itute.com

Online & home tutors Registered business name: itute ABN: 96 297 924 083

Specialist Mathematics

2015

Trial Examination 2
(2 hours)

SECTION 1 Multiple-choice questions

Instructions for Section 1

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1 $(1,1)$ is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The line which is **NOT** an asymptote of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

A. $bx + ay = 0$

B. $x - \sqrt{1-a^2}\, y = 0$

C. $\sqrt{1+b^2}\, x - y = 0$

D. $x - \sqrt{a^2 - 1}\, y = 0$

E. $\sqrt{b^2+1}\, x + y = 0$

Question 2 The range of function f with the rule $f(x) = \cos^{-1}(ax) - \frac{\pi}{2}$ is $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

For $a > 0$, the domain of f is

A. $\left[-\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right]$

B. $\left[-\frac{1}{\sqrt{2}\,a}, \frac{1}{\sqrt{2}\,a}\right]$

C. $\left[-\frac{\sqrt{3}\,a}{2}, \frac{\sqrt{3}\,a}{2}\right]$

D. $\left[-\frac{\sqrt{3}}{2a}, \frac{\sqrt{3}}{2a}\right]$

E. $\left[\frac{\sqrt{3}\,a}{2}, -\frac{\sqrt{3}\,a}{2}\right]$

Question 3 For $b \in R^+$, $\left\{z : \text{Arg}(z) = \frac{\pi}{3}\right\} \cap \left\{z : |z - ib\sqrt{3}| = |z - b|\right\}$ is

A. $\left\{\frac{b}{2} + \frac{ib\sqrt{3}}{2}\right\}$

B. $\left(\frac{b\sqrt{3}}{2}, \frac{b}{2}\right)$

C. $\left\{\frac{b\sqrt{3}}{2} + \frac{ib}{2}\right\}$

D. $\left(\frac{\sqrt{3}}{2}, \frac{i}{2}\right)$

E. $\left\{\frac{1}{2} - \frac{i\sqrt{3}}{2}\right\}$

Question 4 A possible factor of $z^5 + i$ is

A. $z + 1$

B. $z - i$

C. $z - 1$

D. $z^4 - iz^3 - z^2 + iz + 1$

E. $z^4 - iz^3 + z^2 - iz + 1$

Question 5 Given $f : \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \to R$, $f(x) = \sin x$,

A. $f^{-1}(x) = \sin^{-1} x + \frac{\pi}{2}$

B. $f^{-1}(x) = \sin^{-1} x$

C. $f^{-1}(x) = \cos^{-1} x + \frac{\pi}{2}$

D. $f^{-1}(x) = \cos^{-1} x$

E. $f^{-1}(x) = \sin^{-1} x - \frac{\pi}{2}$

Question 6 For non-zero $a, b \in R$, $\frac{d}{dx}\left[\cot^2\left(\frac{a+2x}{b\sqrt{x}+1}\right)-\operatorname{cosec}^2\left(\frac{a+2x}{b\sqrt{x}+1}\right)\right]=$

A. 1

B. $\frac{2}{b}\sin\left(\frac{a+2x}{b\sqrt{x}+1}\right)$

C. 0

D. $\frac{2}{b}\cos\left(\frac{a+2x}{b\sqrt{x}+1}\right)$

E. -1

Question 7 Given $\frac{z+1}{z+i}=i$, $\operatorname{Arg}(z)=$

A. 0.79

B. 2.4

C. -0.79

D. -0.75π

E. -0.25π

Question 8 For $x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, the exact solution(s) to $(\sin x+1)\left(\tan x+\frac{3}{2}\right)=0$ is/are

A. $-\frac{\pi}{2}, \frac{3\pi}{2}, \tan^{-1}\left(-\frac{3}{2}\right)$

B. $\frac{3\pi}{2}, \tan^{-1}\left(-\frac{3}{2}\right)$

C. $-\frac{\pi}{2}, \tan^{-1}\left(-\frac{3}{2}\right)$

D. $-\frac{\pi}{2}, \frac{3\pi}{2}$

E. $\tan^{-1}\left(-\frac{3}{2}\right)$

Question 9 For $a, b, \alpha, \beta \in R^+$, the area of the smallest rectangle which can cover the graph of $\frac{(x+\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1$ is

A. $(a-\alpha)(b+\beta)$

B. $4(a+\alpha)(b-\beta)$

C. $4ab$

D. $4\alpha\beta$

E. $\alpha\beta ab$

Question 10 The line $y=mx$ is a tangent to the curve $y=\tan^{-1}x-\frac{\pi}{4}$.
The value of m is closest to

A. 0.08

B. 0.12

C. 0.16

D. 0.20

E. 0.24

Question 11 A vector makes the same angle θ with three mutually perpendicular vectors $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$.
A possible value of θ is

A. $\frac{\pi}{4}$

B. $\frac{3\pi}{4}$

C. $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

D. $\sin^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

E. $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Question 12 Vectors $\underset{\sim}{a} = \underset{\sim}{i} + \sqrt{2}\,\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = -\sqrt{2}\,\underset{\sim}{i} + 2\,\underset{\sim}{j} + \sqrt{2}\,\underset{\sim}{k}$ are on the same plane. Another possible vector on the plane is

A. $\frac{1}{2}\underset{\sim}{j}$

B. $\frac{1}{2}\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{k}$

C. $\frac{1}{2}\underset{\sim}{i} + \frac{1}{\sqrt{2}}\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}$

D. $-\frac{1}{2}\underset{\sim}{i} + \frac{1}{\sqrt{2}}\underset{\sim}{j}$

E. $-\frac{1}{\sqrt{2}}\underset{\sim}{i}$

Question 13 Given $\overrightarrow{AB} = 4\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\overrightarrow{AC} = -\underset{\sim}{j} + 12\underset{\sim}{k}$, the shortest distance from point C to vector $\overrightarrow{AB}$ is closest to

A. 11.5

B. 12.0

C. 12.5

D. 13.0

E. 13.5

Question 14 Let A be the area of the region enclosed by the x-axis, the line $x = a$ and the curve $y = \tan^{-1}\left(\frac{x}{a}\right)$, where $a \in R^{+}$. Which one of the following statements is correct?

A. $A = \int_{0}^{\frac{\pi}{4}} \tan^{-1}\left(\frac{x}{a}\right)dx$

B. $A = a\left(\frac{\pi}{4} - \int_{0}^{\frac{\pi}{4}} \tan x\, dx\right)$

C. $A = \frac{\pi}{4} - a\int_{0}^{\frac{\pi}{4}} \tan x\, dx$

D. $A = \frac{a\pi}{4} - \int_{0}^{\frac{\pi}{4}} \tan x\, dx$

E. $A = \frac{\pi}{4} - \int_{0}^{\frac{a\pi}{4}} \tan x\, dx$

Question 15 The region enclosed by the curve $y=\sin^{-1}\left(\frac{x}{x^2+1}\right)$ and the x-axis from $x=0$ to $x=4$ is rotated about the x-axis. The volume of the solid of revolution is closest to

A. 0.9

B. 1.2

C. 1.5

D. 1.8

E. 2.1

Question 16 Given $f'(x)=\log_e\sqrt{x^2+1}$ and $f(2)=2$, the value of $f(0)$ is closest to

A. 0.7

B. 0.9

C. 1.1

D. 1.3

E. 1.5

Question 17 Given $f(x)=\log_e\left(\frac{a\cos^{-1}x}{b}\right)$, $a,b\in R^+$, the value of $f'\left(\frac{1}{2}\right)$ is

A. $\frac{6b}{\sqrt{3}a\pi}$

B. $-\frac{6}{a\pi}$

C. $\frac{a\pi}{\sqrt{3}b}$

D. $-\frac{2\sqrt{3}}{\pi}$

E. $\frac{6}{\sqrt{3}\pi}$

Question 18 A particle moves in a plane with position $\tilde{r}(t)$ and velocity $\tilde{v}(t) = 2t\tilde{i} - \tilde{j}$, $t \geq 0$.
The straight-line distance from $\tilde{r}(1)$ to $\tilde{r}(2)$ is closest to

A. $2\sqrt{2}$

B. 3

C. $\sqrt{10}$

D. $\sqrt{11}$

E. $2\sqrt{3}$

Question 19 The position of a particle is given by $\tilde{r} = 4.9t\tilde{i} + 4.9\left(\sqrt{3}t - t^2\right)\tilde{j}$ where $t \geq 0$.
The minimum speed of the particle is

A. 4.9

B. 9.8

C. $4.9\left(1+\sqrt{3}\right)$

D. $9.8\sqrt{3}$

E. 19.6

Question 20 A particle moves with constant acceleration. It has an initial velocity of $10\tilde{i}$ m s^{-1}.
It has a velocity of $(-5)\tilde{i}$ m s^{-1} after a displacement of $\tilde{s} = 7.5\tilde{i}$ m.
When the velocity is $(-5)\tilde{i}$ m s^{-1}, the total distance (m) travelled by the particle is closest to

A. 7.5

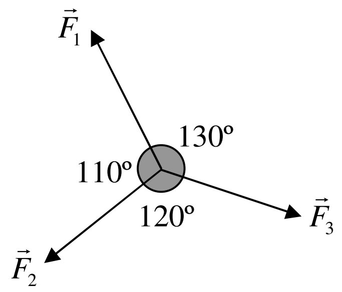
B. 10.0

C. 12.5

D. 15.0

E. 17.5

Question 21 A particle is in **equilibrium** under the action of three forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ (see diagram below).



Which one of the following statements is correct?

A. $\left|\vec{F}_3\right|\cos 110° = \left|F_2\right|\cos 130°$

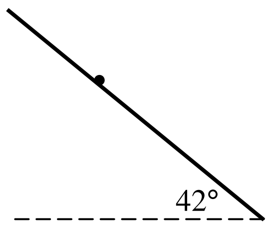
B. $\left|\vec{F}_3\right|\sin 30° = \left|\vec{F}_2\right|\sin 20°$

C. $\left|\vec{F}_3\right|\cos 30° = \left|\vec{F}_2\right|\cos 20°$

D. $\left|\vec{F}_3\right|\sin 50° = \left|\vec{F}_2\right|\sin 70°$

E. $\left|\vec{F}_3\right|\cos 50° = \left|\vec{F}_2\right|\cos 70°$

Question 22 A 0.5-kg particle slides *at constant speed* down a rough plane inclined at $42°$ to the horizontal. The particle exerts a force of F newtons on the plane.



The value of F is closest to

A. 4.9

B. 3.6

C. 3.3

D. 0.5

E. 0.4

SECTION 2 Extended-answer questions

Instructions for Section 2

Answer **all** questions.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m s^{-2}, where $g = 9.8$.

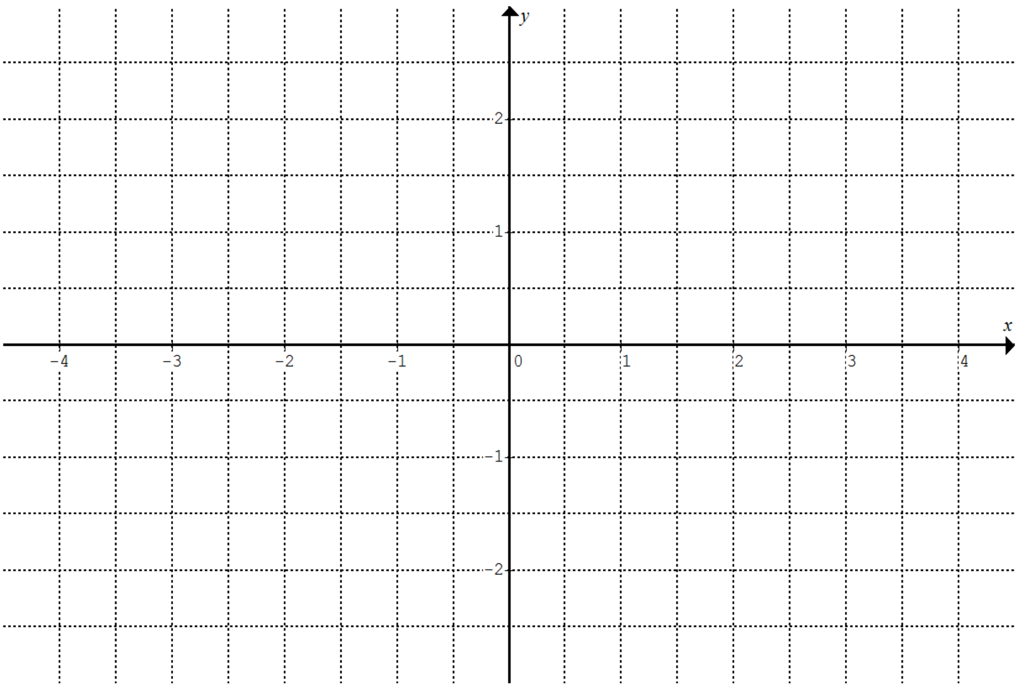
Question 1 Consider relations $x^2 + 4y^2 = 4$ and $4\left(x - \frac{3}{2}\right)^2 - 8y^2 = 1$.

a. The graphs of the two relations intersect at some points. Find the exact coordinates of these points.

3 marks

b. Sketch the graphs of the relations on the same axes. Show and label the coordinates of the axis intercepts and intersections, and the asymptotes with equations.

3 marks



c. Find the equations of the tangents to the graph of $4\left(x-\frac{3}{2}\right)^2-8y^2=1$ at $x=0$. 2 marks

The region enclosed by the tangents in part c and the curve $4\left(x-\frac{3}{2}\right)^2-8y^2=1$ is rotated about the x-axis by $180°$.

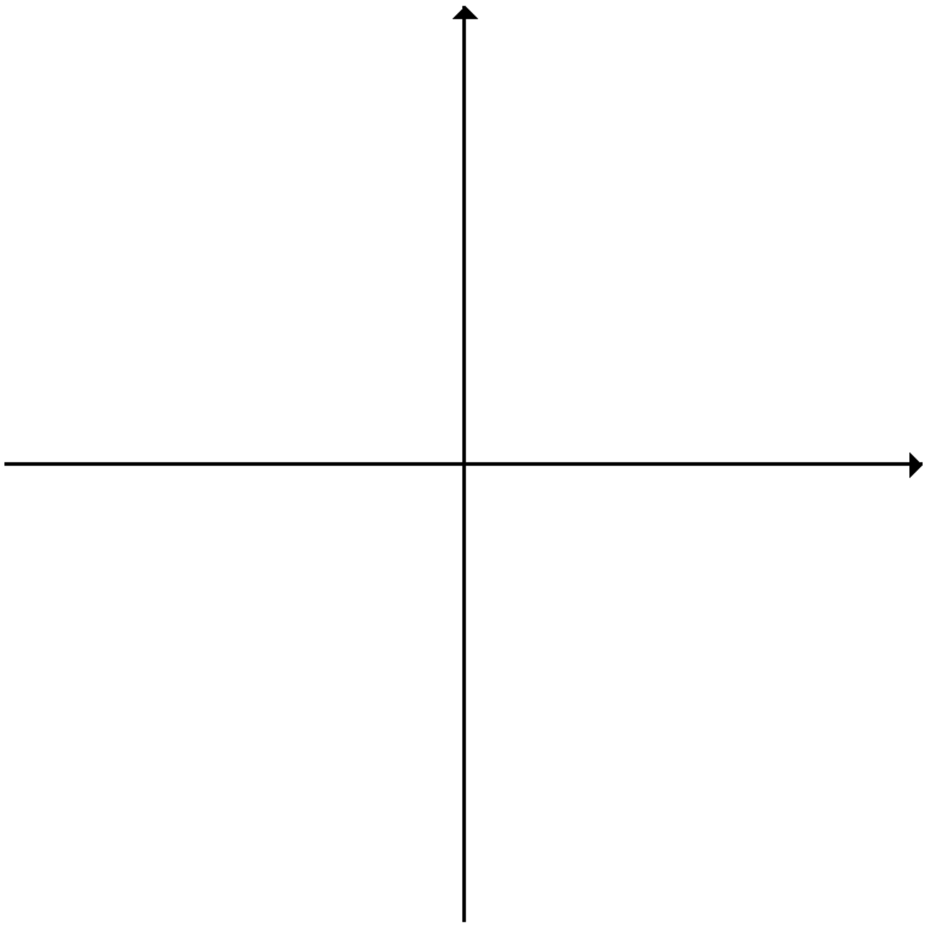
d. Find the exact volume of the solid of revolution. 3 marks

Question 2 The volume of water in a tank is given by $V = \frac{5\pi}{6} - \cos^{-1}\left(h - \frac{\sqrt{3}}{2}\right)$ where V is measured in m^3 and h in metres is the depth of water in the tank.

a. Find the exact values of the maximum depth and maximum volume of the tank. 2 marks

b. Sketch the graph of $V = \frac{5\pi}{6} - \cos^{-1}\left(h - \frac{\sqrt{3}}{2}\right)$.

Label the axes. Show and label the endpoints and point of inflection with their exact coordinates. 3 marks



At time $t = 0$ water is pumped into the empty tank at a rate of 50 litres per minute. t is measured in minutes.

c. Express h in terms of t. 3 marks

d. Find the exact time in minutes required to fill the tank. 1 mark

e. Find the rate of increase of the depth when the water in the tank is $\sqrt{3}$ metres deep. 3 marks

Question 3 Given unit vector $\tilde{i}$ is in the positive x direction, and unit vector $\tilde{j}$ is in the positive y direction. The position of a motorcyclist is given by $\tilde{r}_c(t) = \left(10 - 100\cos\frac{\pi t}{15}\right)\tilde{i} + \left(160 + 150\sin\frac{\pi t}{15}\right)\tilde{j}$ where time $t \geq 0$ and it is measured in seconds.

a. Find the Cartesian equation of the path of the motorcyclist. 2 marks

b. Find the exact time when the motorcyclist first returns to the position $\tilde{r}_c(0)$. 1 mark

c. Find the exact maximum speed of the motorcyclist. 2 marks

A spectator is at position $\underset{\sim}{r}_s = 10\underset{\sim}{i} + \frac{40}{3}\underset{\sim}{k}$.

d i. Find the exact time when the motorcyclist is first closest to the spectator. 3 marks

d ii. Find the exact value of the closest distance. 1 mark

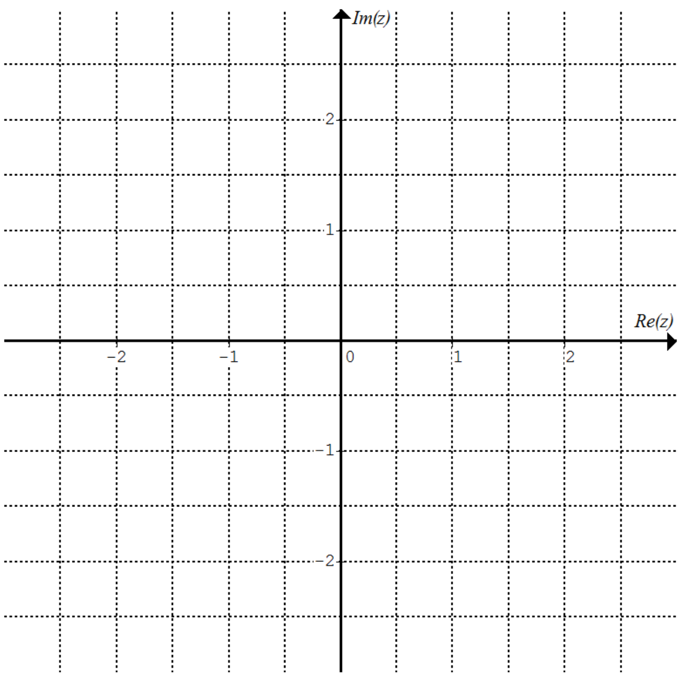
e. Show that the acceleration of the motorcyclist is $\underset{\sim}{a} = k\left(\underset{\sim}{r}_c - \underset{\sim}{r}_0\right)$ where $k \in R$ and $\underset{\sim}{r}_0 = 10\underset{\sim}{i} + 160\underset{\sim}{j}$.
2 marks

Question 4 Consider $z-\dfrac{1}{z}$ where $z=x+yi$.

a. Find the real part and the imaginary part of $z-\dfrac{1}{z}$ in terms of x and y. 2 marks

b i. Find the Cartesian equation of $\left\{z:\ i\,2\,\mathrm{Im}(z)=z-\dfrac{1}{z}\right\}$. 2 marks

b ii. Sketch $\left\{z:\ \left|i\,2\,\mathrm{Im}(z)-\left(z-\dfrac{1}{z}\right)\right|>0\right\}$. 3 marks



c. Find the Cartesian equation of $\left\{z:\ 2\operatorname{Re}(z)=z+\frac{1}{z}\right\}$. 1 mark

d. Simplify $\left\{z:\ \left|i\,2\operatorname{Im}(z)-\left(z-\frac{1}{z}\right)\right|>0\right\}\cap\left\{z:\ 2\operatorname{Re}(z)=z+\frac{1}{z}\right\}$. 1 mark

e. Find the Cartesian equation of $\left\{z:\ 2\operatorname{Re}(z-1-i)=z+\frac{1}{z-1-i}-1-i\right\}$. 1 mark

Question 5 A 1-kg particle is projected vertically upwards with a speed of 20 m s^{-1} at $t = 0$. Air resistance against its motion is $0.01v^2$ newtons where v is the speed of the particle in m s^{-1}.

a. Draw a diagram showing the forces on the particle while it moves upwards. Label the forces. 1 mark

b. Write an appropriate equation of motion by which the maximum height reached by the particle can be calculated. Let the upward displacement from the point of projection be x metres. 1 mark

c i. Use calculus to solve the equation in part b to show $x = 50\log_e\left(\dfrac{13.8}{9.8 + 0.01v^2}\right)$. 3 marks

c ii. Find the maximum height reached by the particle, in metres, correct to 2 decimal places. 1 mark

In the downward motion back to the point of projection, the particle experiences the same air resistance of $0.01v^2$ newtons.

d. Write an appropriate equation by which the speed of the particle after falling x metres can be calculated.

1 mark

e i. Use calculus to solve the equation in part d to show $v^2 = 980\left(1 - e^{-\frac{x}{50}}\right)$.

3 marks

e ii. Find the speed of the particle when it returns to its point of projection, in metres per second, correct to 2 decimal places.

1 mark

f. Find the time taken for the downward motion of the particle, correct to 2 decimal places.

3 marks

End of Exam 2