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SPECIALIST MATHS TRIAL EXAMINATION 1 SOLUTIONS 2014

Question 1 (3 marks)

$$\int \frac{3+x}{4-x^2}dx$$

Let $\frac{3+x}{4-x^2} \equiv \frac{A}{(2-x)} + \frac{B}{(2+x)}$ **(1 mark)**

$$\equiv \frac{A(2+x)+B(2-x)}{(2-x)(2+x)}$$

True iff $3+x \equiv A(2+x)+B(2-x)$

Put $x=-2$, $1=4B$, $B=\frac{1}{4}$

Put $x=2$, $5=4A$, $A=\frac{5}{4}$

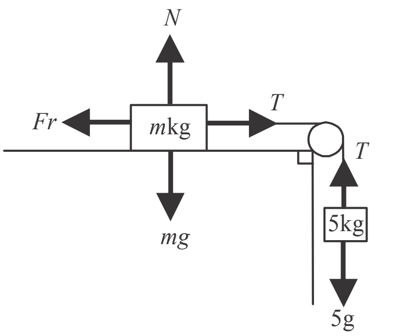
$$\int \frac{3+x}{4-x^2}dx = \int \frac{5}{4(2-x)}dx + \int \frac{1}{4(2+x)}dx$$ **(1 mark)**

$$= -\frac{5}{4}\log_e|2-x| + \frac{1}{4}\log_e|2+x| + c$$ **(1 mark)**

Note: you can go on and simplify this using log laws but you risk losing a mark if you make a mistake. (see 2011 Exam 1 Question 1 Examiners report on VCAA website)

Question 2 (4 marks)

a.



(1 mark) – for 3 correct forces
(1 mark) – for 3 more correct forces

b. At the point of moving $Fr = \mu N$.

<u>Around the 5kg mass:</u> $T = 5g$ **(1 mark)**

<u>Around the *m* kg mass:</u>

$Fr = T$ and $N = mg$

So $\mu N = 5g$

$\frac{1}{5} \times mg = 5g$

$mg = 25g$

$m = 25$ **(1 mark)**

Question 3 (3 marks)

Since $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent $\alpha(\underset{\sim}{i} - 3\underset{\sim}{j} + 3\underset{\sim}{k}) + \gamma(2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}) = x\underset{\sim}{i} + y\underset{\sim}{j}$

(1 mark)

So $\alpha + 2\gamma = x \quad -(1)$

$-3\alpha - \gamma = y \quad -(2)$

$3\alpha + 2\gamma = 0$

$\alpha = \dfrac{-2\gamma}{3} - (3)$ **(1 mark)**

(3) in (1) $-\dfrac{2\gamma}{3} + 2\gamma = x$

$\dfrac{4\gamma}{3} = x$

$\gamma = \dfrac{3x}{4}$

(3) in (2) $-3 \times -\dfrac{2\gamma}{3} - \gamma = y$

$2\gamma - \gamma = y$

$\gamma = y$

So $y = \dfrac{3x}{4}$

So $p = \dfrac{3}{4}$

(1 mark)

Question 4 (3 marks)

$2\cot(x) = -\text{cosec}(x)$

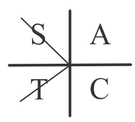
$2\dfrac{\cos(x)}{\sin(x)} = \dfrac{-1}{\sin(x)}, \quad \sin(x) \neq 0$

$2\cos(x)\sin(x) = -\sin(x)$

$2\cos(x)\sin(x) + \sin(x) = 0$

$\sin(x)(2\cos(x) + 1) = 0$ **(1 mark)**

$\sin(x) \neq 0$ (from above) $\quad 2\cos(x) + 1 = 0$



$\cos(x) = -\dfrac{1}{2}$

$x = \dfrac{2\pi}{3} + 2k\pi$ or $x = \dfrac{4\pi}{3} + 2k\pi, \quad k \in Z$

(1 mark) **(1 mark)**

Question 5 (3 marks)

$$a = \frac{-x}{(x^2+1)^2}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{-x}{(x^2+1)^2}$$

$$\frac{1}{2}v^2 = \int \frac{-x}{(x^2+1)^2}\,dx$$ **(1 mark)**

$$= -\frac{1}{2}\int \frac{du}{dx}u^{-2}\,dx \qquad \text{where } u = x^2+1$$

$$= -\frac{1}{2}\int u^{-2}\,du \qquad \frac{du}{dx} = 2x$$

$$= -\frac{1}{2}\times\frac{u^{-1}}{-1}+c$$

$$\frac{1}{2}v^2 = \frac{1}{2(x^2+1)}+c$$

Given $v = 1$ when $x = 0$,

$$\frac{1}{2} = \frac{1}{2}+c$$

$$c = 0$$

So $\frac{1}{2}v^2 = \frac{1}{2(x^2+1)}$ **(1 mark)**

$$v^2 = \frac{1}{x^2+1}$$

$$v = \pm\frac{1}{\sqrt{x^2+1}}$$

Since $v = 1$ when $x = 0$, reject the negative branch.

$$v = \frac{1}{\sqrt{x^2+1}}$$ **(1 mark)**

Question 6 (4 marks)

$$2xy+\frac{\log_e(y)}{x}=k$$

When $y=1$,

$$2x+0=k$$

$$x=\frac{k}{2}$$

(1 mark)

Use implicit differentiation to differentiate the function.

$$2y+2x\frac{dy}{dx}+\frac{x\times\frac{1}{y}\times\frac{dy}{dx}-\log_e(y)}{x^2}=0$$

$$2y+2x\frac{dy}{dx}+\frac{1}{xy}\frac{dy}{dx}-\frac{1}{x^2}\log_e(y)=0$$

(1 mark) – product rule **(1 mark)** – quotient rule

When $y=1, \frac{dy}{dx}=\frac{1}{2}$ (given) and $x=\frac{k}{2}$.

So, $2+2\times\frac{k}{2}\times\frac{1}{2}+\frac{2}{k}\times\frac{1}{2}-\frac{4}{k^2}\times 0=0$

$$2+\frac{k}{2}+\frac{1}{k}=0$$

$$4k+k^2+2=0$$

$$k^2+4k+2=0$$

$$k=\frac{-4\pm\sqrt{16-4\times1\times2}}{2}$$ (quadratic formula)

$$=\frac{-4\pm\sqrt{8}}{2}$$

$$=\frac{-4\pm2\sqrt{2}}{2}$$

$$k=-2\pm\sqrt{2}$$

(1 mark)

Question 7 (4 marks)

a. $\underset{\sim}{r}(t) = (\cos(t) + 2)\underset{\sim}{i} + 4\sin(t)\underset{\sim}{j}, \quad t \in \left[0, \frac{3\pi}{2}\right]$

$$x = \cos(t) + 2 \qquad y = 4\sin(t)$$

(1 mark)

$$x - 2 = \cos(t) \qquad \frac{y}{4} = \sin(t)$$

$$(x-2)^2 = \cos^2(t) \qquad \frac{y^2}{16} = \sin^2(t)$$

$$(x-2)^2 + \frac{y^2}{16} = \cos^2(t) + \sin^2(t)$$

$$(x-2)^2 + \frac{y^2}{16} = 1$$

(1 mark)

b. From part **a.**, we have an ellipse with centre at $(2,0)$, semi-major axis length of 4 and semi-minor axis length of 1.

Since $t \in \left[0, \frac{3\pi}{2}\right]$,

when $t = 0$, $\quad \underset{\sim}{r}(0) = 3\underset{\sim}{i} + 0\underset{\sim}{j}$,

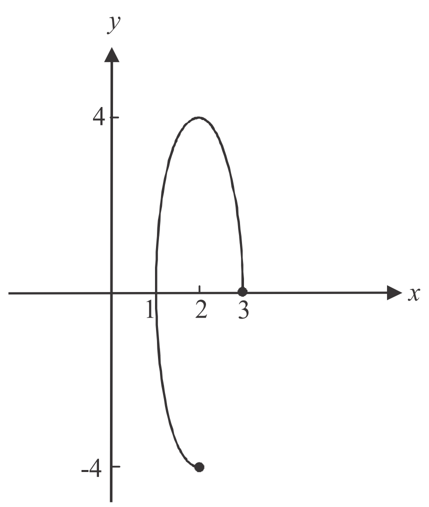
when $t = \frac{\pi}{2}$, $\quad \underset{\sim}{r}\left(\frac{\pi}{2}\right) = 2\underset{\sim}{i} + 4\underset{\sim}{j}$,

when $t = \pi$, $\quad \underset{\sim}{r}(\pi) = \underset{\sim}{i} + 0\underset{\sim}{j}$ and

when $t = \frac{3\pi}{2}$, $\quad \underset{\sim}{r}\left(\frac{3\pi}{2}\right) = 2\underset{\sim}{i} - 4\underset{\sim}{j}$.

So the particle starts at the point (3,0) and follows an elliptical path passing through the points (2,4), and (1,0) before finishing at the point $(2,-4)$.

Note that the endpoints are included.



(1 mark) – correct shape
(1 mark) – correct endpoints

Question 8 (4 marks)

$$z^4-8z^2+49=0$$

$$(z^4-14z^2+49)+6z^2=0 \qquad \text{(complete the square)}$$ **(1 mark)**

$$(z^2-7)^2-\left(i\sqrt{6}z\right)^2=0$$

$$\left(z^2-7-i\sqrt{6}z\right)\left(z^2-7+i\sqrt{6}z\right)=0$$

(1 mark)

$z^2-i\sqrt{6}z-7=0$ or $z^2+i\sqrt{6}z-7=0$

$$z=\frac{i\sqrt{6}\pm\sqrt{-6-4\times1\times-7}}{2} \quad \text{or} \quad z=\frac{-i\sqrt{6}\pm\sqrt{-6-4\times1\times-7}}{2}$$

$$=\frac{i\sqrt{6}\pm\sqrt{22}}{2} \qquad\qquad =\frac{-i\sqrt{6}\pm\sqrt{22}}{2}$$

So $z=\pm\frac{\sqrt{22}}{2}+\frac{\sqrt{6}i}{2}$ **(1 mark)** or $z=\pm\frac{\sqrt{22}}{2}-\frac{\sqrt{6}i}{2}$ **(1 mark)**

Question 9 (5 marks)

a. $g(x)=\frac{4}{\pi}\arcsin\left(\frac{x}{3}-2\right)+1$

Finding the domain:

For g to be defined, $-1\le\frac{x}{3}-2\le1$

$$1\le\frac{x}{3}\le3$$

$$3\le x\le9$$

So $d_g=[3,9]$ **(1 mark)**

Finding the range:

Method 1

The range of the function $y=\arcsin(x)$

is $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

so, $r_g=\left[\frac{-\pi}{2}\times\frac{4}{\pi}+1,\ \frac{\pi}{2}\times\frac{4}{\pi}+1\right]$ **(1 mark)**

$=[-1,\ 3]$

<u>Method 2</u>

$$g(x)=\frac{4}{\pi}\arcsin\left(\frac{x}{3}-2\right)+1$$

So, $\frac{\pi}{4}\left(g(x)-1\right)=\arcsin\left(\frac{x}{3}-2\right)$

The range of the function $y=\arcsin(x)$ is $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

So
$$-\frac{\pi}{2}\le\frac{\pi}{4}\left(g(x)-1\right)\le\frac{\pi}{2}$$
$$-2\le g(x)-1\le 2$$
$$-1\le g(x)\le 3$$

So $r_g=[-1,3]$ **(1 mark)**

b. Let $y=\frac{4}{\pi}\arcsin\left(\frac{x}{3}-2\right)+1$

and let $y=\frac{4}{\pi}\arcsin(u)+1$ where $u=\frac{x}{3}-2$

$\frac{dy}{du}=\frac{4}{\pi\sqrt{1-u^2}}$ $\frac{du}{dx}=\frac{1}{3}$

Now, $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ (Chain rule)

$$=\frac{4}{\pi\sqrt{1-u^2}}\times\frac{1}{3}$$ **(1 mark)**

$$=\frac{4}{\pi\sqrt{1-\left(\frac{x}{3}-2\right)^2}}\times\frac{1}{3}$$

$$=\frac{4}{3\pi\sqrt{1-\left(\frac{x^2}{9}-\frac{4x}{3}+4\right)}}$$ **(1 mark)**

$$=\frac{4}{3\pi\sqrt{-\frac{x^2}{9}+\frac{4x}{3}-3}}$$

$$=\frac{4}{3\pi\sqrt{\frac{-x^2+12x-27}{9}}}$$

$$=\frac{4}{\pi\sqrt{-(x^2-12x+27)}}$$

$$=\frac{4}{\pi\sqrt{-(x-3)(x-9)}}$$

Re-read the question!
So $a=4$, $b=-1$ and $c=3$.

1 mark

Question 10 (7 marks)

a. $y=\frac{1}{x^2+3}$

When $x=0,\ y=\frac{1}{3}$

range $=\left(0,\frac{1}{3}\right]$ **(1 mark)**

b. area $=\int_0^1 \frac{1}{x^2+3}dx$ **(1 mark)**

$=\frac{1}{\sqrt{3}}\int_0^1 \frac{\sqrt{3}}{(\sqrt{3})^2+x^2}dx$

$=\frac{1}{\sqrt{3}}\left[\arctan\left(\frac{x}{\sqrt{3}}\right)\right]_0^1$

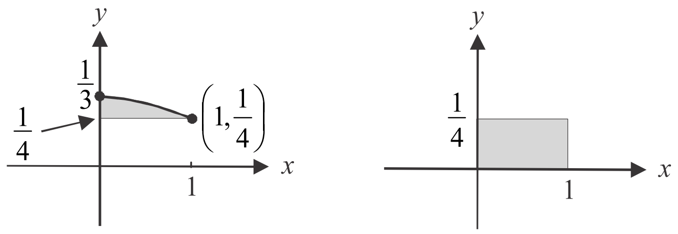
$=\frac{1}{\sqrt{3}}\left(\arctan\left(\frac{1}{\sqrt{3}}\right)-\arctan(0)\right)$

$=\frac{1}{\sqrt{3}}\left(\frac{\pi}{6}-0\right)$

$=\frac{\pi}{6\sqrt{3}}$ square units **(1 mark)**

c. When $x=1,\ \ y=\frac{1}{4}$.

The volume generated can be broken up into two parts.



The first is obtained by rotating the shaded region in the left hand diagram around the y-axis.

This volume $=\pi\int_{\frac{1}{4}}^{\frac{1}{3}} x^2 dy$ **(1 mark)**

Since $y=\frac{1}{x^2+3}$, then $x^2=\frac{1}{y}-3$

so, volume $=\pi\int_{\frac{1}{4}}^{\frac{1}{3}}\left(\frac{1}{y}-3\right)dy$ **(1 mark)**

The second part is obtained by rotating the shaded region in the right hand diagram around the y-axis.

This forms a cylinder with radius 1 and height $\frac{1}{4}$,

$$\text{so volume} = \pi r^2 h \qquad \text{(formula sheet)}$$
$$= \pi \times 1 \times \frac{1}{4}$$
$$= \frac{\pi}{4}$$

(1 mark)

$$\text{Alternatively, volume} = \pi \int_0^{\frac{1}{4}} x^2 \, dy$$
$$= \pi \int_0^{\frac{1}{4}} 1 \, dy \qquad \text{(since we are rotating the line } x = 1 \text{ around the } y\text{-axis)}$$
$$= \pi \left[y \right]_0^{\frac{1}{4}}$$
$$= \frac{\pi}{4}$$

Combining these two volumes we have,

$$\text{total volume} = \pi \int_{\frac{1}{4}}^{\frac{1}{3}} \left(\frac{1}{y} - 3 \right) dy + \frac{\pi}{4}$$
$$= \pi \left[\log_e |y| - 3y \right]_{\frac{1}{4}}^{\frac{1}{3}} + \frac{\pi}{4}$$
$$= \pi \left\{ \left(\log_e \left(\frac{1}{3} \right) - 1 \right) - \left(\log_e \left(\frac{1}{4} \right) - \frac{3}{4} \right) \right\} + \frac{\pi}{4}$$
$$= \pi \left\{ \log_e \left(\frac{4}{3} \right) - \frac{1}{4} \right\} + \frac{\pi}{4}$$
$$= \pi \log_e \left(\frac{4}{3} \right) \text{ cubic units}$$

(1 mark)