E

Free Exam
for 2006-15 VCE study design

Engage
Education
Foundation

Units 3 and 4 Specialist Maths: Exam 1

Practice Exam Solutions

Stop!

Don’t look at these solutions until you have attempted the exam.

Any questions?

Check the Engage website for updated solutions, then email practiceexams@ee.org.au.

Question 1a

If A, B and C are collinear, then $\overrightarrow{OA} + t(\overrightarrow{AB}) = \overrightarrow{OC}$ for some value of t.

$$\Rightarrow\ \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k} + t\boldsymbol{i} - 4t\boldsymbol{j} + 5\boldsymbol{k} = 3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$$
$$\Rightarrow 1 + t = 3 \text{ and } 3 - 4t = 1 \text{ and } -2 - 5t = 2$$

However, there is no value of t for which all three of these equations are true, and so A, B and C are not collinear.

Question 1b

Let $\hat{\boldsymbol{a}}$ be a unit vector in the direction of $\overrightarrow{OA}$, and $\hat{\boldsymbol{c}}$ a unit vector in the direction of $\overrightarrow{OC}$.
Thus, $\hat{\boldsymbol{a}} = \frac{1}{\sqrt{14}}(\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k})$ and $\hat{\boldsymbol{c}} = \frac{1}{\sqrt{14}}(3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$.

A vector which bisects the angle between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\boldsymbol{d} = \hat{\boldsymbol{a}} + \hat{\boldsymbol{c}} = \frac{1}{\sqrt{14}}(4\boldsymbol{i} + 4\boldsymbol{j})$ (or any other multiple of $\boldsymbol{i} + \boldsymbol{j}$). Hence any point D in the form $k(1, 1, 0), k \in \mathbb{R}$ will satisfy the question.

Question 1c

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{\|\overrightarrow{OA}\|\|\overrightarrow{OB}\|}\right)$$
$$= \cos^{-1}\left(\frac{-7}{14}\right)$$
$$= \frac{2\pi}{3}$$

Question 1d

$$\overrightarrow{OC}_{\text{perpendicular to } \overrightarrow{OB}} = \overrightarrow{OC} - \left(\overrightarrow{OC} \cdot \widehat{\overrightarrow{OB}}\right)\widehat{\overrightarrow{OB}}$$
$$= [3 \quad 1 \quad 2] - \frac{11}{14}[2 \quad -1 \quad 3]$$
$$= \frac{10}{7}\boldsymbol{i} + \frac{25}{14}\boldsymbol{j} - \frac{5}{14}\boldsymbol{k}$$

Question 2

Find the gradient of the tangent to the graph at the specified point:

$$\frac{d}{dx}(3x^2y + 9y^2 - 3xy) = \frac{d}{dx}(24x)$$
$$6xy + 3x^2\frac{dy}{dx} + 18y\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 24$$
$$\frac{dy}{dx}(18y + 3x^2 - 3x) = 24 - 6xy + 3y$$
$$\frac{dy}{dx} = \frac{24 - 6xy + 3y}{18y + 3x^2 - 3x}$$

At (3, 2), $\frac{dy}{dx} = \frac{24 - 6(3)(2) + 3(2)}{18(2) + 3(3^2) - 3(3)}$
$$= \frac{-6}{54} = \frac{-1}{9}$$

Hence the gradient of the normal = $\frac{-1}{\left(\frac{-1}{9}\right)} = 9$

Find the equation of the normal, given that it is in the form $y = 9x + c$ and passes through (3, 2):

$2 = 9(3) + c$
$c = -25$
$\Rightarrow y = 9x - 25$

Question 3a

$$
\begin{aligned}
|i\bar{z}-z| &= \left|\text{cis}\left(\frac{\pi}{2}\right)\text{cis}(-\theta)-\text{cis}(\theta)\right| \\
&= \left|\text{cis}\left(\frac{\pi}{2}-\theta\right)-\text{cis}(\theta)\right| \\
&= \left|\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}\right)-\cos(\theta)-i\sin(\theta)\right| \\
&= |\sin(\theta)-\cos(\theta)+(\cos(\theta)-\sin(\theta))i| \\
&= \sqrt{(\sin(\theta)-\cos(\theta))^2+(\cos(\theta)-\sin(\theta))^2} \\
&= \sqrt{2-4\sin(\theta)\cos(\theta)} \\
&= \sqrt{2-2\sin(2\theta)}
\end{aligned}
$$

Question 3b

Note: $\theta \in (-\pi, \pi] \Rightarrow 2\theta \in (-2\pi, 2\pi]$

$\text{Arg}(i\bar{z}-z)$ is undefined when $|i\bar{z}-z| = 0$

$$
\begin{aligned}
&\Rightarrow 2-2\sin(2\theta)=0 \\
&\sin(2\theta)=1 \\
&\Rightarrow 2\theta = -\frac{3\pi}{2},\frac{\pi}{2} \\
&\theta = -\frac{3\pi}{4},\frac{\pi}{4}
\end{aligned}
$$

Since domain is in the form $\theta \in (-\alpha, \alpha)$,

$$\alpha = \frac{\pi}{4}$$

Question 3c

$$
\begin{aligned}
\text{Arg}(i\bar{z}-z) &= \text{Arg}(\sin(\theta)-\cos(\theta)+(\cos(\theta)-\sin(\theta))i) \\
&= \tan^{-1}\left(\frac{\cos(\theta)-\sin(\theta)}{\sin(\theta)-\cos(\theta)}\right) \\
&= \tan^{-1}(-1) \\
&= \frac{3\pi}{4}
\end{aligned}
$$

Note: It is possible to ascertain by graphical means that for $\theta \in \left(-\frac{\pi}{4},\frac{\pi}{4}\right)$, $(i\bar{z}-z)$ lies in the 2nd quadrant, not the 4th quadrant.

Question 4

$$\int_3^7 \frac{3x^2 + 2x^3 - 12x}{\sqrt{x-3}}\,dx = \int_0^4 \frac{3(u+3)^2 + 2(u+3)^3 - 12(u+3)}{\sqrt{u}}\,du \text{ where } u = x - 3$$

$$= \int_0^4 \frac{3(u+3)^2 + 2(u+3)^3 - 12(u+3)}{\sqrt{u}}\,du$$

$$= \int_0^4 \frac{(3u^2 + 18u + 27) + (2u^3 + 18u^2 + 54u + 54) - (12u + 36)}{\sqrt{u}}\,du$$

$$= \int_0^4 \frac{2u^3 + 21u^2 + 60u + 45}{\sqrt{u}}\,du$$

$$= \int_0^4 2u^{\frac{5}{2}} + 21u^{\frac{3}{2}} + 60u^{\frac{1}{2}} + 45u^{-\frac{1}{2}}\,du$$

$$= [2\left(\frac{2}{7}\right)u^{\frac{7}{2}} + 21\left(\frac{2}{5}\right)u^{\frac{5}{2}} + 60\left(\frac{2}{3}\right)u^{\frac{3}{2}} + 45(2)u^{\frac{1}{2}}]_0^4$$

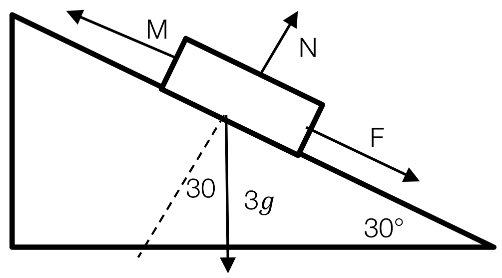
$$= \left[\frac{4u^{\frac{7}{2}}}{7} + \frac{42u^{\frac{5}{2}}}{5} + 40u^{\frac{3}{2}} + 90u^{\frac{1}{2}}\right]_0^4$$

$$= \frac{4}{7}(2^7) + \frac{42}{5}(2^5) + 40(2^3) + 90(2)$$

$$= \frac{512}{7} + \frac{21 \cdot 64}{5} + 320 + 180$$

$$= 500 + \frac{512}{7} + \frac{1344}{5}$$

Question 5a



Question 5b

In this question, redefine $\boldsymbol{i}$ and $\boldsymbol{j}$ in line with the hill. Splitting equations of movement we have:

$N - 3g\cos(30°) = 0$

$N = 3g\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}g}{2}$

Similarly, in the other direction we have:

$M - \mu N - 3g\sin(30°) = 0$

$M = \mu N + 3g\sin(30°)$

$M = \frac{1}{\sqrt{3}} \cdot \left(\frac{3\sqrt{3}g}{2}\right) + 3g\left(\frac{1}{2}\right)$

$M = 3g = 29.4\text{N}$

Question 5c

Movement in $\boldsymbol{i}$ direction would then be:

$3g - 3g\sin(30°) = ma$

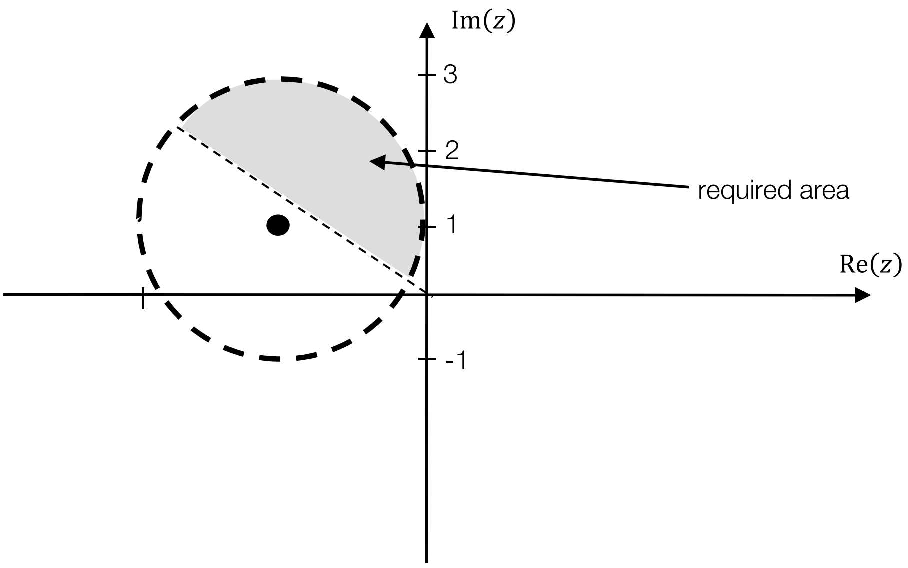
$3g - \frac{3g}{2} = 3a$

$a = \frac{g}{2} = 4.9\text{m/s}^2$

Question 6

$$V = \pi \int_0^2 y^2 dx$$
$$= \pi \int_0^2 \frac{(x+3)^2}{x^2+1} dx$$
$$= \pi \int_0^2 (\frac{x^2+1}{x^2+1} + \frac{6x}{x^2+1} + \frac{8}{x^2+1}) dx$$
$$= \pi \left[\int_0^2 1\, dx + 3 \int_0^2 \frac{2x}{x^2+1} dx + 8 \int_0^2 \frac{1}{x^2+1} dx \right]$$
$$= \pi \left[[x]_0^2 + 3 \int_1^5 \frac{1}{u} du + 8[\tan^{-1}(x)]_0^2 \right] \text{ where } u = x^2+1$$
$$= \pi[2 + 3[\ln|u|]_1^5 + 8\tan^{-1}(2)]$$
$$= \pi(2 + 3\ln(5) + 8\tan^{-1}(2))$$

Question 7



Question 8

The two particles will collide when they are in the same position at the same time.

Equating $\boldsymbol{i}$ components:

$$12 - 2t = 5t - t^2$$
$$t^2 - 7t + 12 = 0$$
$$(t-3)(t-4) = 0$$
$$\Rightarrow t = 3 \text{ or } t = 4$$

Equating $\boldsymbol{j}$ components:

$$6t - 2 = t^2 + 6$$
$$t^2 - 6t + 8 = 0$$
$$(t-2)(t-4) = 0$$
$$\Rightarrow t = 2 \text{ or } t = 4$$

Hence the particles collide at $t = 4$ seconds.