NAME: ___________________________

UNITS 3 & 4 Practice Examination

VCE® Specialist Mathematics

Written examination 2

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	5	5	58
			80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 28 pages.
- 4 page formula booklet.
- Answer sheet for multiple-choice questions.

Instructions
- Write your name in the space provided above on this page.
- Write your name on the multiple-choice answer sheet
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $\text{g} = 9.8$.

Question 1

The **domain** of the function with the rule $f(x) = \arccos(2x)$ is

A. $[-1,1]$

B. $[-2,2]$

C. $\left[-\frac{1}{2},\frac{1}{2}\right]$

D. $[0,\pi]$

E. $\left[0,\frac{\pi}{2}\right]$

Question 2

The rule of the relation determined by parametric equations $x = 3\cot(t) + 1$ and $y = 2\operatorname{cosec}(t) - 1$ is

A. $\frac{(x+1)^2}{3} + \frac{(y-1)^2}{2} = 1$

B. $\frac{(x+1)^2}{9} - \frac{(y-1)^2}{4} = 1$

C. $\frac{(x-1)^2}{9} - \frac{(y+1)^2}{4} = 1$

D. $\frac{(y+1)^2}{4} + \frac{(x-1)^2}{9} = 1$

E. $\frac{(y+1)^2}{4} - \frac{(x-1)^2}{9} = 1$

Question 3

The graph of $y = \frac{1}{ax^2+bx+c}$ has asymptotes at $x = -2$ and $x = 4$ and $y = 0$. Given that the graph has one stationary point with a y −coordinate of $-\frac{1}{18}$, it follows that

A. $a = 2, b = -4, c = -16$

B. $a = 2, b = 4, c = -16$

C. $a = 1, b = 2, c = -8$

D. $a = 1, b = -2, c = -8$

E. $a = 2, b = -12, c = 16$

Question 4

If $y = \cot\left(x - \frac{\pi}{2}\right)$, which of the following statements is **not** true?

A. $y = 1/\tan\left(x - \frac{\pi}{2}\right)$

B. $y = \tan(\pi - x)$

C. $y = \cot(x + \frac{\pi}{2})$

D. $y = 1/\tan\left(\frac{\pi}{2} - x\right)$

E. $y = -\tan(x)$

Question 5

The region of the complex plane inside the circle of radius a centred at the origin is given by the set of points z, where $z \in \mathbb{C}$, such that

A. $z\bar{z} < a$

B. $z\bar{z} < a^2$

C. $\left(\text{Re}(z)\right)^2 + \left(\text{Im}(z)\right)^2 < a$

D. $|z| < a^2$

E. $z\bar{z} > a$

Question 6

The $z = \text{cis}\left(\frac{5\pi}{6}\right)$. The imaginary part of $z + i$ is

A. $\frac{i}{2}$

B. $\frac{1}{2}$

C. $\frac{\sqrt{3}}{2}$

D. $\frac{3}{2}$

E. $\frac{3i}{2}$

Question 7

If $z = r\text{cis}(\theta)$, then $\frac{(\bar{z})^2}{z}$is equivalent to

A. $r^3\text{cis}(-3\theta)$

B. $r^3\text{cis}(\theta)$

C. $2\text{cis}(-3\theta)$

D. $r^3\text{cis}(-\theta)$

E. $r\text{cis}(-3\theta)$

Question 8

The principal arguments of the solution to the equation $z^2 = 1 - i$ are

A. $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$

B. $-\frac{\pi}{8}$ and $\frac{7\pi}{8}$

C. $\frac{3\pi}{8}$ and $\frac{11\pi}{8}$

D. $-\frac{\pi}{8}$ and $-\frac{9\pi}{8}$

E. $-\frac{\pi}{4}$ and $-\frac{5\pi}{4}$

Question 9

The definite integral $\int_{\pi/4}^{\pi/2} e^{-\cot(x)}(\cot^2(x)+1)\,dx$ can be written in the form $\int_a^b \frac{1}{e^u}\,du$

A. $u = \cot(x)$, $a = 1$and $b = 0$

B. $u = \cot(x)$, $a = 0$ and $b = 1$

C. $u = -\cot(x)$, $a = 1$and $b = 0$

D. $u = -\cot(x)$, $a = 0$ and $b = 1$

E. $u = \tan(x)$, $a = 1$and $b = \infty$

Question 10

The region bounded by the lines $x = 0, y = 2$ and the graph of $y = x^{\frac{5}{3}}$, where $x \geq 0$ is rotated about the $y-$axis to form a solid of revolution. The volume of this solid is

A. $\frac{5\pi}{2^{7/5}}$

B. $\frac{20}{11}2^{1/5}\pi$

C. $\frac{3\pi}{2^{1/3}}$

D. $\frac{10}{11}2^{8/25}\pi$

E. $\frac{12}{13}2^{3/5}\pi$

Question 11

Euler's formula is used to find y_2, where $\frac{dy}{dx} = \tan(x)$, $x_0 = 0, y_0 = 1$ and $h = 0.1$

The value of y_2 correct to four decimal places is

A. 1.0000 and this is an overestimate of $y(0.2)$

B. 1.0100 and this is an underestimate of $y(0.2)$

C. 1.0100 and this is an overestimate of $y(0.2)$

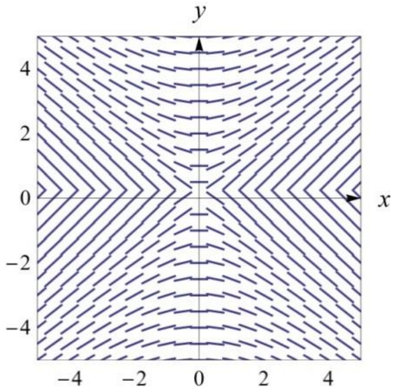
D. 1.0303 and this is an underestimate of $y(0.2)$

E. 1.0100 and this is an overestimate of $y(0.2)$

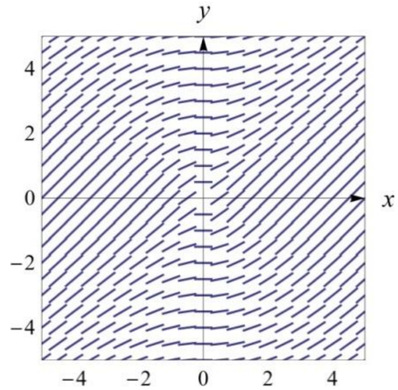
Question 12

Which diagram best represents the direction field of the differential equation $\frac{dy}{dx} = x/y$?

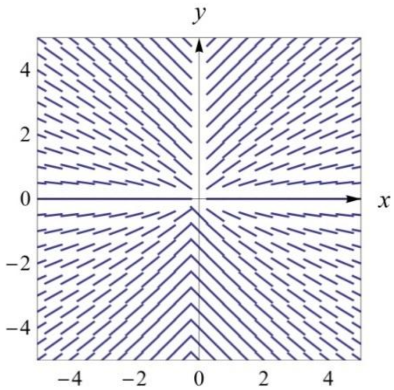
A.



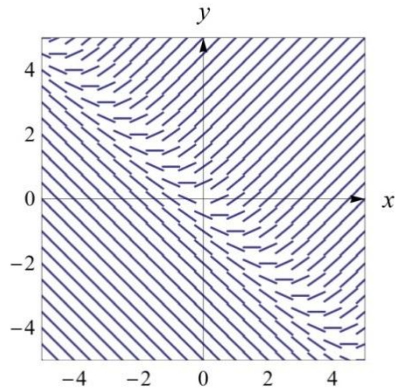
B.



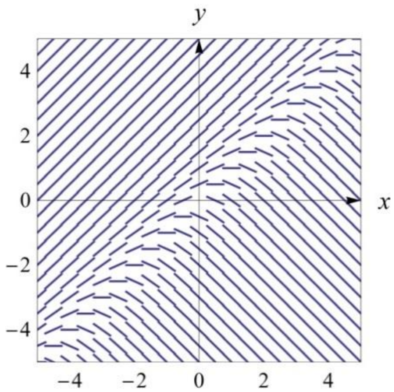
C.



D.



E.



Question 13

Which of the following expressions is equivalent to $\ddot{x}dx$

A. vdv

B. $v\dfrac{dv}{dx}$

C. $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

D. v^2dv

E. $\dfrac{dv}{dt}$

Question 14

Consider the four vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{c} = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$, $\mathbf{d} = 3\mathbf{i} + 3\mathbf{k}$.

Which one of the following is a linearly dependent set of vectors?

A. $\{\mathbf{a}, \mathbf{b}\}$

B. $\{\mathbf{a}, \mathbf{c}\}$

C. $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$

D. $\{\mathbf{b}, \mathbf{c}, \mathbf{d}\}$

E. $\{\mathbf{a}, \mathbf{c}, \mathbf{d}\}$

Question 15

Find the distance between point $P(3,0,4)$ and point $Q(0,-2,1)$.

A. 4

B. 22

C. $\sqrt{22}$

D. 30

E. $\sqrt{30}$

Question 16

Water containing 5 grams of salt per litre flows at a rate of 8 litres per minute into a tank that initially contained 100 litres of pure water. The concentration of salt in the tank is kept uniform by stirring and the mixture flows out of the tank at a rate of 10 litres per minute. If m grams is the amount of salt in the tank t minutes after the water begins to flow, the differential equation for m in terms of t is

A. $\frac{dm}{dt} = 5 - \frac{m}{100}$

B. $\frac{dm}{dt} = 5 - \frac{m}{98}$

C. $\frac{dm}{dt} = 5 - \frac{m}{100 - 2t}$

D. $\frac{dm}{dt} = 40 - \frac{m}{100 + 2t}$

E. $\frac{dm}{dt} = 40 - \frac{10m}{100 - 2t}$

Question 17

Let $\mathbf{u} = 2\mathbf{i} + 3\mathbf{k}$, $\mathbf{v} = -3\mathbf{i} + 2\mathbf{k}$ and $\mathbf{w} = -2\mathbf{i} + 10\mathbf{k}$.

Which one of the following statements is **not** true?

A. $\mathbf{u} \cdot \mathbf{v} = 0$

B. $(\mathbf{u} + \mathbf{w}) \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w}$

C. $\mathbf{w} \cdot \mathbf{w} = |\mathbf{w}|^2$

D. $\mathbf{u} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{v}$

E. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} = |\mathbf{u}|$

Question 18

A box of mass m kg rests on the floor of a lift. Starting from rest, the lift moves downwards with a constant acceleration a ms^{-2}, so that after time t, the box has traveled x m downward and is travelling with a downward velocity of v ms^{-1}. Let F_N denote the normal reaction force of the lift floor on the box. Which of the following statements is **not** true?

A. $ma = m\text{g} - F_N$

B. $mv^2 = 2x(m\text{g} - F_N)$

C. $m\dfrac{dv}{dt} = m\text{g} - F_N$

D. $mg = -ma + F_N$

E. $x = \dfrac{1}{2}(\text{g} - \dfrac{F_N}{m})t^2$

Question 19

A ball of mass m is projected vertically with an upwards velocity $v(x)$ m/s, where x is the distance measured upwards from ground level in metres at time t seconds. The motion of the ball is retarded by an air resistance of magnitude kv^2 newtons. The equation of motion of the ball is

A. $m\dfrac{dv}{dt} = mg + kv^2$

B. $m\dfrac{d^2x}{dt^2} = -mg + kv^2$

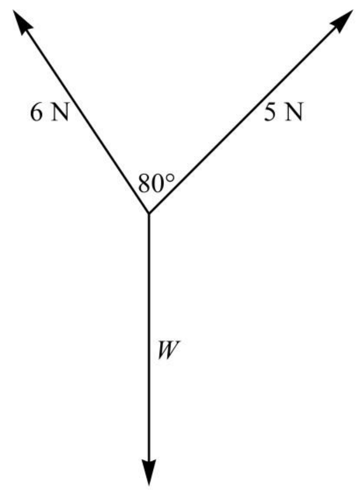
C. $\dfrac{dv^2}{dx} = 2g - 2\dfrac{kv^2}{m}$

D. $\dfrac{d(v^2/2)}{dx} = mg + kv^2$

E. $mv\dfrac{dv}{dx} = -mg - kv^2$

Question 20

Forces of magnitude 5 N, 6 N and W N act on a particle that is in equilibrium, as shown in the diagram below.



The magnitude of W, in newtons, can be found by evaluating

A. $\sqrt{5^2 + 6^2 - 2 \times 5 \times 6\cos(80°)}$

B. $5^2 + 6^2 - 2 \times 5 \times 6\cos(80°)$

C. $5^2 + 6^2 - 2 \times 5 \times 6\cos(100°)$

D. $\sqrt{5^2 + 6^2 - 2 \times 5 \times 6\cos(100°)}$

E. $\sin(80°) \times \dfrac{6}{\sin(140°)}$

Question 21

An ice hockey puck of mass $\frac{1}{5}$ kg, slides across a perfectly level, perfectly smooth ice rink with an initial speed of $10\ \text{ms}^{-1}$. A constant frictional force acts on the hockey puck, so that after travelling 9 m, the puck is travelling at $8\ \text{ms}^{-1}$. The magnitude of the frictional force, in newtons, is

A. $\frac{2}{45}$

B. $\frac{2}{9}$

C. $\frac{2}{5}$

D. 2

E. 10

Question 22

The velocity $v\ \text{ms}^{-1}$ of a body which is moving in a straight line, when it is x m from the origin, is given by $v = \arctan(x)$. The acceleration of the body in ms^{-2} is given by

A. $\sec^2(x)$

B. $\frac{\arctan(x)}{x^2+1}$

C. $\tan(x)\sec^2(x)$

D. $\frac{1}{x^2+1}$

E. $\frac{1}{2}\left(\frac{2\arcsin(x)^2}{\sqrt{1-x^2}\arccos(x)^3} + \frac{2\arcsin(x)}{\sqrt{1-x^2}\arccos(x)^2}\right)$

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
Unless otherwise specified an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g $\mathrm{m/s^2}$, where $\mathrm{g} = 9.8$.

Question 1 (11 marks)

A curve is defined by parametric equations

$$x = 2\sec(t) + 1$$
$$y = 3\tan(t) - 2$$

for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

a. Find the Cartesian equation of the curve. 2 marks

b. Find the values of t for which the gradient of the curve is 3. 2 marks

Consider the **different** relation $\frac{x^2}{9} + \frac{(y-2)^2}{4} = 1$.

c. Find the value(s) of y if the tangent line is vertical 2 marks

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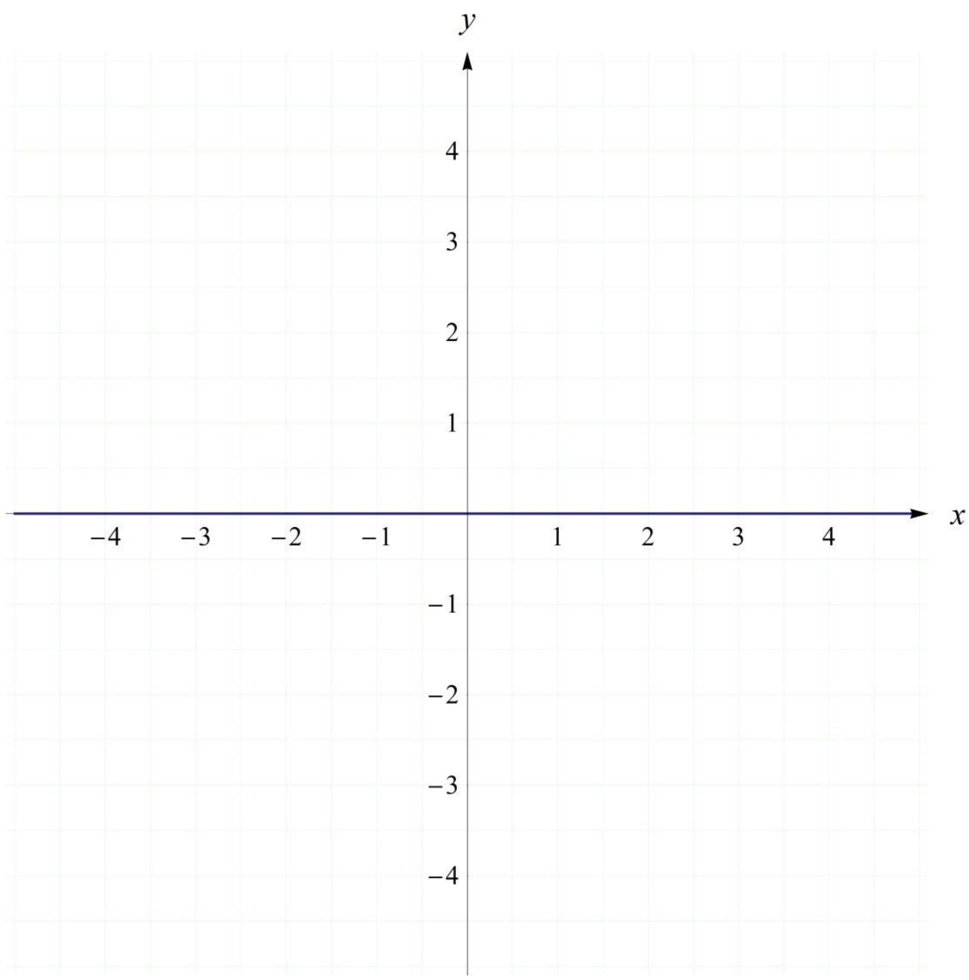
d. Sketch the graph of $\frac{x^2}{9} + \frac{(y-2)^2}{4} = 1$, showing the coordinates of any relevant vertices.

2 marks



The region in the first quadrant enclosed by the graph of $\frac{x^2}{9}+\frac{(y-2)^2}{4}=1$, the y −axis, and the lines $y=1$ and $y=3$ is rotated about the y −axis to form a solid of revolution.
Consider a **different** relation $\frac{x^2}{9}+\frac{(y-2)^2}{4}=1$.

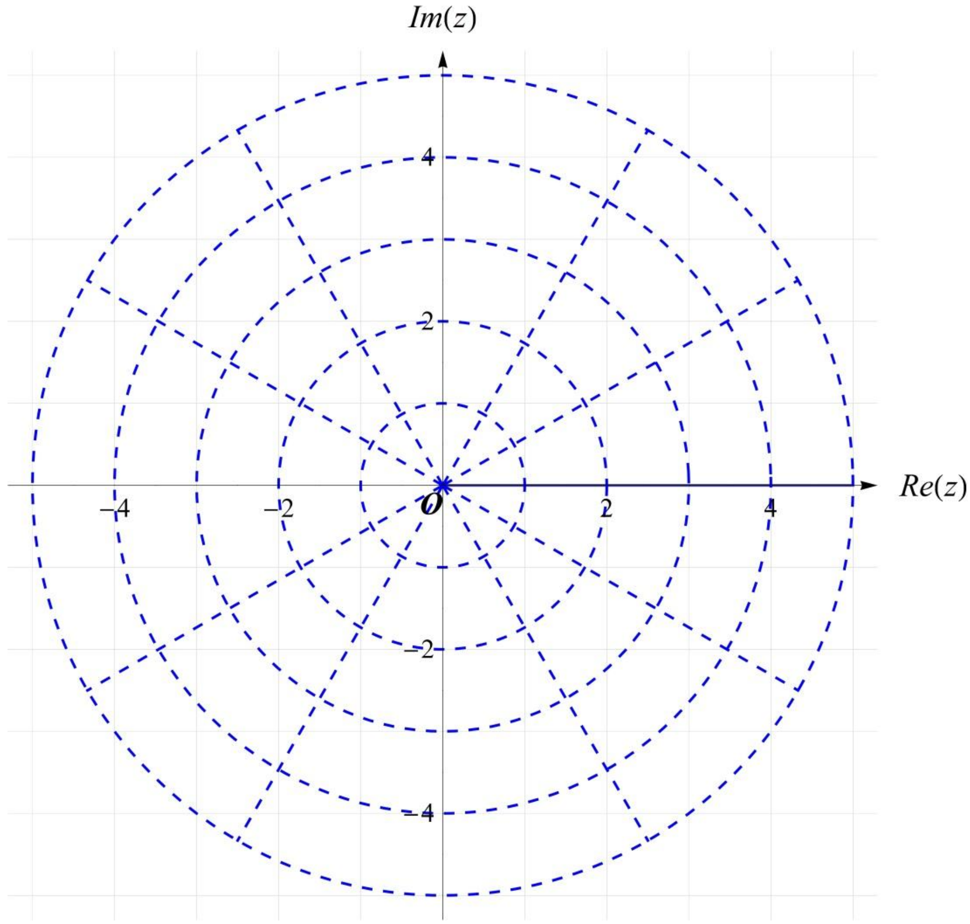
e. **i.** Write down a definite integral, in terms of y, that gives the volume of this solid of revolution.

2 marks

ii. Find the volume of this solid of revolution. 1 mark

Question 2 (12 marks)

a. On the Argand diagram below, sketch $\{z: |z| = 3, z \in \mathbb{C}\}$ **and** sketch $\{z: |z + \sqrt{3} - i| = |z - \sqrt{3} + i|, z \in \mathbb{C}\}$. 3 marks



b. Find all the elements of $\{z: |z| = 3, z \in \mathbb{C}\} \cap \{z: |z + \sqrt{3} - i| = |z - \sqrt{3} + i|, z \in \mathbb{C}\}$, expressing your answer(s) in the form $a + ib$. 2 marks

One of the roots of $z^3 + 27 = 0$ is $z = \frac{3}{2} + i\frac{3\sqrt{3}}{2}$

c. Write down the other roots in cartesian form. 2 marks

d. Plot and label all these roots on the Argand diagram provided in **part a.** 1 mark

e. Express $z^3 + 27$ as a product of linear factors in terms of z. 1 mark

f. On the diagram provided in **part a.**, shade the region defined by $\{z: |z| \leq 3, z \in \mathbb{C}\} \cap \{z: \text{Arg}(z) > \frac{\pi}{3}, z \in \mathbb{C}\}$ 1 mark

g. Find the area of the region shaded in **part e.** 2 marks

Question 3 (11 marks)

The number of individuals n (**measured in thousands**) in a large city that have been infected by a flu virus, t months since the beginning of the year is modelled by $\log_e(n) = 5 - 4e^{-6t/5},\ t \geq 0$.

a. Verify that $\log_e(n) = 5 - 4e^{-6t/5}$ satisfies the differential equation

$\frac{1}{n}\frac{dn}{dt} = \frac{6}{5}(5 - \log_e(n))$ 2 marks

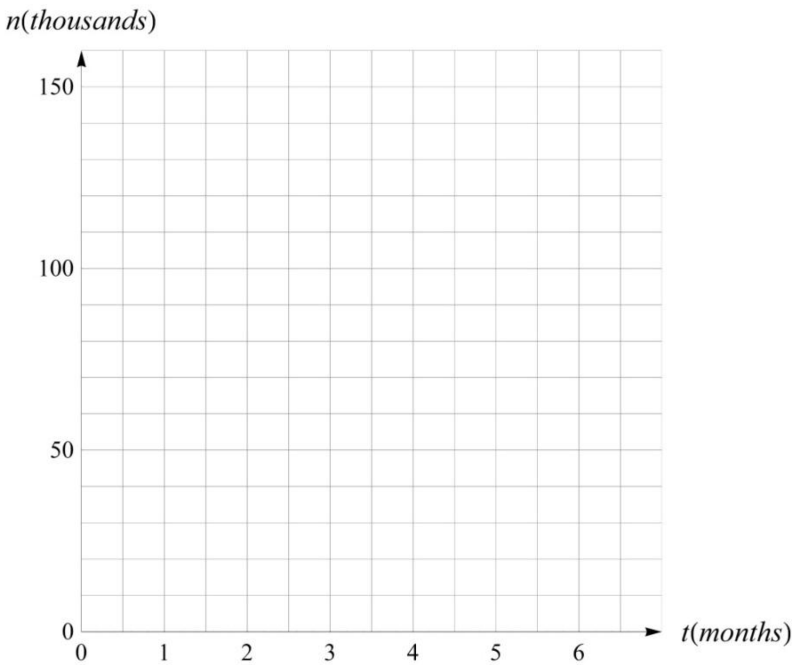
b. Find the initial number of infected individuals in the city. Express your answer correct to the nearest individual. 1 mark

c. Find the limiting number of individuals that would eventually become infected with the flu virus. Express your answer correct to the nearest individual. 2 marks

d. **i.** Show that $\frac{d^2n}{dt^2} = \frac{36}{25}\, n(4 - \log_e(n))(5 - \log_e(n))$. 2 marks

ii. The graph of n as a function t has a point of inflection. Find the exact coordinates of this point. 2 marks

e. Plot the graph of n as a function of t on the axes below for $0 \le t \le 6$ showing any relevant axis intercepts, points of inflexion and/or asymptotes. 2 marks



Question 4 (12 marks)

Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{m} = 18\mathbf{i} + 18\mathbf{j} + 0\mathbf{k}$.

a. Resolve **a** into two vector components, one is parallel to **m** and one is perpendicular to **m**. 2 marks

b. Find the acute angle θ between vectors **a** and **m**. 2 marks

c. Find the value of real number β such that $\mathbf{b} = 4\mathbf{i} + \beta\mathbf{j} - 4\mathbf{k}$ makes an angle $\frac{\pi}{4}$ with vector $\mathbf{m}$ where $\mathbf{b} \neq \mathbf{a}$. 2 marks

d. Find α if $\mathbf{m} = 6\mathbf{a} + \alpha\mathbf{b}$. 2 marks

Consider three new vectors $\mathbf{p}$, $\mathbf{r}$ and $\mathbf{q} = |\mathbf{r}|\mathbf{p} + |\mathbf{p}|\mathbf{r}$.

e. Using properties of the dot product, show that the cosine of the angle between vectors $\mathbf{p}$ and $\mathbf{q}$ is $\frac{|\mathbf{r}||\mathbf{p}|+\mathbf{r}\cdot\mathbf{p}}{|\mathbf{q}|}$

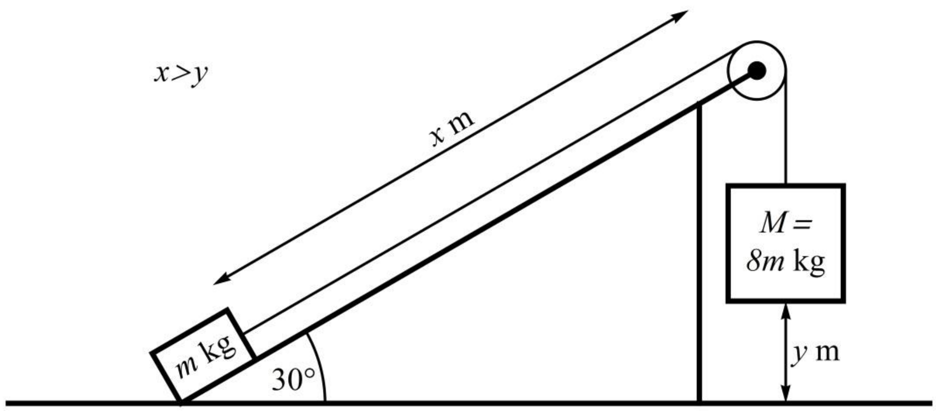
2 marks

f. Hence, show that vector $\mathbf{q} = |\mathbf{r}|\mathbf{p} + |\mathbf{p}|\mathbf{r}$ is the **angle bisector** of vectors $\mathbf{p}$ and $\mathbf{r}$. (Hint: Find the cosine of the angle between vectors $\mathbf{r}$ and $\mathbf{q}$).

2 marks

Question 5 (12 marks)

Two blocks of stone of mass m kg and $M = 8m$ kg are connected by a light, strong, inextensible rope. The rope passes over a smooth pulley at the top of a rough ramp elevated at $30°$ to the horizontal. The smaller mass m is x metres from the pulley, and the heavier mass M, is initially held in place with its bottom y metres above the ground, as shown in the figure below. You may assume that $x > y$.



The coefficient of friction between the ramp and the smaller block is $\frac{1}{\sqrt{3}}$. At time $t = 0$ seconds, the heavier block is released, and the system begins to move.

a. **Find** a pair of equations for the initial acceleration a ms^{-2} and rope tension τ newton, over the time interval before M reaches ground level. 3 marks

b. **Hence**, find the acceleration a up the plane (leave answer in terms of g if necessary). 1 mark

c. Find the speed V ms^{-1} of the heavier block M, just before it hits the ground (Express your answer in terms of y). 2 marks

d. Find the speed v ms^{-1} of the lighter block m, just before it reaches the pulley at the top of the ramp. 4 marks

e. If block m comes to rest instantaneously as it reaches the pulley, show that $x = \frac{16}{9}y$ 2 marks

End of Examination

SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.

This formula sheet is provided for your reference.

Specialist Mathematics Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$
cosine rule:	$c^2=a^2+b^2-2\,ab\sin C$

Coordinate geometry

ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

hyperbola: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Circular (trigonometric) functions

$\cos^2(x)+\sin^2(x)=1$

$\sin(2x)=2\sin(x)\cos(x)$

$1+\tan^2(x)=\sec^2(x)$

$\cot^2(x)+1=\operatorname{cosec}^2(x)$

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$

$\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$

$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$

$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$

$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

$\tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

function	sin^{-1}	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1,1]$	$[-1,1]$	$\mathbb{R}$
range	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0,\pi]$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + iy = r(\cos\theta + i\sin\theta) = r\text{ cis }\theta$

$|z| = \sqrt{x^2 + y^2} = r$

$-\pi < \text{Arg}(z) \le \pi$

$z_1 z_2 = r_1 r_2 \text{cis }(\theta_1 + \theta_2)$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} cis\ (\theta_1 - \theta_2)$

$z^n = r^n \text{cis }(n\theta)$ (de Moivre's theorem)

Calculus

$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$

$\int x^n \mathrm{d}x = \frac{1}{n+1}x^{n+1} + c, n \ne -1$

$\frac{\mathrm{d}}{\mathrm{d}x}(e^{ax}) = ae^{ax}$

$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$\frac{\mathrm{d}}{\mathrm{d}x}(\log_e(x)) = \frac{1}{x}$

$\int \frac{1}{x}\,\mathrm{d}x = \log_e|x| + c$

$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(ax)) = a\cos(ax)$

$\int \sin(ax)\,\mathrm{d}x = \frac{1}{a}\cos(ax) + c$

$\frac{\mathrm{d}}{\mathrm{d}x}(\cos(ax)) = -a\sin(ax)$

$\int \cos(ax)\,\mathrm{d}x = -\frac{1}{a}\sin(ax) + c$

$\frac{\mathrm{d}}{\mathrm{d}x}(\tan(ax)) = a\sec^2(ax)$

$\int \sec^2(ax)\,\mathrm{d}x = \frac{1}{a}\tan(ax) + c$

$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$

$\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = \sin^{-1}(ax) + c, a > 0$

$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}}$

$\int \frac{-1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \cos^{-1}(ax) + c, a > 0$

$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1}(x)) = \frac{1}{1+x^2}$

$\int \frac{a}{a^2+x^2}\,dx = sin^{-1}(ax) + c$

product rule: $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}}{\mathrm{d}x}(v) + v\frac{\mathrm{d}}{\mathrm{d}x}(u)$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Euler's method: $\frac{dy}{dx} = f(x), x_0 = a \text{ and } y_0 = b$
$\Rightarrow x_{n+1} = x_n + h,\ \ y_{n+1} = y_n + hf(x_n)$

acceleration: $a = \frac{d^2x}{dx^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at, s = ut + \frac{1}{2}at^2, v^2 = u^2 + 2as, s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$|\mathbf{r}| = \sqrt{x^2+y^2+z^2} = r$

$\mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2\cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$

$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$

Mechanics

momentum: $\mathbf{p} = m\mathbf{v}$

equation of motion: $\mathbf{R} = m\mathbf{a}$

friction: $F \le \mu N$

Multiple Choice Answer Sheet

Student Name:

Shade the letter that corresponds to each correct answer.

Question	A	B	C	D	E
Question 1	[ ]	[ ]	[ ]	[ ]	[ ]
Question 2	[ ]	[ ]	[ ]	[ ]	[ ]
Question 3	[ ]	[ ]	[ ]	[ ]	[ ]
Question 4	[ ]	[ ]	[ ]	[ ]	[ ]
Question 5	[ ]	[ ]	[ ]	[ ]	[ ]
Question 6	[ ]	[ ]	[ ]	[ ]	[ ]
Question 7	[ ]	[ ]	[ ]	[ ]	[ ]
Question 8	[ ]	[ ]	[ ]	[ ]	[ ]
Question 9	[ ]	[ ]	[ ]	[ ]	[ ]
Question 10	[ ]	[ ]	[ ]	[ ]	[ ]
Question 11	[ ]	[ ]	[ ]	[ ]	[ ]
Question 12	[ ]	[ ]	[ ]	[ ]	[ ]
Question 13	[ ]	[ ]	[ ]	[ ]	[ ]
Question 14	[ ]	[ ]	[ ]	[ ]	[ ]
Question 15	[ ]	[ ]	[ ]	[ ]	[ ]
Question 16	[ ]	[ ]	[ ]	[ ]	[ ]
Question 17	[ ]	[ ]	[ ]	[ ]	[ ]
Question 18	[ ]	[ ]	[ ]	[ ]	[ ]
Question 19	[ ]	[ ]	[ ]	[ ]	[ ]
Question 20	[ ]	[ ]	[ ]	[ ]	[ ]
Question 21	[ ]	[ ]	[ ]	[ ]	[ ]
Question 22	[ ]	[ ]	[ ]	[ ]	[ ]

Solution Pathway

Below are sample answers. Please consider the merit of alternative responses.

Specialist Mathematics Exam 2: SOLUTIONS

Section 1: Multiple-choice Answers

1. C	**2. E**	**3. A**	**4. D**	**5. B**
6. D	**7. E**	**8. B**	**9. B**	**10. B**
11. B	**12. A**	**13. A**	**14. C**	**15. C**
16. E	**17. E**	**18. D**	**19. E**	**20. D**
21. C	**22. B**			

Section 1: Multiple-choice Solutions

Question 1

$Domain\ arccos = [-1,1] \Rightarrow -1 \le 2x \le 1 \Rightarrow x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ $\textbf{Answer: C}$

Question 2

$\text{cosec}^2(t) - \cot^2(t) = 1 \Rightarrow \frac{(y+1)^2}{4} - \frac{(x-1)^2}{9} = 1$ $\textbf{Answer: E}$

Question 3

The denominator $ax^2 + bx + c$ has intercepts at $x = -2$ and $x = 4$ and a t.p. of -18 at $x = 1$

$\Rightarrow a(x+2)(x-4)|_{x=1} = -18 \Rightarrow a = 2$

Finally, expanding gives $ax^2 + bx + c = 2x^2 - 4x - 16 \Rightarrow a = 2, b = -4\ and\ c = -16$ $\textbf{Answer: A}$

Question 4

$\frac{1}{\tan\left(\frac{\pi}{2} - x\right)} = \frac{1}{\cot(x)} \neq \cot(x)$ $\textbf{Answer: D}$

Question 5

A region of the complex plane inside the circle of radius a centred at the origin is

$|z| < a \Rightarrow \sqrt{z\bar{z}} < a \Rightarrow z\bar{z} < a^2$ **Answer: B**

Question 6

$\text{Im}\left(\text{cis}\left(\frac{5\pi}{6}\right) + i\right) = \text{Im}\left(\cos\left(\frac{5\pi}{6}\right) + i\left(\sin\left(\frac{5\pi}{6}\right) + 1\right)\right) = \frac{3}{2}$ **Answer: D**

Question 7

$\frac{(\bar{z})^2}{z} = \frac{\left(\overline{r\text{cis}(\theta)}\right)^2}{r\text{cis}(\theta)} = \frac{(r\text{cis}(-\theta))^2}{r\text{cis}(\theta)} = \frac{r^2\text{cis}(-2\theta)}{r\text{cis}(\theta)} = r\text{cis}(-3\theta)$ **Answer: E**

Question 8

$z^2 = 1 - i \Rightarrow r^2\text{cis}(2\theta) = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4} + 2k\pi\right), k = 0,1$ Solving for θ

$\theta = \frac{1}{2}\left(-\frac{\pi}{4} + 2k\pi\right), k = 0,1 \Rightarrow \theta \in \left\{-\frac{\pi}{8}, \frac{7\pi}{8}\right\}$ **Answer: B**

Question 9

$e^{-\cot(x)} = \frac{1}{e^{\cot(x)}} \Rightarrow u = \cot(x), \frac{du}{dx} = -\text{cosec}^2(x) = -(\cot^2(x) + 1)$

$when\ x = \frac{\pi}{4}, u = 1\ and\ when\ x = \frac{\pi}{2}, u = 0$

$\therefore \int_{\pi/4}^{\pi/2} e^{-\cot(x)}(\cot^2(x) + 1)\,dx = -\int_1^0 \frac{1}{e^u}\,du = \int_0^1 \frac{1}{e^u}\,du$ **Answer: B**

Question 10

$y = x^{\frac{5}{3}} \Rightarrow x = y^{\frac{3}{5}}$

$V = \pi\int_0^2 x^2(y)\,dy = \pi\int_0^2 y^{\frac{6}{5}}\,dy = \pi\frac{20}{11}2^{1/5}$ **Answer: B**

Question 11

Exact value $y(0.2) = 1 + \int_0^{0.2} \tan(x)\ \mathrm{d}x = 1 - \log_e(\cos(0.2)) \approx 1.020$

Euler's estimate $y(0.2) = 1 + 0.1\tan(0) + 0.1\tan(0.1) \approx 1.0100$ **Answer: B**

Question 12

If $f(x,y) = x/y$ then $f(-x,y) = -f(x,y)$ and $f(x,-y) = -f(x,y) \Rightarrow$ **Answer: A**

Question 13

$\ddot{x}dx = \frac{dv}{dt}\frac{dx}{dt}dt = \frac{dx}{dt}\frac{dv}{dt}dt = vdv$ **Answer: A**

Question 14

By inspection $\mathbf{a} + \mathbf{b} = \mathbf{c} \Rightarrow$ **Answer: C**

Question 15

If (3,0,4) , $Q(0,-2,1)$ then $\overrightarrow{PQ} = -3\mathbf{i} - 2\mathbf{j} - 3\mathbf{k} \Rightarrow |PQ| = \sqrt{22}$ **Answer: C**

Question 16

$\frac{dm}{dt} = 5 \times 8 - \frac{m}{100 + (8-10) \times t}$ **Answer: E**

Question 17

Clearly, $\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$ **Answer: E**

Question 18

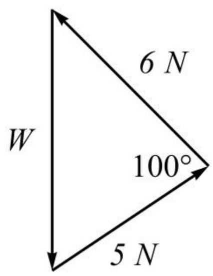
Weight is in same direction as acceleration, and normal reaction force is in the opposite direction to acceleration. **Answer: D**

Question 19

Up is positive, weight and air resistance are down, so they are negative in Newton's Law.

$\Rightarrow ma = -mg - kv^2, but\ a = v\frac{dv}{dx} \Rightarrow$ **Answer: E**

Question 20



Application of the cosine rule to the triangle above gives

$W = \sqrt{5^2 + 6^2 - 2 \times 5 \times 6\cos(100°)}$ **Answer**: **D**

Question 21

$a = \dfrac{v^2 - u^2}{2 \times s} = \dfrac{8^2 - 10^2}{18} = -2\text{ms}^{-2}$

magnitude of the frictional force $= m|a| = \dfrac{2}{5}$ **Answer**: **C**

Question 22

acceleration $= v\dfrac{dv}{dx} = \dfrac{\arctan(x)}{x^2 + 1}$ **Answer**: **B**

Section 2: Extended Answer Solutions

Question 1 (11 marks)

a. (2 marks)

$\sec(t) = \dfrac{x-1}{2}$ *and* $\tan(t) = \dfrac{y+2}{3}$ (M1)

$\sec^2(t) - \tan^2(t) = 1 \Rightarrow \dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{9} = 1$ (A1)

b. (2 marks)

$\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{3}{2\sin(t)}$ (M1)

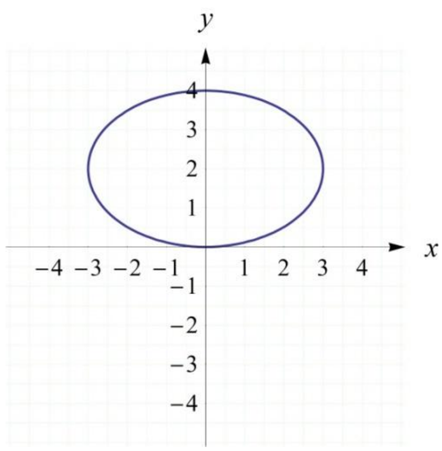
$\dfrac{dy}{dx} = \dfrac{3}{2\sin(t)} = 3 \Rightarrow t = \dfrac{\pi}{6}$ (A1)

c. (2 marks)

$\frac{dy}{dx} = \frac{4x}{9(y-2)}$ (A1)

$\frac{dy}{dx} = \infty \Rightarrow y = 2$ (A1)

d.



Shape (A1)

Correct Vertices (A1)

e. i. (2 marks)

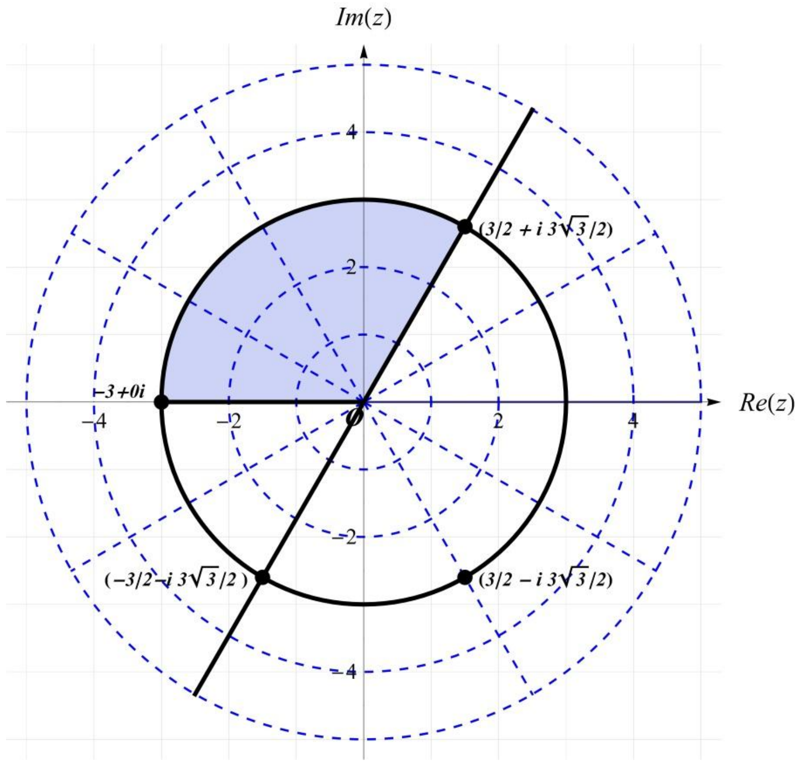
$$V = \pi \int_1^3 9(1-(y-2)^2/4)\,dy$$ Integrand(A1) Limits (A1)

ii. (1 mark)

$\frac{33\pi}{2}$ (A1)

Question 2 (12 marks)

a. (3 marks)



Circle − correct centre and radius (A1)

Straight Line Shape(A1) Slope (A1)

b. (2 marks)

$\frac{3}{2} + i\frac{3\sqrt{3}}{2}$ and $-\frac{3}{2} - i\frac{3\sqrt{3}}{2}$ (A2)

c. (2 marks)

-3 and $\frac{3}{2} - i\frac{3\sqrt{3}}{2}$ (A2)

d. (1 mark)
Totally correct (A1)

e. (1 mark)
$(z+3)\left(z-\frac{3}{2}-i\frac{3\sqrt{3}}{2}\right)\left(z-\frac{3}{2}+i\frac{3\sqrt{3}}{2}\right)$ (A1)

f. (1 mark)
Totally correct (A1)

h. (2 marks)
$\text{Area} = \frac{\left(\frac{2\pi}{3}\right)}{2\pi} \times \pi \times 3^2$ (M1)
$\text{Area} = 3\pi$ square units (A1)

Question 3 (11 marks)

a. (2 marks)
Differentiating both sides with respect to time and using the chain rule
$\frac{\mathrm{d}}{\mathrm{d}t}\log_{\mathrm{e}}(n) = \frac{\mathrm{d}}{\mathrm{d}t}\left(5-4e^{-6t/5}\right) \Rightarrow \frac{1}{n}\frac{dn}{dt} = \frac{24}{5}e^{-6t/5}$ (M1)
but $\frac{24}{5}e^{-6t/5} = \frac{6}{5}(5-\log_{\mathrm{e}}(n)) \Rightarrow \frac{1}{n}\frac{dn}{dt} = \frac{6}{5}(5-\log_{\mathrm{e}}(n))$ (M1)

b. (1 mark)
$t=0 \Rightarrow \log_{\mathrm{e}}(n(0)) = 5-4e^{-6\times 0/5} \Rightarrow n(0)=e \Rightarrow 1000\times e \approx 2718$ individuals (A1)

c. (2 marks)
$\lim_{t\to\infty}\log_{\mathrm{e}}(n) = \lim_{t\to\infty}\left(5-4e^{-6t/5}\right) = 5$ (M1)
Number of individuals eventually infected $1000\times e^5 \approx 148413$ (A1)

d. **i.** (2 marks)
Differentiating both sides of $\frac{dn}{dt} = \frac{6}{5}n(5-\log_{\mathrm{e}}(n))$ with respect to t and using the chain rule gives
$\frac{d^2n}{dt^2} = \frac{d}{dn}\left(\frac{6}{5}n(5-\log_{\mathrm{e}}(n))\right)\frac{dn}{dt} = \frac{6}{5}(4-\log_{\mathrm{e}}(n))\frac{dn}{dt}$ (M1)
Substituting $\frac{dn}{dt} = \frac{6}{5}n(5-\log_{\mathrm{e}}(n))$ in the preceding equation gives the required result (M1)

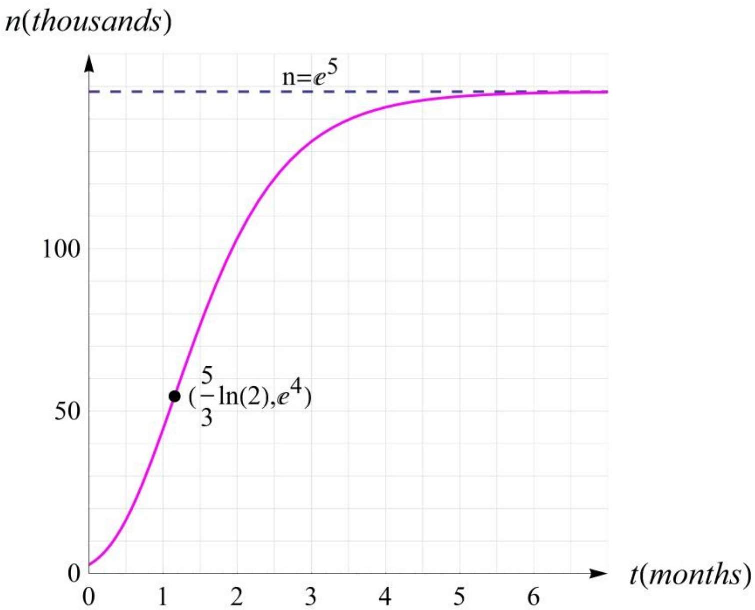
ii. (2 marks)
At a point of inflection, $\frac{d^2n}{dt^2} = \frac{6}{5}n(4-\log_{\mathrm{e}}(n))(5-\log_{\mathrm{e}}(n))$ changes sign

0	$< n <$	e^4	$< n <$	e^5
$\frac{6}{5}n$		$(4-\log_{\mathrm{e}}(n))$		$(5-\log_{\mathrm{e}}(n))$
$\frac{d^2n}{dt^2}=0$	$\frac{d^2n}{dt^2}>0$	$\frac{d^2n}{dt^2}=0$	$\frac{d^2n}{dt^2}<0$	$\frac{d^2n}{dt^2}=0$

Or using a sign diagram shows that $\frac{d^2n}{dt^2}$ changes sign at $n=e^4$ (M1)
Solving $\log_{\mathrm{e}}(n) = 5-4e^{-6t/5}$ for t when $n=e^4$ gives $t=\frac{5}{3}\log_{\mathrm{e}}(2)$

$\therefore (t, n) = \left(\frac{5}{3}\log_e(2), e^4\right)$ (A1)

e. (2 marks)



asymptote (A1)
shape (A1)

Question 4 (12 marks)

a. (2 marks)

Vector component parallel to $\mathbf{m}$ is $\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$ (A1)

Vector component perpendicular to $\mathbf{m}$ is $-\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}$ (A1)

b. (2 marks)

$\theta = \arccos\left(\frac{\mathbf{a}\cdot\mathbf{m}}{\sqrt{\mathbf{a}\cdot\mathbf{a}}\sqrt{\mathbf{m}\cdot\mathbf{m}}}\right)$ (M1)

$\theta = \frac{\pi}{4}$ (A1)

c. (2 marks)

$\frac{\mathbf{b}\cdot\mathbf{m}}{\sqrt{\mathbf{a}\cdot\mathbf{a}}\sqrt{\mathbf{m}\cdot\mathbf{m}}} = \frac{4+\beta}{\sqrt{2}\sqrt{32+\beta^2}} = \cos\left(\frac{\pi}{4}\right)$ (M1)

Solving for β gives $\beta = 2$ (A1)

d. (2 marks)

$18\mathbf{i} + 18\mathbf{j} + 0\mathbf{k} = 6(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) + \alpha(4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$ (M1)

Equating components on both sides gives $\alpha = 3$.

e. (2 marks)

cosine of the angle between vectors $\mathbf{p}$ and $\mathbf{q}$ is

$\frac{\mathbf{p}\cdot(|\mathbf{r}|\mathbf{p}+|\mathbf{p}|\mathbf{r})}{|\mathbf{p}||(|\mathbf{r}|\mathbf{p}+|\mathbf{p}|\mathbf{r})|}$ (M1)

$= \frac{(|\mathbf{r}|\mathbf{p}\cdot\mathbf{p}+|\mathbf{p}|\mathbf{p}\cdot\mathbf{r})}{|\mathbf{p}||(|\mathbf{r}|\mathbf{p}+|\mathbf{p}|\mathbf{r})|} = \frac{(|\mathbf{r}||\mathbf{p}|+\mathbf{p}\cdot\mathbf{r})}{|(|\mathbf{r}|\mathbf{p}+|\mathbf{p}|\mathbf{r})|}$ (A1)

f. (2 marks)

cosine of the angle between vectors **r** and **q** is

$\frac{\mathbf{r}\cdot(|\mathbf{r}|\mathbf{p}+|\mathbf{p}|\mathbf{r})}{|\mathbf{r}||(|\mathbf{r}|\mathbf{p}+|\mathbf{p}|\mathbf{r})|} = \frac{(|\mathbf{r}|\mathbf{r}\cdot\mathbf{p}+|\mathbf{p}|\mathbf{r}\cdot\mathbf{r})}{|\mathbf{r}||(|\mathbf{r}|\mathbf{p}+|\mathbf{p}|\mathbf{r})|} = \frac{(|\mathbf{r}||\mathbf{p}|+\mathbf{p}\cdot\mathbf{r})}{|(|\mathbf{r}|\mathbf{p}+|\mathbf{p}|\mathbf{r})|}$ (M1)

cosine of the angle between vectors **p** and **q**= cosine of the angle between vectors **r** and **q**

∴ **q** bisects the angle between vectors **p** and **r** (A1)

Question 5 (12 marks)

a. (3 marks)

Applying Newtons 2nd Law to the smaller mass, taking positive in the direction up the plane gives

$\tau - mg\sin(30°) - \frac{mg}{\sqrt{3}}\cos(30°) = m\,a$ (M1)

Simplifying

$\tau - mg = m\,a$ ① (M1)

Applying Newtons 2nd Law to the heavier mass, taking positive in the downward direction gives

$8mg - \tau = 8ma$ ② (A1)

b. (1 mark)

Solving equations ① and ② gives

$a = \frac{7}{9}\text{g ms}^{-2}$ (A1)

c. (2 marks)

Attempt to use the constant acceleration formula $V^2 = u^2 + 2\,a\,y$ (M1)

$u = 0, a = \frac{7}{9}\text{g} \Rightarrow \text{V} = \frac{\sqrt{14gy}}{3}$ ms^{-1} downwards (A1)

d. Find the speed v ms^{-1} of the lighter block m, just before it reaches the pulley at the top of the ramp (4 marks)

From equation ①, $a = -\text{g}$ after the heavier block reaches ground level (A1)

Attempt to use the constant acceleration formula $v^2 = V^2 + 2a(x-y)$ (M1)

Attempt to substitute $a = -\text{g}$ and $\text{V} = \frac{\sqrt{14gy}}{3}$ in preceding equation (M1)

Simplifying

$v = \sqrt{\frac{14\text{g}y}{9} - 2\,\text{g}\,(x-y)}$ (A1)

e. (2 marks)

If block comes to rest at top of ramp, $v = 0$ in preceding equation (M1)

Solving

$\frac{14\text{g}y}{9} - 2\,\text{g}\,(x-y) = 0 \Rightarrow x = \frac{16y}{9}$ (M1)

