquad \frac{du}{dv} = 4v \Rightarrow \frac{1}{2}\frac{du}{dv} = 2v$$
$$= \frac{1}{2}\log_e |2v^2 - 1| + c$$
$$1 = \frac{1}{2}\log_e(1) + c \Rightarrow c = 1$$
$$x = \frac{1}{2}\log_e |2v^2 - 1| + 1$$
$$v = \sqrt{\frac{1}{2}\left(e^{2(x-1)} + 1\right)}$$

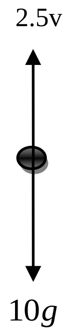
Since $v = 1$ when $x = 1$ we take the **positive** square root

Question 22

Answer: A

Explanation

The equation of motion is



$$2.5v - 10g = -10a$$

$$a = \frac{10g - 2.5v}{10}$$

$$\frac{dv}{dt} = \frac{10g - 2.5v}{10}$$

$$\frac{dt}{dv} = \frac{10}{10g - 2.5v}$$

$$t = \int_0^{20} \frac{10}{10g - 2.5v}\, dv$$

$$= 2.85 \text{ seconds}$$

SECTION 2

Question 1

a.

From CAS

$(\cos\theta + i\sin\theta)^3 = \cos\theta\left(4\cos^2\theta - 3\right) + i\sin\theta\left(4\cos^2\theta - 1\right)$ [A1]

b.

$\text{cis}(3\theta) = \cos(3\theta) + i\sin(3\theta)$

Since $\left(\text{cis}(\theta)\right)^3 = \text{cis}(3\theta)$, then

$\cos(3\theta) + i\sin(3\theta) = \cos\theta\left(4\cos^2\theta - 3\right) + i\sin\theta\left(4\cos^2\theta - 1\right)$ [M1]

Equating coefficients,

$$\begin{aligned}\sin(3\theta) &= \sin\theta\left(4\cos^2\theta - 1\right)\\ &= \sin\theta\left(4\left(1-\sin^2\theta\right) - 1\right)\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$ [M1]

c.

From CAS and simplifying

$(\cos\theta + i\sin\theta)^5 = \cos\theta\left(1 + 4\sin^2\theta - 16\sin^2\theta\cos^2\theta\right) - i\sin\theta\left(\left(16\sin^2\theta - 4\right)\cos^2\theta - 1\right)$ [A1]

d.

$\text{cis}(5\theta) = \cos(5\theta) + i\sin(5\theta)$

$\cos(5\theta) + i\sin(5\theta) = \cos\theta\left(1 + 4\sin^2\theta - 16\sin^2\theta\cos^2\theta\right) - i\sin\theta\left(\left(16\sin^2\theta - 4\right)\cos^2\theta - 1\right)$ [M1]

Equating coefficients,

$$\begin{aligned}\sin(5\theta) &= -\sin\theta\left(\left(16\sin^2\theta - 4\right)\cos^2\theta - 1\right)\\ &= -\sin\theta\left(\left(16\sin^2\theta - 4\right)\left(1 - \sin^2\theta\right) - 1\right)\\ &= -\sin\theta\left(16\sin^2\theta - 16\sin^4\theta - 5 + 4\sin^2\theta\right)\\ &= -\sin\theta\left(20\sin^2\theta - 16\sin^4\theta - 5\right)\\ &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta\end{aligned}$$ [M2]

e.

$$\sin(5\theta)+\sin(3\theta)+\sin\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta + 3\sin\theta - 4\sin^3\theta + \sin\theta$$
$$= 16\sin^5\theta - 24\sin^3\theta + 9\sin\theta$$
$$= \sin\theta\left(16\sin^4\theta - 24\sin^2\theta + 9\right)$$
$$= \sin\theta\left(4\sin^2\theta - 3\right)^2$$

$$\sin(5\theta)+\sin(3\theta)+\sin\theta = 0 \Rightarrow \sin\theta\left(4\sin^2\theta-3\right)^2 = 0$$

It follows,

$$\sin\theta = 0 \text{ or } \sin\theta = \pm\frac{\sqrt{3}}{2}$$

Solving for θ

$$\theta = 0 \text{ or } \theta = \pm\frac{\pi}{3}$$

[M2] [A1]

f.

$$16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = 0$$
$$\sin\theta\left(16\sin^4\theta - 20\sin^2\theta + 5\right) = 0$$

Since $\sin\theta \neq 0$, then

$$16\sin^4\theta - 20\sin^2\theta + 5 = 0$$
$$\sin^2\theta = \frac{20\pm\sqrt{400-320}}{32}$$
$$= \frac{5\pm\sqrt{5}}{8}$$
$$\sin\theta = \pm\sqrt{\frac{5\pm\sqrt{5}}{8}}$$

[M2]

It follows,

$$\sin(5\theta) = 0$$
$$5\theta = n\pi,\ n\in Z$$
$$\theta = \frac{n\pi}{5}$$

[M1]

Therefore, $\sin\dfrac{2\pi}{5} = \sqrt{\dfrac{5+\sqrt{5}}{8}}$

Question 2

a.

$x=0 \Rightarrow y=2$ giving $(0,\ 2)$

$y=0 \Rightarrow x^2-2=0 \Rightarrow x=\pm\sqrt{2}$ giving $\left(-\sqrt{2},\ 0\right)$ and $\left(\sqrt{2},\ 0\right)$ [A1]

b.

$$f(x)=\frac{\frac{1}{2}\left(2x^2-1\right)-\frac{3}{2}}{2x^2-1}$$

$$=\frac{1}{2}-\frac{3}{2\left(2x^2-1\right)}$$

[M1]

Therefore, the equations of the asymptotes are $x=-\frac{\sqrt{2}}{2}$, $x=\frac{\sqrt{2}}{2}$ and $y=\frac{1}{2}$ [A1]

c.

$$f(x)=\frac{1}{2}-\frac{3}{2\left(2x^2-1\right)}$$

$$=\frac{1}{2}-\frac{3}{2}\left(2x^2-1\right)^{-1}$$

$$f'(x)=\frac{3}{2}\left(2x^2-1\right)^{-2}\times 4x$$

$$=\frac{6x}{\left(2x^2-1\right)^2}$$

$f'(x)=0 \Rightarrow x=0$ and $y=2$

$x<0,\ f'(x)<0$ and $x>0,\ f'(x)>0$

[M2]

Therefore, $(0,\ 2)$ is a local minimum point [A1]

d.

$$f'(x) = \frac{6x}{(2x^2 - 1)^2}$$

From CAS or otherwise,

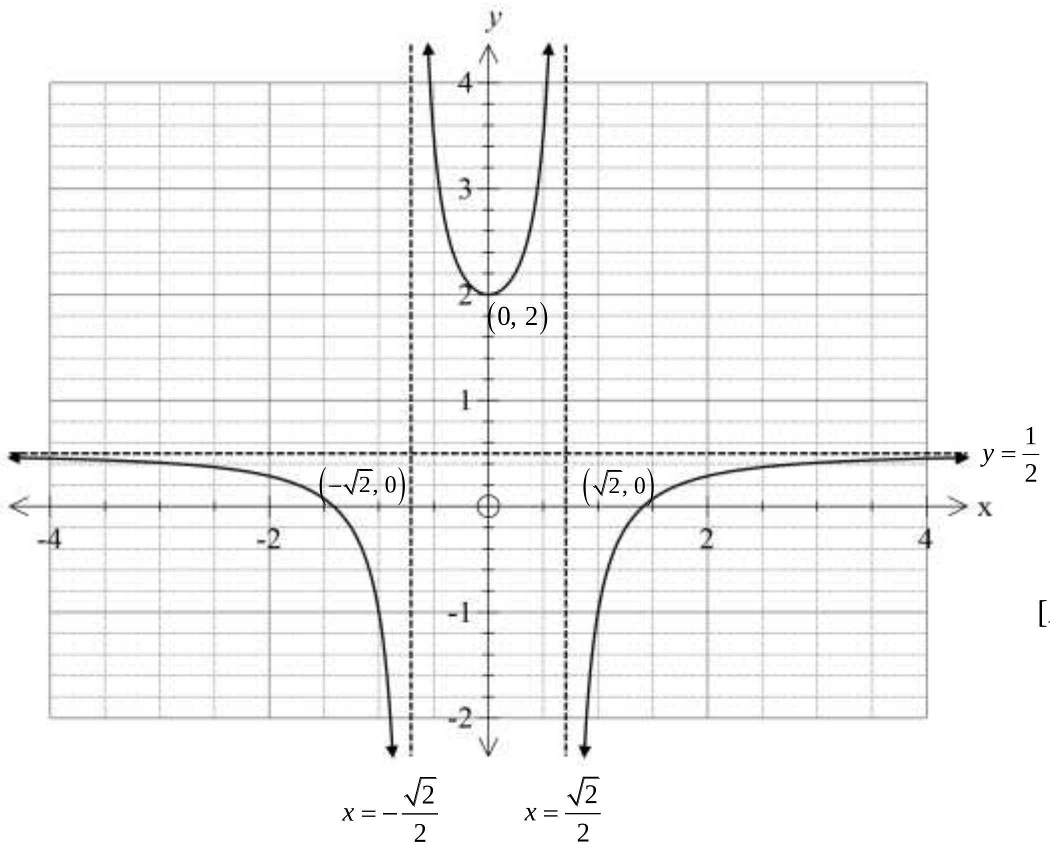
$$f''(x) = -\frac{6(6x^2 + 1)}{(2x^2 - 1)^3}$$

Since $f''(x) \neq 0$ for all $x \in R$

then the curve $y = f(x)$ does not have any points of inflection

[M1] [A1]

e.



[A2]

f.

At the points of intersection,

$$\frac{x^2-2}{2x^2-1}=2+\log_e(x+3) \qquad \text{[M1]}$$

From CAS,

The relevant points of intersection are $(-0.4389,\ 2.9404)$ and $(0.4763,\ 3.2460)$

Therefore, the volume of the solid of revolution, V is equal to

$$V=\pi\int_{-0.4389}^{0.4763}\left(\left(2+\log_e(x+3)\right)^2-\left(\frac{x^2-2}{2x^2-1}\right)^2\right)dx$$

$$=12.35\text{ unit}^3$$

[M2] [A1]

Question 3

a.

$$\overrightarrow{AB}=700\underset{\sim}{i}+400\underset{\sim}{j}$$

$$\overrightarrow{BC}=350\underset{\sim}{i}-1550\underset{\sim}{j}$$

$$\overrightarrow{AC}=1050\underset{\sim}{i}-1150\underset{\sim}{j}$$

[A2]

b.

$$\left|\overrightarrow{AB}\right|=\sqrt{700^2+400^2}=806.23$$

$$\left|\overrightarrow{BC}\right|=\sqrt{350^2+(-1550)^2}=1589.02$$

$$\left|\overrightarrow{AC}\right|=\sqrt{1050^2+(-1150)^2}=1557.24$$

[M1]

Therefore, the perimeter of the paddock $=806.23+1589.02+1557.24=3952.49$.

Correct to the nearest metre, the perimeter is 3952 m. [A1]

c.

Let $\angle BAC = \theta$

It follows,

$$\cos\theta = \frac{\left(700\underset{\sim}{i} + 400\underset{\sim}{j}\right).\left(1050\underset{\sim}{i} - 1150\underset{\sim}{j}\right)}{\sqrt{650000} \times \sqrt{2425000}}$$

$$= \frac{275000}{\sqrt{650000} \times \sqrt{2425000}}$$

$$\theta = 77.35°$$

[M1] [A1]

The area of the paddock, A is equal to

$$A = \frac{1}{2} \times \sqrt{650000} \times \sqrt{2425000} \sin 77.35°$$

$$= 612506 \text{ m}^2$$

$$= 61.3 \text{ ha}$$

[A1]

d.

<u>One of a number of possible methods</u>

A unit vector in the direction of $\overrightarrow{AC}$ is $\dfrac{1050\underset{\sim}{i} - 1150\underset{\sim}{j}}{\sqrt{2425000}}$

Let $\underset{\sim}{v} =$ the vector projection of $\overrightarrow{AB}$ in a direction perpendicular to $\overrightarrow{AC}$.

It follows,

$$\underset{\sim}{v} = 700\underset{\sim}{i} + 400\underset{\sim}{j} - \left(700\underset{\sim}{i} + 400\underset{\sim}{j}\right).\left(\frac{1050\underset{\sim}{i} - 1150\underset{\sim}{j}}{\sqrt{2425000}}\right) \times \frac{1050\underset{\sim}{i} - 1150\underset{\sim}{j}}{\sqrt{2425000}}$$

$$= 700\underset{\sim}{i} + 400\underset{\sim}{j} - \frac{275000}{2425000}\left(1050\underset{\sim}{i} - 1150\underset{\sim}{j}\right)$$

$$= 580.928\underset{\sim}{i} + 530.412\underset{\sim}{j}$$

Therefore, the minimum distance from B to $[AC]$ is

$$\left|\underset{\sim}{v}\right| = \sqrt{580.928^2 + 530.412^2} = 786.65 \text{ m}$$

[M2] [A1]

e.

Therefore, the area of the paddock, A is equal to

$$A = \frac{1}{2} \times \left|\overrightarrow{AC}\right| \times \left|\underset{\sim}{v}\right|$$
$$= \frac{1}{2} \times 1557.24 \times 786.65$$
$$= 612501 \text{ m}^2$$
$$= 61.3 \text{ ha}$$

[A1]

f.

$$\overrightarrow{AB} = 700\underset{\sim}{i} + 400\underset{\sim}{j}$$

$$\overrightarrow{OD} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AC}$$
$$= -200\underset{\sim}{i} + 800\underset{\sim}{j} + \frac{2}{3}\left(1050\underset{\sim}{i} - 1150\underset{\sim}{j}\right)$$
$$= 500\underset{\sim}{i} + \frac{100}{3}\underset{\sim}{j}$$

$$\overrightarrow{OE} = \overrightarrow{OB} + \frac{2}{3}\overrightarrow{BC}$$
$$= 500\underset{\sim}{i} + 1200\underset{\sim}{j} + \frac{2}{3}\left(350\underset{\sim}{i} - 1550\underset{\sim}{j}\right)$$
$$= \frac{2200}{3}\underset{\sim}{i} + \frac{500}{3}\underset{\sim}{j}$$

[M2]

Therefore,

$$\overrightarrow{DE} = \frac{2200}{3}\underset{\sim}{i} + \frac{500}{3}\underset{\sim}{j} - \left(500\underset{\sim}{i} + \frac{100}{3}\underset{\sim}{j}\right)$$
$$= \frac{700}{3}\underset{\sim}{i} + \frac{400}{3}\underset{\sim}{j}$$
$$= \frac{1}{3}\left(700\underset{\sim}{i} + 400\underset{\sim}{j}\right)$$
$$= \frac{1}{3}\overrightarrow{AB}$$

[A1]

Since $\overrightarrow{DE} = k \times \overrightarrow{AB}$, where $k = \frac{1}{3}$ then the fence $[DE]$ is parallel to the boundary $[AB]$

Question 4

a.

$$2\frac{dv}{dt}+v^2+1=0$$

$$\frac{dv}{dt}=-\frac{\left(v^2+1\right)}{2}$$

$$\frac{dt}{dv}=-\frac{2}{v^2+1}$$

$$t=-\int\frac{2}{v^2+1}dv$$

$$=-2\int\frac{1}{v^2+1}dv$$

$$=-2\tan^{-1}v+c$$

[M2]

$v=1$ when $t=0$

It follows,

$$0=-2\tan^{-1}v+c\Rightarrow c=2\tan^{-1}1=2\times\frac{\pi}{4}=\frac{\pi}{2}$$

Therefore,

$$t=\frac{\pi}{2}-2\tan^{-1}v$$

$$2\tan^{-1}v=\frac{\pi}{2}-t$$

$$\tan^{-1}v=\frac{\pi}{4}-\frac{t}{2}$$

[M1]

$$v=\tan\left(\frac{\pi}{4}-\frac{t}{2}\right)$$

b.

$$\frac{\pi}{4}-\frac{t}{2}=\frac{\pi}{2}$$

$$t=-\frac{\pi}{2}+n\times 2\pi,\ n\in Z$$

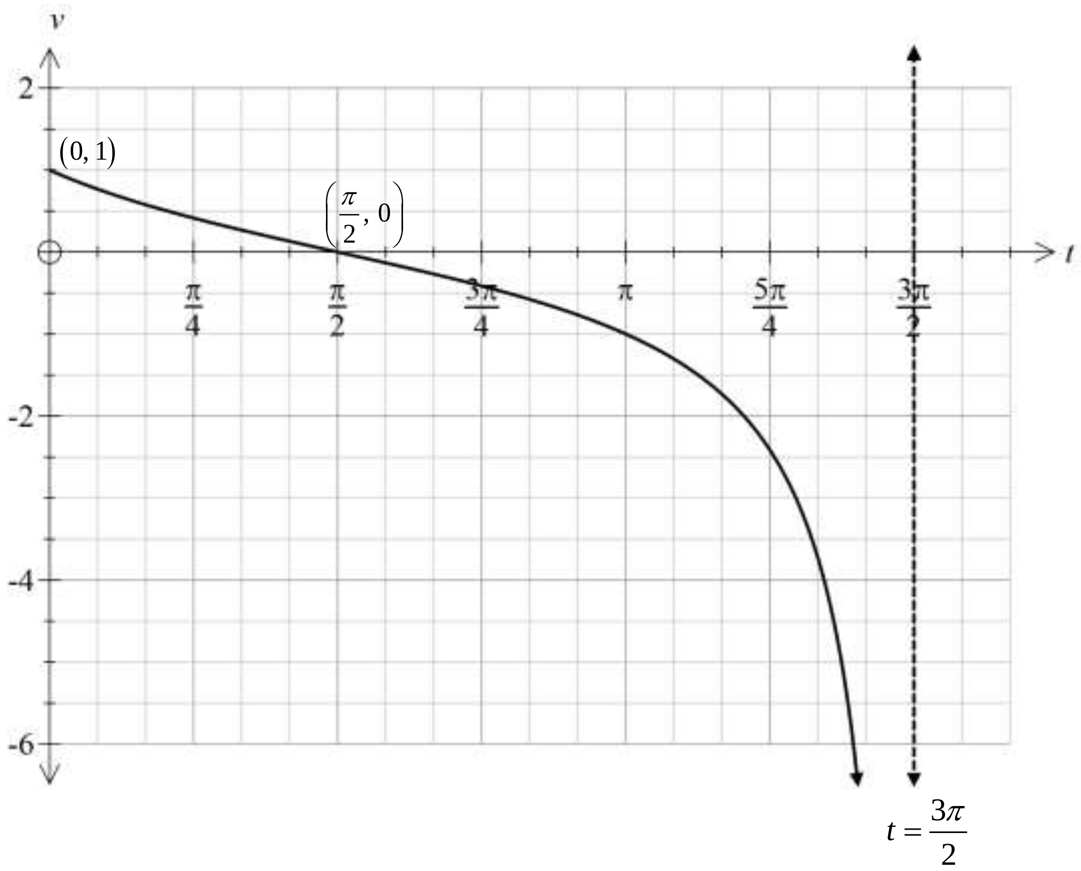
Since $\pi\le a\le 2\pi$

$$n=1\Rightarrow t=\frac{3\pi}{2}\Rightarrow a=\frac{3\pi}{2}$$

[M1]

$t=0\Rightarrow v=1$ and $v=0\Rightarrow t=\frac{\pi}{2}$

The vertical asymptote is $t=\frac{3\pi}{2}$



[A2]

c.

From CAS,

$$\text{Distance} = \int_0^{\frac{5\pi}{4}} \left|\tan\left(\frac{\pi}{4} - \frac{t}{2}\right)\right| dt = 2.61 \text{ metres} \quad \text{[A1]}$$

d.

Let $T =$ the time taken to travel a distance of 0.75 metres

$$\int_0^T \left|\tan\left(\frac{\pi}{4} - \frac{t}{2}\right)\right| dt = 0.75$$

Solving for T

$T = 2.05$ sec

[M2] [A1]

e.

$$v=\tan\left(\frac{\pi}{4}-\frac{t}{2}\right)$$

$$\frac{dx}{dt}=\tan\left(\frac{\pi}{4}-\frac{t}{2}\right)$$

$$x=\int\tan\left(\frac{\pi}{4}-\frac{t}{2}\right)dt$$

$$=\int\frac{\sin\left(\frac{\pi}{4}-\frac{t}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{t}{2}\right)}dt \qquad u=\cos\left(\frac{\pi}{4}-\frac{t}{2}\right)$$

$$\frac{du}{dt}=\frac{1}{2}\sin\left(\frac{\pi}{4}-\frac{t}{2}\right)\Rightarrow\sin\left(\frac{\pi}{4}-\frac{t}{2}\right)=2\frac{du}{dt}$$

$$=\int\frac{2}{u}du$$

$$=2\log_e\left|\cos\left(\frac{\pi}{4}-\frac{t}{2}\right)\right|+c$$

[M2]

$x=0$ when $t=0$

It follows,

$$0=2\log_e\left|\cos\left(\frac{\pi}{4}\right)\right|+c\Rightarrow c=-2\log_e\cos\left(\frac{\pi}{4}\right)=-2\log_e\frac{1}{\sqrt{2}}$$

Therefore,

$$x=2\log_e\left|\cos\left(\frac{\pi}{4}-\frac{t}{2}\right)\right|-2\log_e\frac{1}{\sqrt{2}}$$

[M1]

$$=2\log_e\left|\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{t}{2}\right)\right|$$ [A1]

f.

$$x=2\log_e\left|\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{5\pi}{8}\right)\right|$$

$$=-1.23$$

Therefore, the particle is 1.23 metres to the left of the starting point. [A1]

g.

One of a number of possible methods

$$2\frac{dv}{dt}+v^2+1=0$$

$$\frac{dv}{dt}=-\frac{\left(v^2+1\right)}{2}$$

$$v\frac{dv}{dx}=-\frac{\left(v^2+1\right)}{2}$$

$$\frac{dv}{dx}=-\frac{v^2+1}{2v}$$

$$\frac{dx}{dv}=-\frac{2v}{v^2+1}$$

$$x=-\int\frac{2v}{v^2+1}dv \qquad u=v^2+1,\quad \frac{du}{dv}=2v$$

$$=-\int\frac{1}{u}du$$

$$=-\log_e\left(v^2+1\right)+c$$

[M2]

$v=1$ when $x=0$

It follows,

$$0=-\log_e(2)+c\Rightarrow c=\log_e 2$$

Therefore,

$$x=\log_e 2-\log_e\left(v^2+1\right)$$

$$=\log_e\left(\frac{2}{v^2+1}\right)$$

[M1]