D

Free Exam
for 2006-15 VCE study design

Engage Education Foundation

Units 3 and 4 Specialist Maths: Exam 1

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Any questions?

Check the Engage website for updated solutions, then email practiceexams@ee.org.au.

Section A – Multiple-choice questions

Question 1

a. For f(x) to be a probability density function, the total area under the curve has to be 1.

$$\int_{-a}^{a}\frac{3}{16}x^2dx = 1$$

$$\left[\frac{1}{16}x^3\right]_{-a}^{a} = 1$$

$$\frac{1}{16}a^3 + \frac{1}{16}a^3 = 1 \text{ [1]}$$

$$a^3 = 8$$

$$a = 2 \text{ [1]}$$

b. Mean $E(\bar{X}) = \mu = E(X) = 0$ (the probability density function is symmetrical) [1/2]

Variance $Var(\bar{X}) = \frac{\sigma^2}{n} = \frac{E(X^2)-E^2(X)}{100} = \frac{\int_{-2}^{2}x^2f(x)dx}{100} = \frac{\int_{-2}^{2}\frac{3}{16}x^4dx}{100} = \left[\frac{3}{80}x^5\right]_{-2}^{2} = \frac{3}{40}\times 2^5 = \frac{3\times 32}{40} = \frac{12}{5}$ [1]

Hence, $\bar{X}\sim N\left(0,\frac{12}{5}\right)$ (due to central limit theorem) [1/2]

Question 2

Let W be the weight of a given individual mice.

$W \sim N(\mu, 196)$ where μ is the true mean of the population

$\bar{W} = \frac{1}{n}\sum W_i \sim N\left(\mu, \frac{256}{n}\right)$ since it is a some of identical independent normal random variables

Thus,

$$Z = \frac{\bar{W}-\mu}{\left(\frac{14}{\sqrt{n}}\right)} \sim N(0,1)$$

$$\Pr(-1.96 < Z < 1.96) = 0.95$$

$\Rightarrow Z \in (-1.96, 1.96)$ with 95% confidence $\left[\frac{1}{2}\right]$

$$\Rightarrow \frac{\bar{W}-\mu}{\left(\frac{14}{\sqrt{n}}\right)} \in (-1.96, 1.96)$$

$$\Rightarrow \mu \in \left(\bar{H} - 1.96\left(\frac{14}{\sqrt{n}}\right), \bar{H} + 1.96\left(\frac{14}{\sqrt{n}}\right)\right)$$

Substituting the given numbers:

$$\mu \in \left(197 - 1.96\left(\frac{14}{10}\right), 197 + 1.96\left(\frac{14}{10}\right)\right)$$

$\mu \in (194, 200)$ with 95% confidence [1/2]

Since the prediction, 176cm is an element of the 95% confidence interval, the prediction cannot be rejected at the 0.05 significance level. [1]

Explanation of significance: if the sampling were to be repeated multiple times, 95% of the times, the confidence interval would contain the **predicted** population mean of 200 g. [1]

Question 3

$$\frac{d}{dx}(x^2y + \log_e y + x) = \frac{d}{dx}(2)$$

$2xy + x^2\frac{dy}{dx} + \frac{1}{y}\times\frac{dy}{dx} + 1 = 0$ good attempt at implicit differentiation [1]

$$\frac{dy}{dx}\left(x^2 + \frac{1}{y}\right) = -1 - 2xy$$

$\therefore \frac{dy}{dx} = \frac{-1-2xy}{x^2+\frac{1}{y}}$ correct answer [1]

Question 4

$P(z) = z^2 + 5z + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 8$

$= \left(z + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{8 * 4}{4}$

$= \left(z + \frac{5}{2}\right)^2 + \frac{7}{4}$ successful completion of the square [1]

$= \left(z + \frac{5}{2}\right)^2 - \left(\frac{\sqrt{7}}{2} i\right)^2$ (other methods also ok, eg. quadratic formula)

$= \left(z + \frac{5}{2} + \frac{\sqrt{7}}{2} i\right)\left(z + \frac{5}{2} - \frac{\sqrt{7}}{2} i\right)$

$= 0$

So, by the null factor theorem,

$z + \frac{5}{2} + \frac{\sqrt{7}}{2} i = 0 \ or \ z + \frac{5}{2} - \frac{\sqrt{7}}{2} i = 0$

$\therefore z = -\frac{5}{2} \pm \frac{\sqrt{7}}{2} i$ both answers correct [2]

Question 5

$area = \left|\int_{-2}^{-1} \frac{7x+1}{x^2+2x-8}\,dx\right|$ correct integral for area [1]

$$\frac{7x+1}{x^2+2x-8} = \frac{7x+1}{(x+4)(x-2)} \equiv \frac{A}{x+4} + \frac{B}{x-2}$$

$7x + 1 \equiv A(x-2) + B(x+4)$

Let $x = -4$

$7 \times -4 + 1 = A(-4-2) + B(-4+4)$

$\therefore A = \frac{9}{2}$

Let $x = 2$

$7 \times 2 + 1 = A(2-2) + B(2+4)$

$\therefore B = \frac{5}{2}$

$\Rightarrow \int_{-2}^{-1} \frac{7x+1}{x^2+2x-8}\,dx = \int_{-2}^{-1} \frac{99}{2(x+4)} + \frac{5}{2(x-2)}\,dx$ correct splitting of fraction [1]

$= \left[\frac{9}{2}\log_e|x+4| + \frac{5}{2}\log_e|x-2|\right]_{-2}^{-1}$ correct integration [1]

$= \left[\frac{9}{2}\log_e|-1+4| + \frac{5}{2}\log_e|-1-2|\right] - \left[\frac{9}{2}\log_e|-2+4| + \frac{5}{2}\log_e|-2-2|\right]$

$= \frac{9}{2}\log_e(3) + \frac{5}{2}\log_e(3) - \frac{9}{2}\log_e(2) - \frac{5}{2}\log_e(4)$

$= 7\log_e(3) - \frac{9}{2}\log_e(2) - \frac{5}{2}\log_e(4)$ answer [1]

Question 6a

$x = \sec^2(t)\ \ and\ y^2 = \tan^2(t)$

Using the trigonometric identity $1 + \tan^2(x) = \sec^2(x)$: [1]

$1 + y^2 = x$

$y^2 = x - 1$

$\therefore y = \pm\sqrt{x-1}$ but $y > 0$ in the provided domain,

so y = $\sqrt{x-1}$ answer [1]

Question 6b

$x = sec^2(t)$ for $t \in \left\{t: 0 \le t \le \frac{\pi}{4}\right\}$

$\sec^2(0) = \left(\frac{1}{\cos(0)}\right)^2 = 1$

$\sec^2\left(\frac{\pi}{4}\right) = \left(\frac{1}{\cos\left(\frac{\pi}{4}\right)}\right)^2 = 2$

Hence $1 \le x \le 2$.

$y = \tan(t)$ for $t \in \left\{t: 0 \le t \le \frac{\pi}{4}\right\}$

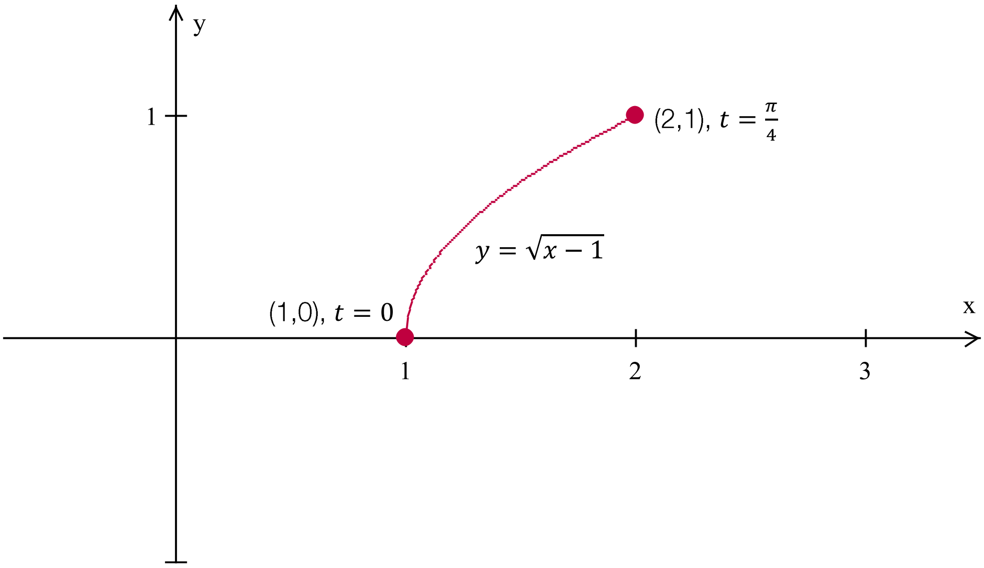
$\tan(0) = 0$

$\tan\left(\frac{\pi}{4}\right) = 1$

Hence $0 \le y \le 1$.

$dom = [1,2]$ and $ran = [0,1]$ answers [2]

Question 6c



graph shape [1], intercept with coordinate and t value [1], closed endpoints with coordinates and t values [1]

Question 7

$cosec\left(\frac{5\pi}{12}\right) = \frac{1}{\sin\left(\frac{5\pi}{12}\right)}$ use of reciprocal circular identities [1]

$\frac{5\pi}{12} = \frac{4\pi}{12} + \frac{\pi}{12} = \frac{4\pi}{12} + \left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{3} + \frac{\pi}{3} - \frac{\pi}{4} = \frac{2\pi}{3} - \frac{\pi}{4}$

$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\therefore \sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$ correct selection of double angle formula [1]

$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

$\therefore cosec\left(\frac{5\pi}{12}\right) = \frac{4}{\sqrt{6}+\sqrt{2}}$

$= \frac{4(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}$

$= \frac{4(\sqrt{6}-\sqrt{2})}{6-2}$ working leading to correct answer [1]

$= \frac{4(\sqrt{6}-\sqrt{2})}{4} = \sqrt{6} - \sqrt{2}$, as required.

Question 8

$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{5}\right)\right) = \frac{5}{25+x^2}$

Let $u = \tan^{-1}\left(\frac{x}{5}\right), \frac{du}{dx} = \frac{5}{25+x^2}$ correct substitution [1]

$\int \frac{\tan^{-1}\frac{x}{5}}{25+x^2}dx = \frac{1}{5}\int \tan^{-1}\frac{x}{5} * \frac{5}{25+x^2}dx$

$= \frac{1}{5}\int u\frac{du}{dx}dx$

$= \frac{1}{5}\int udu$ successful working [1]

$\frac{1}{5} \times \frac{1}{2}u^2 + c$ where $c \in \mathbb{R}$

$\therefore \int \frac{\tan^{-1}\frac{x}{5}}{25+x^2}dx = \frac{1}{10}\left(\tan^{-1}\frac{x}{5}\right)^2 + c$ answer [1]

Question 9

From formula sheet:

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\Rightarrow \cos^2(x) = \frac{\cos(2x)+1}{2}$ and $\sin^2(x) = \frac{1-\cos(2x)}{2}$

$\therefore \int_0^\pi cos^2(x)sin^2(x)dx = \int_0^\pi \left(\frac{\cos(2x)+1}{2}\right)\left(\frac{1-\cos(2x)}{2}\right)dx$

$= \frac{1}{4}\int_0^\pi 1 - \cos^2(2x)\,dx$ correct use of double-angle formula [1]

$\cos(4x) = 2\cos^2(2x) - 1 \Rightarrow \cos^2(2x) = \frac{1+\cos(4x)}{2}$

$\therefore \frac{1}{4}\int_0^\pi 1 - \cos^2(2x)\,dx = \frac{1}{4}\int_0^\pi 1 - \frac{1+\cos(4x)}{2}dx$ second correct use of double-angle formula [1]

$= \frac{1}{8}\int_o^\pi 1 + \cos(4x)\,dx$

$= \frac{1}{8}\left[x + \frac{1}{4}\sin(4x)\right]_0^\pi$ correct integral found [1]

$= \frac{1}{8}\left[\left(\pi + \frac{1}{4}\sin(4\pi)\right) - \left(0 + \frac{1}{4}\sin(0)\right)\right]$

$= \frac{1}{8}(\pi + 0 - 0)$

$= \frac{\pi}{8}$ answer [1]

Question 10a

$x = t^3 \Rightarrow t = x^{\frac{1}{3}}$

$y = \log_e(t) = \log_e(x^{\frac{1}{3}})$ answer [1]

Question 10b

$\boldsymbol{v}(t) = \frac{d}{dx}(\boldsymbol{r}(t))$

$= 3t^2\boldsymbol{i} + \frac{1}{t}\boldsymbol{j}$ derivative [1]

$speed = |\boldsymbol{v}(t)|$

$= \sqrt{(3t^2)^2 + \left(\frac{1}{t}\right)^2}$ [1]

$= \sqrt{9t^4 + \frac{1}{t^2}}$ answer [1]

Question 10c

Speed is a minimum when $\frac{d}{dx}(|\boldsymbol{v}(t)|) = 0$

stating derivative of speed should be zero for speed to be at a minimum [1]

$\frac{d}{dx}\left(\sqrt{9t^4 + \frac{1}{t^2}}\right) = \frac{1}{2}(9t^4 + t^{-2})^{-\frac{1}{2}}(36t^3 - 2t^{-3}) = 0$ evaluation of derivative [1]

$\left(9t^4 + \frac{1}{t^2}\right)^{-\frac{1}{2}} \neq 0$ due to the negative power

$36t^3 - 2t^{-3} = 0$ by the null factor theorem.

$36t^3 = \dfrac{2}{t^3}$

$t^6 = \dfrac{1}{18}$

$\therefore t = \left(\frac{1}{18}\right)^{\frac{1}{6}}$ answer [1]